# BULLETIN

OF THE

# AMERICAN MATHEMATICAL SOCIETY

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J. R. KLINE
VIRGIL SNYDER
J. L. DOOB
H. W. KUHN
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J. A. SHOHAT
MORGAN WARD

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. DAVIS
. KLINE
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J. L. DOOB
H. W. KUHN
G. A. HEDLUND
J. A. SHOHAT
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R. G. D. Richardson

ROLAND GEORGE DWIGHT RICHARDSON

By order of the Council of the Society,
this volume is dedicated to

ROLAND GEORGE DWIGHT RICHARDSON

in appreciation of his devoted service to the
Society especially during his term of
office as Secretary from
1921 to 1940

CALCUTTA UNIVERSITY CENTRAL LIBRARY

# BULLETIN
# OF THE
# AMERICAN MATHEMATICAL SOCIETY

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## MATHEMATICAL REVIEWS OFFERS A READING MACHINE FOR MICROFILM

When the microfilm service of *Mathematical Reviews* was introduced, it was realized that its usefulness would depend to a large extent upon the availability of reading machines. The Committee on Scientific Aids to Learning, a committee of National Research Council, is promoting, among other things, the use of microfilm. As a result of its efforts, a reading machine is being manufactured which will be sold at a retail price of $32.00. A grant from the Committee on Scientific Aids to Learning has made it possible for *Mathematical Reviews* to distribute a limited number of these machines on the following terms.

**Terms of offer.** A reading machine for microfilm will be given—as long as the available supply lasts—to any person who has paid his subscription, *at the rate to which he is entitled*, to *Mathematical Reviews* in advance for three years beginning January, 1941. The person who receives a reading machine must pay express charges and import duty, if any, from Buffalo, New York. Until January 1 this offer was made only to the present subscribers to *Mathematical Reviews*. Since that date, however, it is extended to new subscribers also. Because only a limited number of machines is available, anyone who desires one should place an order early.

The purpose of the Committee on Scientific Aids to Learning in affording the financial support making possible the distribution of the readers in connection with subscriptions to *Mathematical Reviews* was jointly the promotion of microfilm and aid to *Mathematical Reviews*.

**History of Students Microfilm Reader.** The history of the reader, known as the Students Microfilm Reader, begins in the fall of 1939. At that time an advisory group on microphotography to the Committee on Scientific Aids to Learning, composed of Mr. Keyes D. Metcalf (chairman), Director of the Harvard University Library, Professors Ralph D. Bennett and Ernest I. Huntress of Massachusetts Institute of Technology, Dr. Vernon D. Tate of the National Archives, and Dr. Irvin Stewart (ex officio), Director of the Committee on Scientific Aids to Learning, was requested to consider the possibilities of designing and making available a simple, inexpensive microfilm reading machine for the use of the individual scholar. The problem was discussed at length and did not appear insoluble. Several designs were suggested, and three models were constructed. Each of these models was thoroughly tested both in the laboratory and in actual use; a set of plans and specifications embodying the final accepted design was prepared for distribution to manufacturers specializing in equipment of this type.

Bids for the manufacture of the reading machine were received from a number of companies, and the Spencer Lens Company was authorized to build a pilot model. It was built, tested and inspected, and the Committee on Scientific Aids to Learning has now signed a contract for a number of these machines. In addition, they will be placed on the market by the Spencer Lens Company.

**Purpose of Students Microfilm Reader.** Emphasis has been placed throughout on suitability for the purpose for which the reader was designed: simplicity and low cost. No claims are made for extreme convenience, beauty, ready-portability, or universality.

The benefits of microphotography in assembling research data of all types are well known. Facilities exist in the principal libraries, archives, and other institutions for the reproduction of their holdings. Many individuals have secured equipment and microphotographed extensive files. The greatest difficulty to date has not been to secure material on microfilm but rather to obtain adequate utilization equipment. In sponsoring the development of a simple, inexpensive microfilm reader, the Committee on Scientific Aids to Learning has considered solely the requirements of the individual. Excellent equipment developed primarily for commercial and library use is already available on the market. In most cases, it is entirely satisfactory (except for price) for individual use. The Students Microfilm Reader is not intended to compete with any existing reader equipment. It was developed specifically to permit the individual scholar or scientist to utilize in his own study or laboratory microphotographic copies which he may have made personally or procured from one of the existing sources of supply.

G. B. Price

## THE OCTOBER MEETING IN NEW YORK

The three hundred seventy-second meeting of the American Mathematical Society was held at Columbia University on Saturday, October 26, 1940. The attendance included the following one hundred sixty-eight members of the Society:

Leon Alaoglu, C. B. Allendoerfer, Warren Ambrose, R. G. Archibald, K. J. Arrow, M. F. Becker, E. E. Betz, Gertrude Blanch, R. P. Boas, H. F. Bohnenblust, W. M. Bond, Samuel Borofsky, D. G. Bourgin, C. B. Boyer, A. T. Brauer, H. C. Brodie, A. A. F. Brown, A. B. Brown, Jewell H. Bushey, J. H. Bushey, S. S. Cairns, W. D. Cairns, J. W. Calkin, A. D. Campbell, E. W. Cannon, Achille Capecelatro, J. A. Clarkson, T. F. Cope, Byron Cosby, Richard Courant, H. B. Curry, M. D. Darkow, Norman Davids, M. R. Demers, C. H. Dix, Jesse Douglas, J. E. Eaton, M. L. Elveback, H. T. Engstrom, Paul Erdös, C. J. Everett, G. M. Ewing, J. M. Feld, Aaron Fialkow, E. J. Finan, G. E. Forsythe, R. M. Foster, J. S. Frame, K. O. Friedrichs, Orrin Frink, T. C. Fry, G. N. Garrison, H. P. Geiringer, B. P. Gill, J. W. Givens, Saul Gorn, S. L. Greitzer, H. M. Griffin, D. W. Hall, F. C. Hall, N. A. Hall, Hans-Karl Hammer, D. C. Harkin, Philip Hartman, M. H. Heins, Edward Helly, J. G. Herriot, E. H. C. Hildebrandt, Einar Hille, Banesh Hoffmann, Lulu Hofmann, T. R. Hollcroft, E. A. Hoy, R. Y. Hoy, E. M. Hull, Seymour Jablon, R. L. Jeffrey, S. A. Joffe, R. A. Johnson, Edward Kasner, L. S. Kennison, J. R. Kline, R. F. Koch, B. O. Koopman, Arthur Korn, A. W. Landers, M. K. Landers, Solomon Lefschetz, C. A. Lester, Howard Levi, Madeline Levin, Marie Litzinger, L. H. Loomis, E. R. Lorch, A. N. Lowan, Brockway McMillan, C. C. MacDuffee, H. F. MacNeish, Dorothy Maharam, P. T. Maker, A. J. Maria, D. H. Maria, Walther Mayer, A. E. Meder, H. L. Mintzer, Don Mittleman, E. C. Molina, Deane Montgomery, Marston Morse, Z. I. Mosesson, C. A. Nelson, Philip Newman, Oystein Ore, J. C. Oxtoby, Gordon Pall, W. F. Penney, B. J. Pettis, E. L. Post, Walter Prenowitz, C. J. Rees, M. S. Rees, H. J. Riblet, R. G. D. Richardson, J. F. Ritt, I. F. Ritter, H. E. Robbins, S. L. Robinson, Benjamin Rosenbaum, J. E. Rosenthal, Arthur Sard, Abraham Schwartz, I. E. Segal, Hyman Serbin, Stephan Serghiesco, L. W. Sheridan, Seymour Sherman, Max Shiffman, James Singer, Abraham Sinkov, L. L. Smail, P. A. Smith, Abraham Spitzbart, Wolfgang Sternberg, C. N. Stokes, M. H. Stone, W. C. Strodt, Alvin Sugar, M. M. Sullivan, W. G. Swope, J. D. Tamarkin, J. M. Thomas, Arthur Tilley, E. W. Titt, C. B. Tompkins, W. J. Trjitzinsky, J. W. Tukey, Annita Tuller, G. C. Webber, Louis Weisner, M. E. Wells, William Wernick, A. P. Wheeler, O. L. Wheeler, A. L. Whiteman, H. P. Wirth, Jack Wolfe, Leo Zippin, Antoni Zygmund.

The meeting opened Saturday morning with two sessions: Analysis, Professors Einar Hille and J. R. Kline presiding; and Algebra and Geometry, Professor S. S. Cairns presiding. At the afternoon session, at which Vice President T. C. Fry presided, Professor H. B. Curry gave an address entitled *Some aspects of the problem of mathematical rigor.*

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1 to 8 were read before the section for Analysis; those numbered 9 to 13 before the section for Algebra and Geometry; and those numbered 14 to 21 were read by title.

1. G. E. Forsythe: *On Riesz summability methods of order r, for* $\mathcal{R}(r) < 0$. Preliminary report. (Abstract 46-11-488.)

2. Jenny E. Rosenthal: *Generating functions and properties of certain orthogonal polynomials*. II. (Abstract 46-11-501.)

3. I. E. Segal: *The space of Besicovitch almost periodic functions.* (Abstract 46-11-505.)

4. A. B. Brown: *On the number of independent parameters.* (Abstract 46-11-485.)

5. P. T. Maker: *A topological characterization of monotone functions.* Preliminary report. (Abstract 46-11-496.)

6. E. R. Lorch: *The integral representation of weakly almost-periodic transformations in reflexive vector spaces.* (Abstract 46-7-361.)

7. Einar Hille: *A class of differential operators of infinite order.* II. (Abstract 47-1-31.)

8. L. H. Loomis: *The decomposition of meromorphic functions into rational functions of univalent functions.* (Abstract 46-11-495.)

9. Saul Gorn: *Homomorphisms and modular functionals.* (Abstract 46-11-489.)

10. N. A. Hall: *The solution of the quadrinomial equation in infinite series by the method of iteration.* (Abstract 46-11-490.)

11. A. T. Brauer: *On a problem of partitions.* (Abstract 46-11-484.)

12. Aaron Fialkow: *The conformal theory of a hypersurface.* (Abstract 46-11-487.)

13. William Wernick: *Distributive properties of set operators.* (Abstract 46-11-506.)

14. R. S. Martin: *Minimal positive harmonic functions.* (Abstract 47-1-37-*t*.)

15. Francisco Perez: *A generalization of the theory of invariant factors and similar matrices.* (Abstract 46-11-498-*t*.)

16. L. B. Robinson: *A functional equation with negative exponent.* (Abstract 46-11-500-*t*.)

17. L. B. Robinson: *A functional equation with a singular line.* (Abstract 46-11-499-*t*.)

18. Peter Scherk: *On real closed curves of the order $n+1$ in the projective n-space.* Preliminary report. (Abstract 46-11-502-*t*.)

19. H. S. Wall: *Real power series bounded in the unit circle.* (Abstract 46-9-440-*t*.)

20. William Wernick: *Functional dependence in the calculus of propositions.* (Abstract 46-11-507-*t*.)

21. Hassler Whitney: *The mappings of a 3-complex into a space with vanishing fundamental group.* (Abstract 46-11-508-*t*.)

T. R. HOLLCROFT,
*Associate Secretary*

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## THE NOVEMBER MEETING IN DETROIT

The three hundred seventy-third meeting of the American Mathematical Society was held at Wayne University, Detroit, Michigan, on Friday and Saturday, November 22–23, 1940. Ninety-five persons registered including the following sixty-nine members of the Society:

Max Astrachan, J. V. Atanasoff, W. L. Ayres, I. A. Barnett, E. F. Beckenbach, R. A. Beth, W. M. Borgman, H. K. Brown, R. S. Burington, R. V. Churchill, C. J. Coe, A. H. Copeland, Max Coral, C. C. Craig, J. M. Dobbie, M. J. Dunford, Ben Dushnik, W. D. Duthie, P. S. Dwyer, Samuel Eilenberg, A. G. Fleiger, K. W. Folley, Morris Friedman, H. H. Goldstine, V. G. Grove, O. G. Harrold, B. A. Hausmann, G. E. Hay, E. D. Hellinger, T. H. Hildebrandt, J. D. Hill, C. C. Hurd, F. B. Jones, Samuel Kaplan, Wilfred Kaplan, D. K. Kazarinoff, J. L. Kelley, E. D. McCarthy, A. V. Martin, D. D. Miller, E. W. Miller, D. C. Morrow, S. B. Myers, A. L. Nelson, C. J. Nesbitt, J. W. Odle, E. W. Paxson, E. C. Pixley, V. C. Poor, J. E. Powell, G. Y. Rainich, E. D. Rainville, C. E. Rickart, Arthur Rosenthal, J. B. Rosser, W. T. Scott, H. A. Simmons, B. M. Stewart, Otto Szász, Leonard Tornheim, A. W. Tucker, W. A. Vezeau, H. S. Wall, I. T. Weinberg, E. T. Welmers, K. P. Williams, L. A. Wolf, M. C. Wolf, J. W Zimmer.

The meeting opened Friday afternoon with a session for short papers, followed by an address by Professor J. B. Rosser of Cornell University entitled *Many-valued logics.* On Saturday morning Professor H. S. Wall of Northwestern University delivered an address under the title *Some recent developments in the theory of continued fractions*, and the meeting closed with a second session for short papers. Professor A. L. Nelson presided on Friday afternoon and Professor K. P. Williams on Saturday morning.

On Friday evening a dinner was held at the Belcrest Hotel with sixty-six persons attending. Professor A. L. Nelson presided at the dinner and Dean W. W. Whitehouse of Wayne University welcomed the Society in behalf of the institution. At the Saturday session Professor E. F. Beckenbach presented a resolution of appreciation to Wayne University and its Department of Mathematics for their hospitality.

The titles of papers read at this meeting follow, papers 1–6 being read Friday afternoon and 7–10 Saturday morning. Papers 11–13, whose abstract numbers are followed by the letter *t*, were read by title. Paper 2 was read by Professor Miller, 3 by Professor Wall, 4 by Dr. Scott.

1. K. W. Folley: *A property of a simply ordered set.* (Abstract 47-1-23.)

2. Ben Dushnik and E. W. Miller: *On the dimension of a partial order.* (Abstract 47-1-89.)

3. W. T. Scott and H. S. Wall: *Linear manifolds of Hausdorff means.* (Abstract 46-11-504.)

4. W. T. Scott and H. S. Wall: *A geometrical method in the theory of continued fractions.* (Abstract 46-11-503.)

5. G. Y. Rainich: *Postulates for fields.* Preliminary report. (Abstract 47-1-7.)

6. A. W. Tucker: *Barycentric mappings.* (Abstract 47-1-110.)

7. F. B. Jones: *Totally discontinuous linear functions whose graphs are connected.* (Abstract 47-1-93.)

8. J. W. Odle: *Non-separating and non-alternating transformations modulo a family of sets.* (Abstract 47-1-102.)

9. Samuel Eilenberg: *Linear measure and convexity.* (Abstract 47-1-22.)

10. H. A. Simmons: *Maximum numbers associated with a symmetric Diophantine equation in n reciprocals.* (Abstract 47-1-86.)

11. A. N. Milgram: *Extensible and inextensible decompositions.* Preliminary report. (Abstract 47-1-96-*t*.)

12. M. M. Day: *Reflexive Banach spaces which cannot be made uniformly convex.* (Abstract 46-9-396-*t*.)

13. M. M. Day: *Some more uniformly convex spaces.* (Abstract 46-11-486-*t*.)

W. L. AYRES,
*Associate Secretary*

# BOOK REVIEWS

*Advances and Applications of Mathematical Biology.* By N. Rashevsky. Chicago, University Press, 1940. 18+214 pp. $2.00.

As the title indicates, the book under discussion deals with the more recent progress made by Rashevsky and his group in the systematization of what they have termed physico-mathematical biology. The problems with which the author is primarily concerned are those already delineated in his fundamental work published about two years ago, viz.: (a) cellular metabolism and growth, (b) conduction of nerve impulses and (c) neuropsychological reactions. In the present volume one finds that the analytical method first employed has been somewhat simplified and additional examples are given of the agreement between actual observations and some of the conclusions derived from mathematical reasoning. In the case of cellular diffusion, for example, the author has succeeded in developing formulations which are not restricted by the postulate that the cells are spherical. The elimination of this restriction obviously opens the way to a more general application of the author's rationalizations. Thus, he is enabled to outline with greater precision his concepts regarding the effects of diffusion forces on cell division.

This and the preceding book will probably interest the mathematician mainly because they reveal the kinds of important and vital problems that can be tackled with rather simple analytical tools. The biologist will undoubtedly be stimulated by the fresh ideas emerging from some of the mathematical developments, even though the unverifiable nature of some of the postulates will for the present hinder the experimental or observational evaluation of all the deductions that can be reached.

ANTONIO CIOCCO

*Sur les Valeurs Exceptionnelles des Fonctions Méromorphes et de Leurs Dérivées.* By Georges Valiron. (Actualités Scientifiques et Industrielles, no. 570.) Paris, Hermann, 1937. 55 pp.

One of the outstanding methods of the present century in the theory of functions of a complex variable is that of normal families of functions, a method which has well known applications to the study of convergence of sequences of analytic functions, the distribution of functional values of functions analytic or meromorphic in a circle or in the entire finite plane, conformal mapping, the theorems of Picard, Landau, Schottky, and so on. Each new condition for normality of a sequence of functions has yielded new information on one or more of these topics.

At least two decades ago, Montel conjectured that a family of functions $f(z)$ analytic in a region is normal there provided $f(z)$ fails to assume the value zero and $f'(z)$ fails to assume the value unity. This conjecture was discussed at least orally by many mathematicians, and was finally proved in 1935 by Miranda. Miranda had been preceded by Bureau, who obtained some related theorems by the use of the Nevanlinna theory of meromorphic functions but who did not establish Montel's conjecture. Miranda's work was based also on the Nevanlinna theory and on Bureau's results. Valiron subsequently introduced the direct methods that he had previously employed in the study of the theorems of Picard and Borel to obtain wide generalizations of Miranda's theorem.

The fascicle before us is devoted to a systematic exposition of this body of material, together with many new generalizations and complements. There are close connections with the theorems of Picard, Landau, Schottky, and Bloch. The proofs are far more than pure existence proofs, for they involve specific inequalities of a numerical nature on the modulus of the functions $f(z)$.

The treatment is clear, pleasing, suggestive—an admirable exposition of a field of current interest and importance. May there soon be made in this country systematic provision for the encouragement of the writing of similar essays and for their publication!

J. L. WALSH

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*Complex Variable and Operational Calculus with Technical Applications.* By N. W. McLachlan. Cambridge, University Press, 1939. 11+355 pp. $6.50.

Since the time of Heaviside several books on operational calculus have appeared on the market. The majority of them deal almost exclusively with the electrical applications. This is due, no doubt, to its origin in the pioneer work of Heaviside. With very few exceptions these books treat the subject in a formal and heuristic manner leaving much to be desired from the standpoint of mathematical rigor. Several mathematicians have helped to place the subject on a more substantial basis and thereby extend its usefulness. The author has based his treatment of the subject on the Laplace transform and the Mellin inversion theorem in line with modern developments. Except in a few places a sufficient degree of rigor has been maintained throughout the book to meet the needs of the engineer and applied mathematician.

The book is divided into three parts and an appendix. Part I is a treatment of sections of complex variable theory with a view to applications. The second part deals with the operational calculus. In the third part we find the applications to many types of problems encountered in practice. This is followed by an appendix dealing with some theoretical questions and an extensive bibliography.

The first chapter plunges the reader into the theory of complex variables. No time is wasted before the introduction of singular points. An essential singularity is defined as a pole of infinite order which is rather loose. The principal part of the Laurent expansion provides a clear as well as a rigorous definition of the singularities of a single valued function, but this is postponed until the second chapter. This is an example where suggestive language does not lead to clarity but where rigor does not imply a lack of vigor. A few slight lapses like this one mar an otherwise admirable presentation. Branch points receive a very good treatment from the point of view of their use in the applications. The Cauchy-Riemann test for analyticity is completely left out. No doubt, all the functions arising in applications are analytic, but the reader ought to be provided with this important criterion. Integration is taken up in the second chapter. Cauchy's theorem is stated without proof. The Cauchy integral formula, Taylor's theorem and the Laurent expansion round out the chapter. Integrals indefinitely close together and infinitesimal gaps in the contours provide another example of loose but suggestive language.

A very thorough but not always rigorous treatment of the calculus of residues and contour integration around branch points is given in chapter three. This leads up to the treatment of the Bromwich-Wagner integral in the next chapter. By the use of Jordan's lemma and deformation of the contour several important cases of the Bromwich integral receive a clear and thorough treatment. The author closes the fourth chapter with some very neat methods to obtain approximate values of integrals needed in the sequel.

Three of the most useful functions are the Gamma, Bessel, and Error functions. In the sixth chapter an adequate treatment of these functions is provided as well as a discussion of asymptotic expansions and analytic continuation. When branch points occur the treatment of the Bromwich integral is very fussy and the author has treated it with some care. A short chapter on differentiation under the integral sign and other iterated limit problems closes the first part of the book.

Part II introduces the operational calculus on the basis of the $p$-multiplied form of the Laplace transform and Mellin inversion theorem. The conditions under which the inversion is possible are narrow but suitable enough for the purpose at hand. The proof is relegated to the appendix. The term "operational form" is used instead of the more customary word "transform." Although the operational forms of the derivatives of functions with arbitrary initial conditions are derived, very little use is made of them except in the case where the function and all its derivatives vanish when $t$ is zero. The shift operator is neatly applied to obtain Fourier expansions in special cases. A treatment of ordinary differential equations with constant coefficients, Heaviside's unit function, and impulses closes the second part.

The technical problems treated in the third part are too numerous to mention. A feature here is the thorough treatment of several problems with the attainment of numerical results. This is after all the final aim of the technologist. The problems involving partial differential equations are not as clearly stated as those connected with ordinary derivatives.

A final criticism of omission must be made. The central theorem known as the "faltung" or convolution theorem is not even treated as a step-child. This theorem is known to the engineer as Borel's theorem. Under this name it receives only passing attention. The publication of this book is a step in the right direction and the author is to be congratulated for having written a book which will be very valuable to the mathematical technologist.

SAMUEL SASLAW

*Théorie Mathématique du Bridge à la Portée de Tous.* By Émile Borel and André Chéron. Paris, Gauthier-Villars, 1940. 17+392 pp. Fr. 175.

Here is a detailed study of shuffling, card distributions, the finesse, and other points of the game of bridge by an eminent mathematician and an authority on bridge. A survey of the context naturally enough begins with their initial discussion of the shuffling of the deck.

Shuffling of type A is that generally used in which the deck is separated into two approximately equal parts and then dove-tailed together in small alternating packets. Let a sequence be defined as two adjacent cards in an initial order. If the average number of sequences in the deck broken by one operation of $A$ is $S$ and $N$ is the number of such operations, then $NS$ greater than 150 will result in a well shuffled deck. Another test of the adequacy of shuffling is the fact that the mean number of unicolor sequences (two adjacent cards of the same color) is 12. Shuffling of type $B$: A packet of $a_1$ cards is taken off the top of the deck. Then $a_2$ more are placed on top of the $a_1$, $a_3$ more at the bottom, etc. In terms of breaking up sequences, both methods can give satisfactory results, but $B$ more slowly than $A$.

The authors' conclusion in this first chapter on card shuffling is that elementary shuffling of a deck of cards in the time allotted while another person deals the second deck, produces effectively a random placement of the individual card. It is to be noted that far short of this produces random composition of the four hands, i.e., a random residual of the ordinal position of a card modulo 4. In fact, it is observed by the authors that simply gathering up the tricks at the end of play in a jumbling fashion vitiates the most grave consequences of the absence of shuffling.

They present an interesting method of calculating the probability of a certain side (côté) having a given 26 cards (p. 56) and this is extended (p. 62 ff.) to calculating the probability of the four hands being of specified compositions. The method uses a small table of values constructed for the purpose and the probability of a single combination such as the 4333, 3433, 3343, 3334 distribution at specified positions. This is called the "method of coefficients." Chapters one and two deal with the a priori probabilities and mathematical expectations of the hands and sides. Seventeen tables are included in this part. A sample table is 24 which gives for specified numbers of cards of a suit in a hand the probability that a blank suit, blank suit or singleton, blank suit or singleton or doubleton are present in at least one of the adversaries' hands. The authors point out that some of the tables probably have results counter to the usual notions of what is believed by card players. A case in point is illustrated in Table 25 displaying the percentage frequency of non-occurrence of "accidents"—that is, blank suit or singleton. The accident is the rule for a side (two hands) rather than the exception.

There is a study of the defensive value of hands against bids of the opposition. From a study of the tables on strengths of long-suited hands for offensive and defensive games the conclusion is obtained that long-suited hands are very strong on the offense and very weak on the defense. On the other hand, the possessor of high cards without a long suit is better on the defensive. Chapter 3 also includes many illustrative situations showing how many tricks on an average the first lead (entame) from a suit such as *KQ*7xxx will lose the leading side.

It is emphasized that the abnormal distribution of one suit, for example, a hand with 9 spades, has no effect in giving an expectation of a consequent abnormal distribution of one of the other suits (p.

159). On the other hand, the "law of attraction" between long and short suits and its converse the "law of repulsion" between longs (and shorts) are real aids in, say, locating a given card in two hidden hands. Thus if west has indicated a long suit in diamonds and five spades including a queen are in the two hidden hands, east will have on the average a longer suit in spades than west and consequently the queen of spades more often than west. Exact probabilities of such situations are given in Tables 72 to 101.

Many of the tables of the book are followed with brief summarizing statements which reduce their implications to a working "rule of thumb." For instance the summary of one table (111 bis) reads: "The bidder ought to run a practically unlimited risk (in number of tricks set—with probability of success equal to $\frac{1}{2}$) in order to try to make a bid of game or slam—with the two following exceptions. If it is a question of a game bid redoubled, or of a game bid doubled when the bidder alone is vulnerable, the failure ought never to exceed three tricks." There is considerable discussion of the conditions under which to choose the safe or risky line of play based on the mathematical expectation of the two methods. The question of whether or not to double and redouble is similarly taken up.

The notes comprising the last 128 pages discuss with more refinement some of the questions raised in the book proper. Note 1 considers the theoretical displacements of points in a line segment by random interchange of randomly chosen segmentations of the line. This is then compared to the similar but discontinuous case of shuffling cards. The observation is made that the displacements (i.e., shuffling) are in actuality better than that of the idealized cases studied. For one thing, instead of an independent sequence of "elementary operations"—choosing two random points and interchanging the three segments—in shuffling a deck of cards by the usual method of half the deck in each hand, the two points are replaced by many points. There is a consequent diminishing of the number of sub-segments (cards) not displaced, for the same total number of division points used.

Note 2 goes into a detailed study of the method used in enumerating the permutations of a given set of four partitions of 13. Thus the four hands might be $AAAB$ where $A$ is the hand (3334) and $B$ is (4441). From these values are obtained the total probability of a given designated deal. Perhaps the most interesting idea to the reader familiar with probability concepts is that of "virtual" probability discussed in Note 3. For the case of dependent probabilities the product rule holds for virtual probabilities of the separate events, as it holds for real probabilities of independent events. The specific situation is: Consider $N$ objects of $K$ types so that $N=A_1+A_2+\cdots+A_K$. Choose by chance $n$ of them. The probability that the composition of $n$ is $a_1, a_2, \cdots, a_K$ of the $K$ types is given by

$$P_{a_1, a_2, \cdots, a_K} = p_1 p_2 \cdots p_K$$

where

$$p_i = C_{A_i}^{a_i}/(C_N^n)^{A_i/N}.$$

$p_i$ is the "virtual" probability that $a_i$ of the $A_i$ objects will be selected. This idea is applied to the selection of bridge hands and is the means of easily calculating through tables (114, 115) the real probability of one or more hands being of specified composition.

Note 4 gives a short discussion on the theory of finesse and on the "squeeze" play, with the observation that actual play generally amounts to choices between the advantages and disadvantages of the finesse and the squeeze play for the hands under consideration. A lengthy discussion with illustrations on the variation of probabilities in the course of play is the province of Note 6. The new probabilities are of course obtained by the use of Bayes' theorem. From the bidding and one or two rounds of play, certain restrictions can be deduced on the distribution of residue of a given suit. For purposes of finesse, the probabilities pertaining to positions of certain high cards must then be revalued as just indicated.

One of the most interesting tables is 129 which lists in order of decreasing frequency the distribution possibilities of the two hidden hands after the dummy has been displayed. The order is arrived at by a method explained in Note 7 as the "complement to the method of coefficients" mentioned previously. Some common errors of reasoning due mostly to misapplications or non-applications of Bayes' formula are elucidated in Note 8. The use of "psychological" probabilities for complete use of the formula is also given here. The final note of the book deals at length with a problem of finesse which again involves psychological probabilities, i.e., how the individual player reacts to given situations (e.g., can he be depended upon to cover the queen with the king of the suit, or to play the lowest card when following suit with a weak card), as well as information obtained from the cards already played.

It may be said that the book is written with the logically minded rather than the mathematically equipped person continually in view. The 134 tables involve a prodigious amount of computation. The fact that the calculating machine with automatic retransmission of the product for a second multiplication was used instead of factorial and logarithm approximations, and that all probabilities were computed in at least two different ways, leads the authors to make highly confident statements as to the accuracy of the figures.

The reviewer's reaction to a first reading of this work was mainly a sense of increased appreciation of the fundamentals and potentialities of the game—much as one gets from a study of a good chess book. Further study of some of the vital points would undoubtedly save many a trick and generally improve the game of most of us.

JOSEPH A. GREENWOOD

*Introduzione al Pensiero Matematico.* By Friedrich Waismann. Translated by Ludovico Geymonat. (Biblioteca di Cultura Scientifica, vol. 111.) Torino, Einaudi, 1939. 325 pp. Lire 20.

This book, originally published in German (Vienna, 1936) and now translated into Italian, is a popular exposition of the fundamental concepts and views underlying modern mathematics. Special attention is paid to those sides of present day mathematics which may seem to involve "philosophical" problems, and a manifold of philosophical suggestions are advanced, mostly credited to an unpublished manuscript by Wittgenstein. The expository part of the book combines, to a rare degree, accuracy and comprehensibility.

After having traced in outline the historical growth of our present numbers system, the author envisages the question, often raised by philosophers: "How can the postulation of 'new' numbers be justified? Do there 'exist,' e.g., any irrational and complex numbers?" It is shown that these questions are equivalent to the questions whether the enlarged mathematical calculus is consistent, and whether it can be given an interpretation. The answer by a reference to geometrical facts is rejected as unsatisfactory, since proof of the consistency of geometry will depend on the assumption that our number system is consistent. The problem of the existence of the non-natural numbers—as far as it can be separated from the consistency problem—is now solved by giving the well known construction of them on the basis of the natural numbers. Following Skolem, the author then shows how elementary arithmetic can be strictly developed on the minimum basis of three undefined notions (natural number, successor, identity) and one inductive definition (of addition) by using the method of complete induction. The present situation of the "Grundlagenforschung" is reviewed, mention being made of theorems by Gödel and Skolem.

The "philosophical" ideas, expressed in the book, are unfortunately presented in such a condensed form that it is difficult to judge their significance. I will here make a few critical comments, which should in no way obscure the high value of the book as an elementary exposition of the foundations of modern mathematics.

Throughout, the author stresses the important distinction between mathematics as a symbolic calculus and the various possible interpretations of this calculus. A strict observation of this distinction, however, should have led to more careful statements at a few points. When talking "philosophically" about what, for example, the natural numbers *are*, our statements must always be referred to some particular interpretation of arithmetic. But, in Chapters IV and V, it is said unconditionally that natural numbers, integers, and rational numbers *are* of three different logical types. This contention is, of course, true if we presuppose—as is done in the present book—the modern construction of integers and rational numbers. No reasons, however, have been given why there should not exist interpretations of another kind, and perhaps one which at once would give the natural numbers their usual meaning and make natural numbers, integers, and rational numbers all belong to the same logical type.

In Chapter VIII, the method of mathematical induction is discussed and a philosophical interpretation, due to Wittgenstein, is presented. This interpretation may be summarized thus. The sentence "Every (natural number) $a$ has (the property) $P$," means, "1 has $P$, and if $b$ has $P$, then $b+1$ has $P$." Following Skolem the author has not made use of the operators "for every $x$" and "for some $x$" in his development of elementary arithmetic. What he wants to interpret must, therefore, be the meaning of a statement, "$a$ has $P$," where generality is expressed by the free variable $a$. The circularity of his explanation is then obvious. The positivistic principle of verification is invoked as a substantiation. This principle, as applied by the author to mathematics, could, I think, be formulated somewhat like this: "The meaning of a theorem $p$ is identical with (the meaning of) the conjunction of those theorems which constitute the proof of $p$." This principle is open to a number of more or less obvious objections. If the words within parentheses in the above formulation are omitted, then it boils down to a purely syntactical definition which does not have anything to do with "meaning" in an interpretative sense. Whether the author would let them stand or not, cannot, however, be gathered from his presentation. But further: The principle leads to the consequence that only provable sentences have meaning, a consequence which seems absurd if "meaning" is taken in the interpretative sense. And what is "*the* proof" of a mathematical theorem? From the principle of verification, by the way, several other conclusions are drawn. One of them is the Brouwerian contention that the negation of a universal sentence is not equivalent to an existential sentence. This conclusion is clearly possible only if we are already arguing on the basis of an axiomatization of mathematics in accordance with Brouwer's view!

In Chapter IX: B the Frege-Russell interpretation of natural numbers is criticized. The discussion is centered around their well known definition of the phrase "the number of elements in the two sets $A$ and $B$ is the same" as meaning "there is a one-one relation of which $A$ is the domain and $B$ the converse domain" ("$A$"). The chief objections made against this definition are: (1) According to the definition, the number of elements in two sets $A$ and $B$ must either be the same or not be the same, whereas in reality there are sets $A$ and $B$ such that neither the sentence "$A$ and $B$ have the same number of elements" ("$B$") nor its verbal negation ("$C$") is true, if we take this sentence with its accepted meaning. A contention like this, of course, cannot be conclusively proved or refuted. But the alleged instance of two such sets, mentioned by the author (p. 156), does not seem convincing. The argument is essentially dependent on the obscure principle of verification, here applied to empirical sentences. However, even if the contention were true, it is still not clear in what way it would be an objection to the Frege-Russell analysis. The contention seems to allow two different interpretations: (i) By $C$, the verbal negation of $B$, we ordinarily understand the logical negation of $B$; but the law of the excluded middle is not universally valid. (ii) When $B$ and $C$, as ordinarily used, are meaningful, then $C$ is the logical negation of $B$; but in the case at hand—by force of the principle of verification—they are both deprived of meaning, and therefore the law of the excluded middle is not relevant. If (i) covers the author's intentions—why could not $A$, too, be a sentence for which the law of the excluded middle does not hold? If (ii) is meant—why could not the principle of verification sometimes make $A$ and non-$A$ meaningless, too? (2) Two sets, $A$ and $B$ might have the same number of elements—the phrase being taken in its usual meaning—in spite of the fact that no one-one relation held between them; the occurrence of such a relation is an accidental empirical fact. Here the term "relation" has obviously been taken in a restricted sense as synonymous with "physical relation," a sense not intended by the Frege-Russell theory. (The question has been discussed by Ramsey in his *Foundations of Mathematics*.)

In a concluding chapter (to be compared with Chapter IX: C), the philosophical standpoint of the book is restated. In opposition to the view that the rules governing the use of a sign follow from the meaning of the sign, it is asserted that the sign obtains its meaning just through the rules. The rules the author has in mind seem to be of the following kinds: (1) the syntactical rules of the mathematical calculus within which the sign occurs; (2) the rules for translating the mathematical formulae into our everyday language (e.g., "two plus two make four" is a translation of "$2+2=4$"); (3) the rules for integrating the mathematical calculus into everyday language (e.g., from "$2+2=4$" we may deduce "two apples plus two apples make four apples"). Given these rules, the meaning of "1," "2," and so on is determined. To that extent, the author's contention that "the rules give meaning to the signs" is obviously true. The words "one," "two," and so on are, in their everyday usage, endowed with a certain meaning, and by the rules, belonging to category (2), this meaning is transferred to "1," "2," and so on. But, on the other hand, if we let "1," "2," $\cdots$ mean what is ordinarily meant by "one," "two," $\cdots$, then we cannot arbitrarily change the rules of categories (1) and (3) and still obtain true mathematical formulae and valid deductions. So far, the view which the author rejects seems to be a commonplace. Possibly, the author would add to the above list of rules a fourth category: rules for deciding when empirical numerical sentences are true. Through these rules, he would say, words like "one," "two" get their meaning, and thus, ultimately, rules of usage constitute the meaning of mathematical terms. But—can we give any rule for the use of the term "two" that is not of the following type: "If and only if the kind $A$ has the property $P$ then the sentence 'There are two A's' is true"? If, as it seems to be, we cannot, then we have reduced the meaning of "two" to the meaning of "$P$" which latter meaning we have not explained in terms of any rules of usage.

It might be worthwhile mentioning that the theory of natural numbers, given in Chapter VIII, suffers from a somewhat subtle logistic fault. Letting his variables range over a domain of objects, called "natural numbers," the author takes the monadic successor function $a+1$ as his undefined concept besides the relation of identity. On this basis he gives the following "inductive" definition of the dyadic additive function: $a+(b+1)=(a+b)+1$. To be in the standard from, this definition should be completed by another equation: $a+1= \cdots$. The omission here suggests a confusion of the successor function $a+1$ (argument $a$) with the additive function $a+1$ (arguments $a$ and 1).

ANDERS WEDBERG

*Reelle Funktionen*, Part I. By C. Carathéodory. Leipzig, Teubner, 1939. 6+182 pages. RM 11.20.

This volume is Part I of a projected three volume treatise on real functions to be distributed under the following titles: Part I, Numbers, Point Sets, Functions; Part II, Theory of Measure and The Integral; Part III, as yet without a title, but presumed to treat of derivatives and applications.

These volumes will replace the author's one volume, *Vorlesungen Über Reelle Funktionen*, first completed in 1917 and later reissued in 1927 with slight changes. It will be recalled that this earlier volume is written in the elegant axiomatic style for which the author is noted. Among its outstanding features are an abstract treatment of measure and a very full treatment of Lebesque integration.

This volume preserves much of the material in the first 230 pages of the older treatise. A considerable rearrangement of material has been made which adds to the attractiveness of the book. But examples are given only when required to show the necessity for the conditions governing certain theorems. The reviewers believe that the addition of illustrative examples would have been appreciated by many readers.

Some of the essential changes and additions will be taken up by chapters.

Chapter I—Real Numbers. Denumerability and non-denumerability treated in Chapter II of the older treatise here follows an axiomatic introduction to real numbers.

Chapter II—The Limit Notion. A discussion of Cantor's diagonal process is added to the older chapter on limits.

Chapter III—Point Sets in Euclidian Space. The treatment of open and closed sets also involves an iteration process of forming closures and interiors. An application is made to boundary theory.

Chapter IV—Normal Covering Sequences and the Theory of Connectedness. A new feature is the treatment of connectedness of arbitrary sets.

Chapter V—Functions. No essential change over the older volume save for omission of a discussion of functions of bounded variation (deferred to Part II) and the inclusion of a section on continuous monotonic functions in which is introduced a transformation function

$$x = \lambda(y) = \frac{1+y+|y|}{2(1+|y|)}$$

which sends the $y$-interval $(-\infty, \infty)$ into the $x$-interval (0, 1). This preserves local limit properties and permits the extension of some theorems on positive bounded functions to more general functions.

Chapter VI—The Distance Function and Applications. The chapter closes with Radó's proof of a theorem on the extension of closed domains of definition of functions. A different proof was given much later in the older volume.

Chapter VII—Sequences of Functions. This chapter contains perhaps the most interesting addition to the corresponding material of the older volume; namely, continuous convergence and normal function families. The theory of continuous convergence is developed in an elegant manner. However, the author might have pointed out the close connection existing between this theory and ordinary continuity of functions. By the simple device of considering the subscript $k$ of the sequence $f_k(P)$ as another coordinate in our number space, continuous convergence of the sequence becomes identical with the existence of a limit at the point $(P_0, \infty)$ of the corresponding function $F(P, k)$ in the extended space. Thus, for example, Theorem V follows immediately when it is noted that the two conditions given there are equivalent to the requirement that the saltus of $F(P, k)$ vanish at $(P_0, \infty)$, while Theorem VII follows directly from the theorem: A function continuous on a bounded and closed set is uniformly continuous on that set.

The final section of the chapter, on normal families of functions, while brief, contains recent results on function sequences of great interest to the advanced student of analysis.

Some readers will find it regrettable that the author retains in this modern work his limitation of the discussion to euclidean spaces, thus leaving out of account the far-reaching extension of real function theory to more general spaces made during the last thirty years; the more so, as much of the theory can be given in this general setting without in any way complicating the reasoning or lengthening the proofs. Bohnenblust's 1938 *Princeton Notes on Real Variables* indicates the possibilities of such a treatment.

Since this volume contains the more elementary material many readers will naturally be more interested in Parts II and III. The known high standards of the author are enough to insure the completed work a prominent place in mathematical literature.

J. W. HURST
H. M. SCHWARTZ

*A History of Geometrical Methods.* By J. L. Coolidge. Oxford, Clarendon Press, 1940. 18+452 pp. $10.00.

Notwithstanding the rich literature on geometry, there has been no comprehensive synopsis of the subject for a long time. The author has set himself the task of presenting in outline the development of geometry from the beginnings to the present, emphasizing in particular the various methods employed, taking for a general model Chasles's *Aperçue Historique,* which first appeared in 1837. The field is so large that not all material could be considered; selections must be made, and no fixed criterion can be set down for eligibility.

The book is divided into three parts: synthetic, algebraic, and differential. It is explicitly stated that topology is excluded from consideration. The first part covers 116 pages. It includes a sketchy summary of practically everything preceding Descartes, all of synthetic projective geometry, non-euclidean geometry in the narrower classic sense, and a partial discussion of descriptive geometry; axonometry is not mentioned, and the treatment of perspective is very brief. Pohlke's theorem is not mentioned. The substance of this part is largely considered as a completed edifice.

The second part, algebraic geometry, covers 200 pages. It is the field of most of the author's own creative activity; many topics are referred to his extensive writings for fuller discussion. After pointing out various usages of coordinates in earlier writings, the classic period of Fermat and Descartes is given in some detail, followed by a discussion of abridged notation, change of space element and extension to higher dimensions. This in turn is followed by a short description of the Clebsch-Aronhold symbolic notation, minimal coordinates, of elliptic coordinates, pentaspherical coordinates, and other systems.

Unusual care is devoted to the study of enumerative geometry, particularly to the Schroeter calculus. The opinion is suggested that this subject has provided more errors in the literature of the subject of algebraic geometry than any other. After a stormy and acrimonious history, the principles of the discipline are now fully established, but in order to apply them properly, the operator must know how to count. The closely allied theory of correspondence between points of associated algebraic loci is discussed from the same point of view.

The chapter on birational geometry furnishes a good introduction to the geometry on algebraic curves and surfaces; rather more emphasis is put on the transcendental treatment than the history of the literature of the subject warrants. The treatment of the uses of higher spaces is well written, but too brief to be of maximum usefulness, except possibly the paragraph on quaternions. The chapter on transformations begins with linear ones (both collineations and correlations), then quadratic ones and the general Cremona transformation. In the plane, this is practically complete. Like synthetic geometry, this subject involves a number of indispensable tools, the use of which must be thoroughly mastered for further progress, without the hope of contributing much from this section that is new. On page 288 is the remark that Mlodziejowski computed the tables of plane Cremona transformations to order 21. Without debating the real merits of this work, the reviewer inclines to the opinion that the Naples dissertation (1909) of Marazzo should also be mentioned, as it provides the tables of orders up to 23.

After a careful and correct discussion of Hudson's fundamental theorem that in space of more than two dimensions a theory of composition analogous to Noether's theorem for the plane does not exist, the author dismissed the theory in more dimensions with the implication that there isn't much there. In the sense of results, this is correct. The author ascribes this state of affairs to the greater difficulty and to the lack of general methods. What greater incentive is needed? During the last few years the question of the existence of irrational involutions of order two in three way space has been discussed in over a hundred papers, and that of a possible new approach to the question of composition has a literature that is nearly as large. It is pointed out that the group concept will probably play an important part in further advances along this line.

The third and last part of the book, differential geometry, covers 100 pages. After a brief glance at the early writers, including Euler and Monge, a detailed exposition of the methods of Cauchy follows, then the use of intrinsic methods and of movable axes. A full analysis of the work of Gauss is given; it plays an important part in much of the later metric differential geometry. Geodesic lines on a general surface are treated, followed by a summary of the essential property of minimum surfaces. A brief discussion of differential properties of line congruences is then given.

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The chapter on projective differential geometry compares the methods of Wilczynski, Fubini, Cartan, and their followers. The name Halphen is not mentioned in this connection, but he was more truly a founder than the others. The range of problems to which these methods are applicable is sharply limited, but within these limitations, important results are obtained with ease and elegance.

The last chapter, on absolute differential geometry, is devoted largely to an explanation of the notation employed. This is not entirely consistent, and the pace is so rapid that its usefulness remains in doubt. Riemannian geometry, parallelism and the geometry of paths are all touched upon.

The book is provided with a subject index and an extensive bibliographical one. The latter includes (most of) the names cited, including the title and a reference to its source, and in many cases the date of birth and of death of those cited. This is a valuable list; its compilation is a difficult and often a thankless task. A real source of confusion arises in some cases in which the information is not complete, especially when the person cited was born after the middle of the nineteenth century. A special symbol to indicate that the entries were not complete in such cases would have avoided the ambiguity. Numerous slips or actual errors were noticed. Six proper names are misspelled; for homaloid the spelling homoloid is everywhere used. In the formula at bottom of page 221 for $i$ read $i-1$. A real source of confusion arises on page 220 line 10 up. The genus defined is not the geometric genus, but the number $p'$ in the author's notation. The discussion on the following page also refers to $p'$. This is what Noether called the Curvengeschlecht.

On page 208 is a footnote concerning the origin of the transformation connecting adjoints of plane curves, and hyperspace. Compare Cayley: *On the transforms of curves*, Proceedings of the London Mathematical Society, vol. 1 (1865). The formula $I+p-\sigma=12p_a+9$, in which $I$ is the invariant of Zuethen-Segre and $\sigma$ the number of exceptional curves in the system, is discussed in the footnote page 227. Compare Noether, Mathematische Annalen, vol. 8 (1875), pp. 495–528.

Professor Coolidge has put a great deal of careful thought in the preparation of this book. A number of concepts have been correctly explained, which have been sources of confusion in the minds of many, especially those interested primarily in other branches of mathematics. The book will serve a real purpose.

VIRGIL SNYDER

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*Festschrift Rudolph Fueter zur Vollendung seines sechzigsten Altersjahres*, 30 VI. 1940. Zürich, Naturforschende Gesellschaft, 1940. 231 pp.

This volume comprises articles by Oystein Ore, Henri Lebesgue, M. Plancherel, N. Tschebotaröw, Paul Montel, W. Scherrer, L. J. Mordell, Francesco Severi, T. Carleman, E. Hecke, H. S. Vandiver, R. Wavre, H. Brandt, C. Carathéodory, Heinrich Jecklin, Eugenio G. Togliatti, Alfred Kienast, Ernst Trost, Ludwig Bieberbach, J. J.

Burckhardt, Paul Finsler, Heinz Hopf, H. Behnke and K. Stein, Elie-Cartan, Andreas Speiser, Max Gut, F. Gonseth.

W. W. Flexner

*Les Probablitités Associées à un Système d'Évènements Compatibles et Dépendants*; I. *Évènements en Nombre Fini Fixe.* By Maurice Fréchet. (Actualités Scientifiques et Industrielles, no. 859.) Paris, Hermann, 1940. 8+80 pp.

This is part one of a series of three, the others being: II. *Cas Particuliers et Applications*, and III. *Evènements en Nombre Très Grand ou Infini.* In this series Professor Fréchet has gathered together the hitherto scattered literature on a problem of rather general interest. The problem may be stated thus. We consider $m$ quite general events $A_1, \cdots, A_m$ and an event $H$ which is a function of these; that is, the occurrence or non-occurrence of $H$ depends solely on which of the $A$'s occur. The probability that $A_{j_1}, \cdots, A_{j_r}$ occur simultaneously is denoted by $p_{j_1 \cdots j_r}$. We wish to find the probability of $H$, granted that we know the values of the $p$'s.

In Chapter I the author states and proves two interesting and powerful theorems due to Broderick. The first of these theorems asserts that the probability of $H$ is a linear function of the $p$'s, with coefficients depending not on the particular nature of the $A$'s, but only on the function $H$. In the second theorem it is shown that if $H$ is considered to be a function of two sets of events, then the probability of $H$ can be obtained by a symbolic multiplication. The utility of these theorems in obtaining elegant solutions of certain classical problems will no doubt be demonstrated in the second volume of the series.

Chapter I also contains certain related formulae on moments, generating functions, and "conditional" probabilities. In Chapter II the author obtains a number of inequalities due to various writers, some in generalized form. In the remainder of the chapter questions of the following type are answered: what are the necessary and sufficient conditions that a set of $2^m$ numbers be the probabilities $p_{j_1 \cdots j_r}$ defined above, for some set of events $A_1, \cdots, A_m$?

The mathematics used throughout is on a quite elementary level so that the book should prove of interest to a wide circle of readers. A defect, in the reviewer's opinion, is the seemingly haphazard manner in which the various topics are arranged.

I. Kaplansky

## NOTES

The editors of the Bulletin wish to make grateful public acknowledgment of the services rendered by the following persons who have refereed papers: R. P. Agnew, Leonidas Alaoglu, Reinhold Baer, R. W. Barnard, I. A. Barnett, Walter Bartky, P. O. Bell, A. A. Bennett, Garrett Birkhoff, G. D. Birkhoff, Henry Blumberg, R. P. Boas, Salomon Bochner, A. T. Brauer, Richard Brauer, H. W. Brinkmann, A. B. Brown, E. W. Chittenden, Alonzo Church, R. V. Churchill, J. A. Clarkson, A. H. Clifford, A. B. Coble, George Comenetz, A. H. Copeland, C. M. Cramlet, M. M. Day, J. J. DeCicco, Nelson Dunford, Ben Dushnik, L. R. Ford, Orrin Frink, Guido Fubini, H. L. Garabedian, J. J. Gergen, D. W. Hall, W. L. Hart, M. R. Hestenes, Einar Hille, Witold Hurewicz, Nathan Jacobson, R. D. James, R. L. Jeffery, B. W. Jones, Mark Kac, E. R. van Kampen, Edward Kasner, S. C. Kleene, Fulton Koehler, E. P. Lane, C. G. Latimer, D. H. Lehmer, D. C. Lewis, E. R. Lorch, N. H. McCoy, J. C. C. McKinsey, E. J. McShane, Saunders MacLane, H. W. March, Morris Marden, Karl Menger, G. M. Merriman, Deane Montgomery, C. B. Morrey, A. F. Moursund, Oystein Ore, W. F. Osgood, Gordon Pall, G. H. Peebles, F. W. Perkins, B. J. Pettis, Hillel Poritsky, H. A. Rademacher, Tibor Radó, W. C. Randels, A. Rappoport, W. T. Reid, J. F. Ritt, J. H. Roberts, M. S. Robertson, R. M. Robinson, A. E. Ross, J. B. Rosser, O. F. G. Schilling, I. J. Schoenberg, Wladimir Seidel, W. E. Sewell, I. M. Sheffer, J. A. Shohat, L. L. Smail, I. S. Sokolnikoff, N. E. Steenrod, Alvin Sugar, Otto Szász, Gabor Szegö, J. H. Taylor, H. P. Thielman, T. Y. Thomas, J. W. Tukey, S. M. Ulam, J. V. Uspensky, R. J. Walker, A. D. Wallace, J. L. Walsh, L. E. Ward, M. S. Webster, Hassler Whitney, Norbert Wiener, L. R. Wilcox, John Williamson, Aurel Wintner, Leo Zippin, and Max Zorn.

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Tarski, J. M. Thomas, A. W. Tucker, J. W. Tukey, G. E. Wahlen, H. S. Wall and Aurel Wintner.

The National Research Council has just issued a report on the *Bibliography on Orthogonal Polynomials* which was prepared by a committee consisting of Professors J. A. Shohat, Einar Hille and J. L. Walsh. The report is bound in cloth, consists of 204 pages and may be purchased from the National Research Council at $3.00 per volume.

The addresses presented at the symposium on Applications of Mathematics in the Earth-Sciences, under the joint auspices of Sections A and E of the American Association for the Advancement of Science, the American Mathematical Society, the Mathematical Association of America and the Geological Society of America, at Columbus, Ohio, December 29, 1939, have been published as Part A of Volume 4 of the Transactions of 1940 of the American Geophysical Union. The addresses tend to further a desirable rapprochement between geophysicists and mathematicians and begin the important task of bringing to each field the knowledge or some suggestions of the contributions which the other field may make. This volume will include also, as Part B, the symposium at Columbus on Hydrologic Problems in the Ohio and Michigan Basins. The volume will make about 100 pages. The regular postpaid price to non-members of the Union is $1.00 but a special price of $0.75 per copy is being made to members of the American Mathematical Society. Orders, indicating membership in the American Mathematical Society, should be sent to 5241 Broad Branch Road, N.W., Washington, D.C.

Professor M. H. Stone of Harvard University has published a planographed edition of a set of notes on *The Theory of Real Functions*: I, *The Real Number System*. II, *The Theory of Classes*. Interested persons should address Professor Stone.

Professor Saunders MacLane of Harvard University has published a planographed edition of a series of lectures on *Algebraic Functions*, which may be obtained from him by interested persons.

Applications for the Benjamin Peirce Instructorships in Mathematics at Harvard University for the year 1941–1942 should be sent to the Chairman of the Department of Mathematics. Candidates should have received the doctorate or have done equivalent work.

The School of Mathematics of the Institute for Advanced Study each year allocates a small number of stipends to gifted young mathematicians and mathematical physicists to enable them to study and to do research work at Princeton. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks for application may be obtained from the School of Mathematics of the Institute, Fuld Hall, Princeton, N. J., and are returnable by February 1, 1941.

Yale University offers each year a number of Sterling research fellowships, some of them open to mathematicians. For the year 1941–1942 the normal stipend will be $1800. Applicants must have received the degree of Doctor of Philosophy or its equivalent from approved universities in the United States or foreign countries. In general, preference is given to applicants of not more than thirty-five years of age, but older persons are eligible. Applications should be received before March 1; blanks may be obtained from the Graduate School office, Yale University.

Dr. K. E. Bullen, lecturer in mathematics at University College, Auckland, New Zealand, has accepted an appointment at the University of Melbourne.

Assistant Professor G. Hoheisel of the University of Greifswald has been appointed to a professorship at the University of Köln.

Dr. F. Lösch of the University of Rostock has been promoted to a professorship.

Professor R. P. Paranjpye of Fergusson College, Poona, India, has retired.

Dr. W. B. Caton has been made head of the department of mathematics at Athens College, Athens, Alabama.

Dr. Claude Chevalley, formerly of the University of Rennes, has been appointed to an assistant professorship at Princeton University.

Dr. P. L. Dressel of Michigan State College has been promoted to an assistant professorship.

Associate Professor E. P. R. Duval of the University of Oklahoma has been promoted to a professorship.

Mr. W. O. Gordon of Pennsylvania State College has been promoted to an assistant professorship.

Assistant Professor Nathan Jacobson of the University of North

Carolina has been appointed visiting associate professor of mathematics at Johns Hopkins University this year.

Dr. J. R. Jenness of the College of the Ozarks has been made head of the department of physics and mathematics at Parsons College, Fairfield, Iowa.

Dr. S. A. Jennings of the University of British Columbia has been appointed lecturer.

Assistant Professor H. A. Jordan of Georgetown University has been promoted to an associate professorship.

Professor L. C. Knight of the College of Wooster has retired after thirty-two years of service.

Mr. T. H. Lee of the University of Wisconsin has been appointed to an associate professorship at the University of South Carolina.

Assistant Professor A. N. Lowan of Yeshiva College has been promoted to an associate professorship.

Associate Professor Morris Marden of the University of Wisconsin at Milwaukee has been granted leave of absence for this academic year. He is lecturing at the University of Wisconsin in Madison.

Assistant Professor W. T. Martin of the Massachusetts Institute of Technology has been granted leave of absence to spend a year in study and research at Princeton University and the Institute for Advanced Study.

Dr. R. J. Michel of the University of Missouri has been appointed head of the department of mathematics at Southeast Missouri State Teachers College.

Professor J. C. Morehead of the Carnegie Institute of Technology has retired with the title emeritus.

Dr. L. F. Ollmann of Texas Technological College has been appointed to an assistant professorship at the College of Wooster.

Professor George Polya who has been at the Technical School in Zurich is to be at Brown University for two years.

Assistant Professor W. C. Randels of the University of Oklahoma has been promoted to an associate professorship.

Associate Professor H. A. Ruger of Teachers College, Columbia University, has retired.

Mr. C. S. Sutton of Tufts College has been appointed to an assistant professorship at The Citadel.

Commander J. C. Van de Carr, U.S.N., retired, has been granted a leave of absence from his duties in the mathematics department at the University of Oklahoma to take charge of the University unit of the Naval R.O.T.C.

Dr. E. T. Welmers of Michigan State College has been promoted to an assistant professorship.

Professor R. L. Wilder of the University of Michigan has been granted leave of absence to spend a year at the University of Texas.

To the list of names of members of the Society working at the Institute for Advanced Study this year, which appears in the November 1940 issue of this Bulletin, should be added those of Assistant Professor G. M. Ewing of the University of Missouri and Dr. H. T. Muhly, National Research Fellow.

Dr. R. J. Duffin of the University of Illinois has been made an associate in mathematics.

The following appointments to instructorships are announced: Bowling Green State University: Dr. Morris Hendrickson; Cornell University: Dr. Michael Golomb; Georgia School of Technology: Dr. D. T. McClay; Long Island University: Mr. E. A. Knobelauch; Massachusetts Institute of Technology: Dr. G. B. Thomas; Michigan State College: Mr. A. C. Cohen, Dr. B. M. Stewart; Montana State College: Dr. H. M. Schwartz; Northwestern University: Dr. J. M. Dobbie, Mr. W. N. Huff; University of Oklahoma: Mr. B. S. Whitney; Pennsylvania State Forestry School, Mont Alto, Pennsylvania: Mr. Morris Bloom; University of the Philippines: Dr. R. A. Favila; Purdue University: Dr. R. A. Leibler; Stanford University: Dr. P. V. Reichelderfer; Agricultural and Mechanical College of Texas: Dr. R. E. Basye; Texas Technological College: Mr. F. D. Rigby; Tri-State College, Angola, Indiana: Dr. M. G. Moore; University of Virginia: Mr. R. C. Morrow; Williams College: Mr. W. D. Wray; Winthrop College, Rock Hill, South Carolina: Miss Katharine E. Hazard; University of Wisconsin: Dr. H. E. Goheen, Dr. Fred Kiokemeister, Mr. J. W. Odle; University of Wisconsin at Milwaukee: Mr. W. L. G. Mitchell; University of Wyoming: Dr. V. J. Varineau.

Dr. Vito Volterra, formerly professor of mathematics at the University of Rome, died October 11, 1940 at the age of eighty years. He had been a member of the Society since 1905.

Dr. J. R. Carson of the Bell Telephone Laboratories died on October 31, 1940. He had been a member of the Society since 1918.

Professor George Rutledge of the Massachusetts Institute of Technology died on September 21, 1940, at the age of fifty-eight years. He had been a member of the Society since 1910.

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# ABSTRACTS OF PAPERS

## SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

### ALGEBRA

### 1. R. H. Bruck: *The structure of the rational representations of a wide class of linear groups.*

Let $\theta$, $\phi = \theta^{-1}$ be fixed automorphisms of the underlying field (of characteristic 0 or $p$). Denote by $A^*$ the transposed of the matrix obtained from $A$ by subjecting its elements to the automorphism $\phi$. Attention is restricted to groups $\mathfrak{G}$ of $n \times n$ matrices with the property that $A$ in $\mathfrak{G}$ implies $A^*$ in $\mathfrak{G}$; by choice of $\phi$ a large number of groups may be brought under review. $\mathfrak{G}$ being taken as a group of transformations of a contravariant vector $x^i$, the $f$th Kronecker product of $\mathfrak{G}$ may be regarded as a group of transformations of the linear vector space $S$ of all forms $F(T) = \sum c_{i_1 i_2 \cdots i_f} T^{i_1 i_2 \cdots i_f}$, where $T^{i_1 i_2 \cdots i_f}$ is a contravariant tensor. The fundamental idea of the paper is to analyse the Kronecker product by means of a scalar product defined on $S$. The scalar product of $F$ and $G = \sum d_{i_1 i_2 \cdots i_f} T^{i_1 i_2 \cdots i_f}$ is taken as $F \circ G = \sum \overset{\theta}{c}_{i_1 i_2 \cdots i_f} d_{i_1 i_2 \cdots i_f}$, and has properties (1) linearity, (2) $(bF) \circ G = F \circ (b^\theta G) = b^\theta (F \circ G)$, (3) $F \circ G = F_{(A^*)} \circ G_{(A^{-1})}$. (Here $F_{(A)}$ is defined by $F(T) = F_{(A)}(T')$ where $x = Ax'$.) The following special result indicates the nature of the paper: If $\mathfrak{G}$ (over the field of rationals) contains the transposed of each group matrix, the Kronecker product representations are completely reducible. (Received November 12, 1940.)

### 2. C. C. Camp: *A root cubing method of solving equations.*

Algebraic equations were solved by Dandelin and Graeffe by the well known root squaring process. Instead of squaring the roots the present paper gives a method for cubing the original roots. No Encke roots are needed and the sign of a real root is preserved on successive cubings. The general rule is derived by two independent methods and in the case of the cubic by three. The cases of complex and repeated roots are treated in the applications together with that of distinct real roots. (Received November 23, 1940.)

### 3. W. D. Duthie: *Segments in ordered sets.*

The notion of "segment" in a linearly ordered set is extended to (partially) ordered sets. Segments in ordered sets are self-dual, and can be used to characterize various special properties of ordered sets such as being a lattice, modular, distributive, or complemented lattice, giving a completely self-dual set of postulates for these lattices. Convex subsets of a lattice are defined in the natural way, and various relations among a lattice and its lattices of segments and convex subsets are discussed, leading to results on imbedding of lattices in complete lattices. The lattices of segments of modular and distributive lattices are special cases of pseudo-modular and pseudo-distributive lattices, respectively; the latter property is of particular interest, since it is applicable to modular lattices, though the application is somewhat restricted by the fact that the only complete complemented pseudo-distributive modular lattices which are not distributive are one-dimensional projective geometries. However in such cases the existence of a dimension function permits replacing pseudo-distributivity by a more general notion of "$m$-distributivity" ($m$ having the range of the dimension function). Application of the general theory of segments to Boolean algebras yields some known results by new and shorter methods. (Received November 25, 1940.)

### 4. C. J. Everett: *An extension theory for rings.*

A ring theoretic analogue of the O. Schreier "Erweiterungstheorie" for groups is developed about the central problem of determining all rings $R$ containing a given ring $N$ as ideal and such that $R/N$ is isomorphic to a given ring $F$, employing factor systems and their equivalence. The Artin "splitting theorem" is proved: If $R$ contains ideal $N$, $R$ has a splitting ring over $N$. Constructive proof is given of the existence of rings whose chains of left and right annihilators for successive powers of the ring have first repetitions at arbitrary positions. "Closed rings" are shown to be direct summands of over-rings containing them as ideals. Equivalence classes of suitably restricted factor systems form an abelian group under addition. (Received November 23, 1940.)

### 5. Irving Kaplansky: *Maximal fields with valuations.*

A field $K$ with a valuation is said to be maximal if no proper extension of $K$ has the same value-group and same residue-class field as $K$. It was shown by Krull that any field with a valuation could be embedded in a maximal field, and he proposed the problem of determining whether this maximal extension is unique. This question is examined in the present paper, and the following result is obtained. The maximal extension is always unique if the residue-class field has characteristic zero; but if the latter has characteristic $p$, it must first be required that the value-group have no extensions of degree $p$, and second that the residue-class field satisfy a certain condition somewhat stronger than algebraic perfection. Under the same hypotheses it is shown that in the equal characteristic case every maximal field is a generalized type of power series field. The chief tool employed is a generalization of Ostrowski's notion of pseudo-convergence, in terms of which it is possible to give new criteria for maximality and completeness. (Received December 2, 1940.)

### 6. Rufus Oldenburger: *The minimal number problem.*

It has been proved that, for a field with at least $n+1$ elements, each binary form $F$ of degree $n$ with a symmetric tensor $A$ of coefficients can be written as a linear combination of $n$th powers of linear forms. The least number of terms for which $F$ can be so written is called the minimal number $m$ of $F$. The elements of $A$ can be arranged in ordinary matrix displays $B_0, \cdots, B_n$ when $B_i$ is the display obtained from $A$ by using $i$ indices of $A$ as column indices. The minimal number problem is the problem of determining the precise relation between the ranks of $\{B_i\}$ and $m$. The ranks of the matrices $\{B_i\}$ are determined by one of them, denoted by $\gamma$. In the present paper the minimal number problem is solved for algebraically closed fields by proving that $m=\gamma$ or $m=n-\gamma+2$. A modification of this result holds for arbitrary fields. The proof is accomplished in part by the solution of the problem of construction of all representations of $F$. This latter problem is the same as the problem of constructing all forms apolar to $F$. Solutions for several major classical problems of algebra follow. (Received November 23, 1940.)

### 7. G. Y. Rainich: *Postulates for fields*. Preliminary report.

Instead of binary operations or functions of two variables, namely, the sum $x+y$ and the product $xy$, in terms of which a field is usually defined, corresponding functions of one variable or transformations, namely, addition $x'=y+a$ and multiplication $x'=xb$, are considered. The problem is to characterize the field by studying the formal properties of these or, rather, of the group $x'=xb+a$ generated by them. This group is postulationally defined in two ways; once as a two-fold transitive group of transformations and then again as an abstract non-abelian group. From the first point of view the additions and multiplications are introduced as transformations without fixed point and with a given fixed point respectively. This point of view is closely related to projective geometry. From the other point of view the subgroups of additions and multiplications are introduced as the commutator subgroup and its factor group respectively. (Received October 25, 1940.)

### 8. N. E. Rutt: *Rectangular arrays of combinations*.

Let $C$ be the collection of different combinations, two at a time, of the objects in a set $D$ of $2n$ different symbols. From the elements of $C$ construct a rectangular arrangement, henceforth to be called an array, such that no element of $D$ occurs twice in the same row of the arrangement, and no element of $C$ occurs twice in the same arrangement. Two arrays are the same if and only if a reordering of their rows or a reordering of the members of $C$ in their rows will make them formally identical. An investigation of the properties of arrays is undertaken in this paper. It is proved that, for each even natural number $2n$, there is an array whose rows contain $n$ elements of $C$ (the maximum possible), and whose rows are in number $2n-1$ (the maximum possible). Features of the internal structure of arrays are considered, among them the existence and inter-relations of the proper sub-arrays of arrays. The behavior of arrays under permutations of the elements of $D$ receives some attention. An attempt is also made to determine the number of different arrays of certain specified types. (Received November 18, 1940.)

### 9. M. F. Smiley: *Measurability and distributivity in the theory of lattices*.

G. Birkhoff's criterion that a metric lattice be distributive (American Mathematical Society Colloquium Publications, vol. 25, p. 81) is quantified so as to provide a second generalization (this Bulletin, vol. 46 (1940), pp. 239–241) of the measurability of Carathéodory (*Vorlesungen über Reelle Funktionen*, 2d edition, p. 246). Closure properties of the set of "measurable" elements are derived as before. The influence of distributivity on measurability and on the measurability of complements of measurable elements is discussed. (Received November 20, 1940.)

### 10. T. L. Wade: *Tensor algebra and Young's symmetry operators*.

A. Young's symmetry operators (Proceedings of the London Mathematical Society, vol. 33 (1900), pp. 97–146) have recently been used by H. Weyl (*The Classical Groups*, Princeton, 1939, pp. 119–131) in the decomposition of tensor space into irreducible subspaces. This paper is concerned with a study of the law of combination satisfied by the indices of the tensor associated with each diagram or partition of the indices of the arbitrary tensor. It is noted that corresponding to each partition there can be associated a specialization of C. M. Cramlet's general invariant tensor (Tôhoku Mathematical Journal, vol. 28 (1927), pp. 242–250), and that an arbitrary tensor can be decomposed by multiplying an appropriate numerical tensor identity by it, and then contracting. (Received November 25, 1940.)

### 11. R. W. Wagner: *The differentials of analytic matrix functions.*

The notions of Hausdorff and Fréchet concerning differentials are applied to analytic matrix functions as defined by the author in a previous paper. The differential of an analytic matrix function is shown to have a simple form when it is written in terms of the characteristic roots and partial idempotent and partial nilpotent elements of the argument. The function enters the differential through divided difference quotients. This simple form of the differential is used to deduce local properties of the mapping defined by the function. (Received November 23, 1940.)

### 12. Morgan Ward: *The fundamental theorem of arithmetic.*

A set of necessary and sufficient conditions for the fundamental theorem of arithmetic to hold in a semi-group is obtained from the theory of residuated lattices and a new inductive proof of the theorem is given for the lattice of positive integers. (Received October 30, 1940.)

### 13. Hermann Weyl: *Theory of reduction for arithmetical equivalence.* II.

Instead of the arithmetically refined method of reduction used in the first paper, the author now resorts to a rougher method which also goes back to Minkowski, and works without the assumption that the class number of ideals is 1. Its generalization to algebraic number fields $F$ with several infinite prime spots is due to Siegel and P. Humbert (Commentarii Mathematici Helvetici, vol. 12 (1939–1940), pp. 263–306). The author follows the same geometric approach as before, including all classes of lattices over $F$ and adding to the case of a field $F$ that of a field quaternion algebra with totally positive norm over a totally real field. (Received November 23, 1940.)

## ANALYSIS

### 14. Warren Ambrose: *Representation of ergodic flows.* Preliminary report.

A flow is a one-parameter group $T_t$ $(-\infty < t < \infty)$ of measure preserving transformations of a space $\Omega$ into itself. It is measurable if the function $T_tP$ is a measurable function on $T\times\Omega$, where $T$ denotes the real line taken with Lebesgue measure. For a measurable ergodic flow it is shown that a necessary and sufficient condition that a group of unitary operators $U_t$ defined by $U_tf(P)=f(T_tP)$ have an eigenvalue (other than the trivial eigenvalue 1) is that the flow be isomorphic (with respect to measure properties) to a flow built on a measure preserving transformation (for definition of such a flow see abstract 46-11-446), thus showing that if an ergodic flow has an eigenvalue the measure on $\Omega$ must be the direct product measure of a cross section measure with Lebesgue measure along the trajectories of the flow. It is also shown that if the flow has no eigenvalues but satisfies certain conditions which are stronger than measurability then the measure on $\Omega$ is such a direct product measure. (Received November 18, 1940.)

### 15. R. P. Boas and D. V. Widder: *Completely convex functions.* Preliminary report.

A function $f(x)$ is defined to be completely convex in an interval $(a, b)$ if it has even derivatives which satisfy the relation $(-1)^n f^{(2n)}(x) \geqq 0$ there. For example, the functions $\sin x$ and $\cos x$ have the property in the intervals $(0, \pi)$ and $(-\pi/2, \pi/2)$ respectively. It was shown by Widder (Proceedings of the National Academy of Sciences, November, 1940) that such a function must be entire of order at most unity (and type at most $\pi/(b-a)$ if the order is one). A simplified proof of this result is given, and an integral representation for completely convex functions is obtained. Inversion formulas for the integral involved in this representation are discussed. (Received November 23, 1940.)

### 16. Russell Cowan: *A method of solving a linear difference equation with polynomial coefficients of degree m.*

Given a linear difference equation $\Delta$ of order $p$ with polynomial coefficients of degree $m$, a linear differential equation $D$ is found whose recursion relation yields $\Delta$. The order of $D$ is the smaller of $p$ and $m$. Under slight restrictions on the coefficients of $\Delta$, the differential equation has $p+2$ regular singular points. The case $p=2$ is studied in detail. By transforming $D$ to Heun's equation, the solution of $\Delta$ is readily obtained. In special cases one of the singular points of $D$ is not regular. But even under these circumstances, $\Delta$ can be solved by utilizing the characteristic exponent $\lambda$ which is found by transforming $D$ into a Riccati equation. (Received November 18, 1940.)

### 17. J. H. Curtiss: *Degree of polynomial approximation on a lemniscate.*

It is known that a function $F(z)$ analytic and single-valued interior to the lemniscate $\Gamma$: $|\omega(z)| = \mu$, $\omega(z) = (z-\alpha_1)(z-\alpha_2) \cdots (z-\alpha_\lambda)$, can be expanded in a series of the form $\sum_0^\infty q_\nu(z)[\omega(z)]^\nu$ convergent to $F(z)$ for $|\omega(z)| < \mu$, where the functions $q_\nu(z)$ are polynomials of degree less than $\lambda$. Let $S_n(z; F)$ denote the $n$th partial sum of this series, let $S_n^{(r)}(z; F)$ denote the $n$th Cesàro mean of order $r$, $0 < r \leqq 1$, and let $J_n(z; F)$ denote the $n$th Jackson mean. The following theorem is proved: If $C$ consists of one or several of the closed contours of $\Gamma$, and if $G(z)$ is a function analytic interior to $C$, continuous in the closed region or regions, and satisfying a Lipschitz condition of order $\eta$ on $C$, $0 < \eta \leqq 1$, then there exists a function $F(z)$ analytic interior to $\Gamma$ such that, uniformly on and interior to $C$, we have $S_n(z; F) - G(z) = O(n^{-\eta/m} \log n)$, $S_n^{(r)}(z; F) - G(z) = O(n^{-r}) + O(n^{-\eta/m})$, $r \neq \eta/m$, $= O(n^{-r} \log n)$, $r = \eta/m$, $J_n(z; F) - G(z) = O(n^{-\eta/m})$, where $m$ is the order of the multiple point on $C$ of highest order. If $C = \Gamma$, then $F(z) \equiv G(z)$. Examples are given to show that these estimates cannot be improved. (Received November 27, 1940.)

### 18. J. H. Curtiss: *Trigonometric interpolation by means of the complex Lagrange polynomial.*

Let $L_n(\theta; F)$ be the unique polynomial in $e^{i\theta}$ of degree at most $n-1$ which coincides with a given function $F(\theta)$ in the points $\theta = 2\pi k/n$, $k = 1, 2, \cdots, n$. Trivial examples show that even if $F(\theta)$ is analytic for all $\theta$, the sequence $\{L_n(\theta; F)\}$ may diverge for all $\theta \neq 0$, mod $2\pi$. This paper studies the convergence of $\{L_n(\theta; F)\}$ for functions $F(\theta)$ which are continuous and are the boundary value functions of functions $f(z)$, $z = re^{i\theta}$, analytic and of class $H$ for $|z| < 1$. The following results are typical: (a) If the Fourier series for $F(\theta)$ converges absolutely for all $\theta$, then $L_n(\theta) \to F(\theta)$ uniformly; (b) if $F(\theta)$ is of bounded variation in an open neighborhood of $\theta_1$, then $L_n(\theta) \to F(\theta)$ uniformly in a closed subneighborhood of $\theta_1$; (c) $L_n(\theta; G_n) \to F(\theta)$ uniformly, where $G_n(\theta) = (n/2\pi) \int_\theta^{\theta+2\pi/n} F(t)dt$; (d) $L_n(\theta; G_n) - F(\theta) = O(n^{-\alpha})$ if $F(\theta)$ satisfies a Lipschitz condition with exponent $\alpha < 1$. These results are then generalized to the case of interpolation on a Jordan curve $C$ in the manner of the author's paper in the Transactions of this Society, vol. 38 (1935), p. 458. (Received November 27, 1940.)

### 19. L. L. Dines: *On the mapping of quadratic forms.*

If $P(z)$ and $Q(z)$ are real quadratic forms in $n$ variables $z_1, z_2, \cdots, z_n$, the transformation $x = P(z)$, $y = Q(z)$ maps the $n$-dimensional $z$-space into a set of points $\mathfrak{M}$ of the $xy$-plane. This paper obtains properties of the map $\mathfrak{M}$, and from them deduces certain results relative to the pair of forms $P(z)$ and $Q(z)$. Among the results is the theorem proved by Albert (this Bulletin, vol. 44 (1938), p. 250) and by Reid (this Bulletin, vol. 44 (1938), p. 437). The paper will appear in the Bulletin. (Received November 14, 1940.)

### 20. Jesse Douglas: *Solution of the inverse problem of the calculus of variations.*

A solution is presented of the classical problem: given a family of curves in 3-space, as defined by differential equations $y'' = F(x, y, z, y', z')$, $z'' = G(x, y, z, y', z')$; determine the existence or nonexistence of a calculus of variations problem of the form $\int \phi(x, y, z, y', z')dx =$ minimum having the prescribed curve family for the totality of its extremals, and, in the affirmative case, to find the most general form of the function $\phi$. Preliminary notes stating the results and describing the methods used have already appeared in the Proceedings of the National Academy of Sciences, vol. 25 (1939), pp. 631–637, and vol. 26 (1940), pp. 215–221. A detailed paper will be published in the Transactions of this Society. (Received November 20, 1940.)

### 21. J. J. Eachus: *Classification of solutions and of pairs of solutions of $y''' + 2p(x)y' + p'(x)y = 0$ by means of boundary conditions.*

G. D. Birkhoff (Annals of Mathematics, (2), vol. 12 (1911), p. 103) has shown that the differential equation $y''' + 2p(x)y' + p'(x)y = 0$ has three types of solutions according to the number and nature of zeros of the solution. He has further shown that there are only a limited number of possible configurations of the zeros of two solutions of the equation. These same conclusions are deduced in this paper in a new manner, one which leads to a method for determining the "type" of a solution $y_1$ from the values of $y_1$ and its first two derivatives at any point $x_0$, and a method for determining the configuration of the zeros of $y_1$ and $y_2$ from the values of $y_1$ and $y_2$ and their first two derivatives at any point $x_0$. (Received November 25, 1940.)

### 22. Samuel Eilenberg: *Linear measure and convexity.*

Let $X$ be a continuum with the metric $\rho$. If the linear measure $L(X, \rho)$ is finite then there is a metric $\rho'$ equivalent with $\rho$ and such that (1) $\rho'$ is convex (2) $\rho' \geqq \rho$ (3) $L(X, \rho') = L(X, \rho)$. The metric $\rho'$ is unique. (Received October 24, 1940.)

### 23. K. W. Folley: *A property of a simply ordered set.*

It is shown in this paper that a method of proof which was employed by Sierpiński to prove a theorem concerning the decomposition of real numbers may be used to prove an analogous theorem concerning the decomposition of a simply ordered set which contains an everywhere dense $\eta_\alpha$ subset. This result leads to a proposition which is proved to be equivalent to the generalized hypothesis of the continuum. (Received October 25, 1940.)

### 24. H. L. Garabedian: *Relations between hypergeometric methods of summation.*

The definition of hypergeometric summability $(H, \alpha, \beta, \gamma)$ is extended and new inclusion and equivalence relations between hypergeometric methods of summation are obtained. The relation $(H, \alpha_1, 1, \alpha_1+\gamma)\approx(H, \alpha_2, 1, \alpha_2+\gamma)$, $\mathfrak{R}(\alpha_1, \alpha_2, \gamma)>0$, is probably the most significant result established. Corollaries to this result are $(H, \alpha, 1, \alpha+\beta)\approx(C, \beta)$, $\mathfrak{R}(\alpha, \beta)>0$; $(H, \alpha, 1, \gamma)\approx(C, \gamma-\alpha)$, $\mathfrak{R}(\alpha, \gamma, \gamma-\alpha)>0$. These statements imply that, given any hypergeometric method of summation of the type $(H, \alpha, 1, \gamma)$, there exists a Cesàro method of summation equivalent to it; and, conversely, given any Cesàro method of summation $(C, \beta)$ there exists a single infinitude of hypergeometric methods of summation of the type $(H, \alpha, 1, \alpha+\beta)$ equivalent to it. (Received November 20, 1940.)

### 25. J. W. Green: *A special type of conformal map.*

Let $G_z$ be the unit circle in the $z$ plane, $O$ an open set of points on the circumference of $G_z$, and $F$ its complement with respect to the circumference. The set $F$ is supposed to be of positive measure. In the case $\bar{O}-O$ is of zero measure, there exists a function $w=w(z)$ which maps $G_z$ conformally on the interior of the unit circle $G_w$ with certain radial slits deleted. This transformation carries an interval of $O$ into all or part of a radial slit, and carries points of $F$ into points of $|w|=1$, almost everywhere on $F$. (Received November 25, 1940.)

### 26. R. E. Greenwood: *Hankel and other extensions of Dirichlet series.*

This paper considers series of the form $g(s)=\sum_{n=1}^{\infty}a_n \exp(-\lambda_n s)G(\lambda_n s)$. If $G(z)\equiv 1$, then the series reduces to the well known Dirichlet series. If $G(z)$ is analytic in the right half-plane, is bounded in the half-plane to the right of $z_0>0$ and has a certain asymptotic expansion, the series above will be convergent in a half-plane, and also absolutely convergent in a half-plane. The abscissae of convergence are the same as the abscissae for the corresponding Dirichlet series. Summation formulae involving the first $m$ coefficients are developed, and these reduce to Perron's formula for Dirichlet's series when $G(z)\equiv 1$. As a particular case such a function $G(z)$ is chosen so that $g(s)$ is a series of Hankel functions of the first kind of fixed order $\nu$ on the imaginary axis, and a few special results are obtained for this series. (Received November 14, 1940.)

### 27. H. J. Hamilton: *On monotonic and convex solutions of certain difference equations.*

The principal result of this paper is reduction of the problems of existence and of uniqueness (excepting an additive constant) of continuous, convex solutions $U$ of the equation $U(x+1)-U(x)=G(x)$ to the same problems for monotone non-decreasing solutions of a related equation of the same form. This reduction extends itself partially to linear difference equations of higher order with constant coefficients. Our conclusions are applied to some results of Fritz John. (Received September 28, 1940.)

### 28. L. B. Hedge: *Transformations of multiple Fourier series.* Preliminary report.

If $(n)=(n_1, n_2, \cdots, n_k)$ is a lattice point of euclidean $k$-space and $(x)=(x_1, x_2, \cdots, x_k)$ is a point of euclidean $k$-space, and if $a_{(n)}$ and $\lambda_{(n)}$ are functions on the lattice points of $k$-space, we consider the series $S=\sum a_{(n)}e^{i(n\cdot x)}$ and its transform $\lambda S=\sum \lambda_{(n)}a_{(n)}e^{i(n\cdot x)}$ where $(n\cdot x)=n_1x_1+n_2x_2+\cdots+n_kx_k$. A classification, analogous to that used for single series, is defined for the series $S$ and $\lambda S$, and transformations $\lambda$ which take one class into another are characterized for interesting cases. The results include and extend the known transformation theory for single trigonometric series. The spherical summation of Bochner and a recent moment problem solution by the author are the principal tools used for the study. (Received December 2, 1940.)

### 29. A. E. Heins: *On the solution of partial difference equations. Two interval "boundary conditions."* Preliminary report.

The solution of the partial difference equation $f(x+1, t)+f(x-1, t)=2f(x, t+1)$ is considered under the following initial and boundary conditions: For $[t]=0, f(x, t)$ is prescribed, $af(x-[x], t)+f(x-[x]+1, t)=0$ and $bf(x-[x]+p, t)+f(x-[x]+p+1, t)=0$. The partial difference equation is first reduced to an ordinary difference equation with the Laplace transform and the resulting ordinary difference equation solved under the given boundary conditions. The solution depends on a finite expansion of sines and cosines of multiples of angles which are roots of a certain transcendental equation. The nature of this rather interesting expansion will be discussed at a later date. (Received November 18, 1940.)

### 30. M. R. Hestenes: *Extension of the range of a differentiable function.*

In the present paper two methods of extending the range of a differentiable function are given. The first method is essentially a reflection across boundaries. By this method it is shown that a function $f(x_1, \cdots, x_n)$ of class $C^m$ ($m$ finite) on a closed set $A$ with a suitable boundary can be extended to be of class $C^m$ over the whole of euclidean $n$-space. The second method is similar to one used by Whitney (Transactions of this Society, vol. 36 (1934), pp. 63–89) to show that a function of class $C^m$ on an arbitrary closed set $A$ can be extended to be of class $C^m$ over the whole space and to be of class $C^\infty$ at the points not in $A$. (Received November 26, 1940.)

### 31. Einar Hille: *A class of differential operators of infinite order.* II.

Let $D_zw=P_0(z)w''+P_1(z)w'+P_2(z)w$. Let $G(z)$ be an entire function of order $\frac{1}{2}$ and minimal type. Then the operator $G(D_z)$ preserves holomorphism in any domain where the $P$'s are holomorphic. The condition on $G(z)$ is also necessary for most operators $D_z$ of practical importance. The equation $G(D_z)\cdot U=0$ can be studied by the methods of Ritt and Valiron. If the $P$'s are single-valued and $a_1, \cdots, a_m$ are the singularities of $D_zw=0$, let $R$ be an unbounded non-ramified covering surface of the plane punctured at $a_1, \cdots, a_m$ on which all solutions of $(D_z-\lambda_n)w=0$ are single-valued, where $\{\lambda_n\}$ are the roots of $G(z)=0$. $U(z)$ is single-valued on $R$. Its domain of existence $E$ is such that any maximal component of $R-E$ has at least one point $a_k$

as limit-point. If $R$ is simply-connected, so is $E$. $U(z)$ has a unique Ritt series in terms of solutions of $(D_z-\lambda_n)w=0$. If $P_0=P_1=1$, $P_2$ entire, and the $\lambda_n$ suitably restricted, this series converges in $E$, and $E$ is convex. The results extend to differential operators $D_z$ of order $n$. (Received October 5, 1940.)

### 32. Fritz John: *On the character of solutions of hyperbolic equations*. Preliminary report.

Let $u(x_1, \cdots, x_n)$ be a solution of a linear, normal-hyperbolic differential equation of second order with analytic coefficients. The solution $u$ is shown to be "pseudo-analytic" in the following sense: Given a family of "time-like" analytic curves $C_\alpha$ depending on a parameter $\alpha$, all $C_\alpha$ having the same two endpoints, then $\int u dt$ is an analytic function of $\alpha$, $t$ being an analytic parameter on $C_\alpha$. It follows that the values of $u$ on any time-like two-dimensional analytic manifold, interior to the domain of regularity of $u$, are not independent. This is a generalization of results previously obtained by the author for a special equation (Mathematische Annalen, vol. 111, pp. 541–555). The proof makes use of the form of the solution obtained by J. Hadamard in his *Lectures on Cauchy's Problem*. (Received November 20, 1940.)

### 33. J. P. LaSalle: *A note on pseudo-normed linear spaces*.

The equivalence of speudo-normed linear spaces (p.l.s.'s) and linear topological spaces (l.t.s.'s) has been shown by Hyers. (See Duke Mathematical Journal, vol. 5 (1939), pp. 628–634.) In the present paper l.t.s.'s are studied as characterized in terms of the pseudo-norm. It is shown that the existence of an open convex set containing the zero element and properly contained in a p.l.s. $T$ is a necessary and sufficient condition for the existence of a non-null linear functional on $T$. An example is given of a p.l.s. on which no non-null linear functional can be defined. Also the set of all linear functions on a p.l.s. $T$ to a p.l.s. $T'$ is shown to be a pseudo-normed linear space. If $T'$ is convex, then the p.l.s. of all linear functions on $T$ to $T'$ is itself convex. Hence the set of all linear functionals on a p.l.s. $T$ is a convex p.l.s. If a p.l.s. $T$ is locally bounded, then the set of all linear functionals on $T$ is shown to be a normed linear space. (Received October 22, 1940.)

### 34. J. P. LaSalle: *Pseudo-normed linear sets over valued rings*. I.

In this paper a generalization of linear spaces is given by replacing real number multipliers by multipliers taken from a valued ring. A valued ring is defined to be a ring $A$ with a unity element such that there is defined on $A$ a real-valued function $M(\alpha)$ where (1) $M(\alpha)\geqq 0$ for all $\alpha\in A$; (2) $M(\alpha\beta)\leqq M(\alpha)M(\beta)$; (3) $M(\alpha+\beta)\leqq M(\alpha)+M(\beta)$; (4) $M(-1)=1$; (5) $M(\alpha)>1$ for some $\alpha\in A$. The set of all formal polynomials with coefficients taken from a ring with a unity element can be shown to be a valued ring, which indicates somewhat the generality of valued rings. A linear set $T$ over a valued ring $A$ is said to be pseudo-normed w.r.t. a strongly partially ordered set $D$ if there exists a real-valued function $n(x, d)$ on $TD$ such that (1) $n(x, d)\geqq 0$; $n(x, d)\leqq 1$ for all $x\in T$ implies $x=\Theta$; (2) $n(\alpha x, d)\leqq M(\alpha)n(x, d)$; (3) given $d\in D$ there exist $e\in D$ such that $n(x, e)\leqq 1$, $n(y, e)\leqq 1$ implies that $n(x+y, d)\leqq 1$; (4) given $x\in T$ and $d\in D$ there exist $\alpha\in A$ and $y\in T$ such that $x=\alpha y$, $n(y, d)\leqq 1$; (5) given $\alpha\in A$ and $d\in D$ there exist $e\in D$ such that $n(x, e)\leqq 1$ implies that $n(\beta x, d)\leqq 1$ for all $M(\beta)\leqq M(\alpha)$; (6) $e\geqq d$ implies that $n(x, e)\geqq n(x, d)$. Another characterization of $T$ is given in terms of a neighborhood system. The topological properties of such a space $T$ are then studied. Later a generalized concept of linear functions and differentials will be introduced for such spaces. (Received October 22, 1940.)

### 35. Walter Leighton: *On the convergence of continued fractions.*

Let $z_1, z_2, \cdots$ be any bounded sequence of complex numbers such that $R(z_n) \geq \epsilon > 0$ ($n=2, 3, \cdots$), where $\epsilon$ may be taken arbitrarily small. The continued fraction $1+K_1^\infty[a_n/1]$ converges, where $a_n=(z_{n-1}+1/2)(z_n+1/2)$ ($n=2, 3, \cdots$; $z_1=1$). (Received November 18, 1940.)

### 36. J. Marcinkiewicz and Antoni Zygmund: *On the behavior of trigonometric and power series.*

A point set $Z$ on the plane is said to be of circular structure with center $z_0$ if whenever a point $\zeta$ belongs to $Z$, so does the whole circle $|z-z_0|=|\zeta-z_0|$. Given a power series $\sum_0^\infty c_n z^n$, let $\sigma_n^\alpha(\theta)$ denote the $\alpha$th Cesàro means of the series $\sum_0^\infty c_n e^{in\theta}$ and let $L^\alpha(\theta)$ denote the set of the limit points of the sequence $\sigma_0^\alpha(\theta), \sigma_1^\alpha(\theta), \cdots, \sigma_n^\alpha(\theta), \cdots$. The chief result of the paper may be stated as follows: If for every $\theta$ belonging to a set $E$ of positive measure the series $\sum_0^\infty c_n e^{in\theta}$ is summable $(C, \alpha+1)$ $(\alpha>-1)$ to sum $t(\theta)$, then for almost every $\theta$ of $E$ the set $L^\alpha(\theta)$ is of circular structure with center $t(\theta)$. (Received October 31, 1940.)

### 37. R. S. Martin: *Minimal positive harmonic functions.*

The idea of a minimal positive harmonic function arises as follows. The Poisson-Stieltjes integral formula for a sphere depends upon (1) a family of positive harmonic functions (namely the kernel $F(S, P)$ where the point $S$ lies on the boundary and is considered as a parameter), and (2) a family of linear operations (namely integration with respect to mass-distributions on the boundary). Given then a general bounded domain $D$, one may ask for suitable generalizations of the situations just described. The idea of the present paper is to consider, as a generalization of the family $F(S, P)$ corresponding to the sphere, the family of those functions which are *minimal*, positive and harmonic in $D$. A function $u(P)$, positive and harmonic in $D$, is called *minimal* for $D$ if it dominates there no positive harmonic functions, except those of the form $cu(P)$, where $c>0$ is a constant. The purpose of the paper is to discuss the theory of representations of harmonic functions based on this idea. (Received October 22, 1940, from Tibor Radó.)

### 38. C. N. Moore: *On the Cesàro and Abel-Poisson summability of the differentiated double Fourier series.*

It has been shown that at points where $f(x)$ has a derivative or generalized derivative the differentiated Fourier series will be summable $(C, r)$ for $r>1$ or will be summable by the Abel-Poisson method. In the present paper these results are extended to the partial derivatives or generalized partial derivatives of functions of two variables and the corresponding differentiated double Fourier series. The result concerning Abel-Poisson summability appears as a corollary of the result concerning Cesàro summability by use of general convergence factor theorems. (Received November 27, 1940.)

### 39. G. D. Nichols: *A sufficient condition for Cesàro summability.*

The series $\sum f(r) \cos rx$ and $\sum f(r) \sin rx$ are shown to be summable $(C, k)$ provided the $\Delta^k f(r)$ is a monotone null sequence. In case $f(r)$ is a polynomial of degree $k-1$, a closed expression is obtained for the Cesàro sum. (Received November 23, 1940.)

### 40. I. E. Perlin: *Indefinitely differentiable functions of several real variables.*

In the present paper the author considers indefinitely differentiable functions of several real variables. Let $M_{ij\cdots k}=\text{l.u.b.}\ |\partial^{i+j+\cdots+k}f(x_1, x_2, \cdots, x_n)/\partial x_1^i\partial x_2^j\cdots\partial x_n^k|$ on $0\leqq x_p\leqq a_p$. Sufficient conditions that there exist a function $f(x_1, x_2, \cdots, x_n)$ with prescribed $M_{ij\cdots k}$ are established. (Received November 25, 1940.)

### 41. Arthur Sard: *The measure of the critical values of differentiable maps of euclidean spaces.*

Consider the map $y^j=f^j(x^1, \cdots, x^m)$, $j=1, 2, \cdots, n$, of a region $R$ of euclidean $m$-space into part of euclidean $n$-space. Suppose each of the functions $f^j$ is of class at least $C^q$ in $R$ ($q\geqq 1$). A critical point of the map is a point in $R$ at which the matrix of first derivatives, $(f_i^j)$, $i=1, 2, \cdots, m$; $j=1, 2, \cdots, n$, is of less than maximum rank. A critical value $y$ is the image $f(x)$ of a critical point $x$. The following theorem is proved: If $m\leqq n$, the set of all critical values is of $m$-dimensional measure zero (in the sense of Hausdorff-Saks) without further hypothesis on $q$; if $m>n$, the set of all critical values is of $n$-dimensional measure zero providing $q\geqq m-n+1$. For $n=1$, this result specializes to a known theorem, surmised by Marston Morse and proved for all $m$ by A. P. Morse (Annals of Mathematics, (2), vol. 40 (1939), pp. 62–70). The proof of the theorem of the present paper depends in part on a result of A. P. Morse. The hypothesis on $q$ cannot be weakened, as can be seen from an example due to H. Whitney (Duke Mathematical Journal, vol. 1 (1935), pp. 514–517). (Received November 25, 1940.)

### 42. H. M. Schwartz: *Convex functions and the law of the mean.*

Let $S$ denote the class of functions of a real variable for which the law of the mean holds throughout a given interval, and let $T$ denote the class of functions which are either strongly convex or strictly concave in that interval. The classical conditions insuring that the function $f$ be in $S$ are for $f$ of $T$ necessary as well as sufficient. The class $U$ common to both $T$ and $S$ is uniquely determined by the requirement that the $\theta$-function occurring in the law of the mean be a single-valued function of its arguments. This $\theta$-function for $f$ of $U$ is studied both as to its dependence on $f$ and as to its dependence on the variable and its increment. It is shown, for instance, that if $f_n\in S$ ($n=1, 2, \cdots$), sequence $f_n'$ converges continuously to $f'$ in the interval, and $f\in U$, then $\theta(f_n)$ converges to $\theta(f)$ in the domain of its arguments. Most of the results are extended to functions of many variables. (Received November 20, 1940.)

### 43. H. M. Schwartz: *Sequences of Stieltjes integrals.*

The paper contains a study of some of the properties of function sequences of the form $\int_a^x f dg_n = s_n$ ($n=1, 2, \cdots$), $a<x\leqq b$ ($a$, $b$ finite or infinite), where $f$, $g_n$ are bounded functions for which the integrals exist in the sense of Riemann-Stieltjes. When $g_n$ are of bounded variation an obvious sufficiency condition for the convergence of $s_n$ in $(a, b)$ can be stated in terms of the integrability conditions of $f$ with respect to $g_n$, but it is of interest to find such conditions in terms of the convergence properties of the sequence $g_n$. This problem is solved here only partially. Assuming that $g_n$ are uniformly of bounded variation in $(a, b)$, the convergence of $s_n$ is proved for special classes of $f$ and ordinary convergence of the sequence $g_n$, and generally for special modes of convergence of this sequence. A typical result is as follows: $s_n$ converges in

$(a, b)$ (assumed finite) if $g_n \rightarrow g_0$ and if $\int f dg_n$ $(n=0, 1, 2, \cdots)$, $\int f dv$ exist, where $v$ is any of the limit functions of the sequence $v_n$, $v_n(x)$ being the total variation of $g_n$ in $(a, x)$. (Received November 25, 1940.)

### 44. Hyman Serbin: *Upper bounds for the remainder of certain power series.*

This note exemplifies a method of obtaining an upper bound of the remainders of power series in which the coefficients are connected by a recurrence relation with a constant number of terms. The function considered is wave mechanical $\exp(-ix) \cdot F(m+1-ai; 2m+2; 2ix)$ where $F(\alpha; \gamma; z)$ is the confluent hypergeometric function. A majorant of the remainder, suggested by the recurrence relation, is expressed in closed form. The upper bound so obtained is not explicitly given in terms of the parameters but depends on two successive coefficients. However, for the purpose of computation, the result is satisfactory. The above results were obtained in the course of work done by the Project for the Computation of Mathematical Tables, conducted by the Work Projects Administration of New York City. (Received October 30, 1940.)

### 45. W. S. Snyder: *On independence of the path for line integrals of continuous functions.*

Let $p(x, y)$ and $q(x, y)$ be defined and continuous in an oriented rectangle $R$ of the $xy$-plane. A necessary and sufficient condition that $\int p dx + q dy$ be independent of the path in $R$ is derived in terms of the double integrals of the functions $p$ and $q$. The author also formulates and answers a more general form of a question raised by Menger (Proceedings of the National Academy of Sciences, vol. 25 (1939), p. 623), showing that the conditions of Menger and Fubini (Proceedings of the National Academy of Sciences, vol. 26 (1940), p. 199) cannot be satisfied for "undotted" nets. (Received November 23, 1940.)

### 46. Abraham Spitzbart: *Approximation in the sense of least pth powers by polynomials with a single auxiliary condition of interpolation.*

Let $F_0(z)$ be the function, analytic interior to the analytic Jordan curve $C$, of class $E_p$ $(p>0)$ in that region, and satisfying the condition $F_0(\alpha)=A$ ($z=\alpha$ a point interior to $C$, $A$ an arbitrary contant), which minimizes the integral $\int_C |F(z)|^p |dz|$. Let $P_n(z)$ be the minimizing polynomial of degree $n$ for the integral $\int_C |P_n(z)|^p |dz|$, $P_n(\alpha)=A$. It is proved that the sequence $P_n(z)$, $n=0, 1, 2, \cdots$, converges maximally to the function $F_0(z)$ on the closed set $\Gamma$ consisting of $C$ plus its interior (for definition see J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, American Mathematical Society Colloquium Publications). The results extend to the case where a suitable norm function $N(z)$ is introduced into the integral to be minimized. (Received November 25, 1940.)

### 47. Otto Szász: *On the partial sums of harmonic developments and related power series.*

Let $T$ denote the class of all harmonic sine developments $H(r, \theta)=\sum C_\nu r^\nu \sin \nu\theta$, convergent for $0<r<1$, and non-negative for $0<\theta<\pi$. There is a largest $R^n$ such that for every $H$ in $T$, $\sum_1^n C_\nu r^\nu \sin \nu\theta$ is non-negative for $0<r<R_n$, $0<\theta<\pi$. Then $R_n$ is characterized algebraically, which yields an asymptotic estimate. The result is applied to Fourier series of convex functions and to related power series. (Received November 26, 1940.)

### 48. A. E. Taylor: *Conjugations of complex Banach spaces.*

Let $E$ be a complex Banach space. An operation $T$ on $E$ to $E$ is called a conjugation if it is additive, and if $Taz=\bar{a}Tz$, $T^2z=z$. A set $M$ in $E$ is called an essentially real linear manifold (ess. r.l.m.) if it is closed under addition, and under multiplication by real numbers, and if $x$, $ix$ in $M$ implies $x=0$. An ess. r.l.m. is maximal if it is not a proper subset of any ess. r.l.m. If $M$ is a maximal ess. r.l.m. every element of $E$ is uniquely representable in the form $z=x+iy$, where $x$, $y$ are in $M$. The operation $T(x+iy)=x-iy$ is a conjugation. $T$ is continuous if and only if $M$ is closed. Also, if $T$ is any conjugation, the set of elements $x$ such that $Tx=x$ is a maximal ess. r.l.m. It may be shown that every $E$ has a maximal ess. r.l.m. The author does not know if there always exists a closed, maximal ess. r.l.m., but when one does exist it is possible to re-norm the space, without altering the topology, in such a way that $\|x+iy\|=\|x-iy\|$. (Received October 26, 1940.)

### 49. J. M. Thomas: *Orderly differential systems.*

Riquier's existence theorem for orthonomic systems is extended to a larger class of systems called orderly. Substantially the defining property of an orderly system is that it can be decomposed into orthonomic components. The proof of the existence theorem given is new, even for the case of orthonomic systems. Its chief feature is the elimination of the integers called cotes by Riquier. They are replaced by two systems of inequalities which it is thought arise more naturally and have simpler nature than the systems of inequalities among the cotes. (Received November 20, 1940.)

### 50. W. J. Trjitzinsky: *Analytic theory of singular elliptic partial differential equations.*

This work presents an extensive analytic theory of the equation $F(u)=\sum a_{i,k}\cdot\partial^2u/dx_i\partial x_k+\sum b_i\partial u/\partial x_i+(c+\lambda)u=f$ $(i,\ k=1,\ 2,\ \cdots,\ m)$, where $\lambda$ is a parameter and the coefficients are functions of $(x_1, \cdots, x_m)$ continuous in an open domain $D$, the first partials of the $b_i$ and a number of partials of the $a_i$, $k$ are continuous in $D$, and the functions, involved, may become infinite near the frontier of $D$. The investigation was suggested by certain analytic aspects of Schrödinger wave equations. The equation is transformed into an integral equation with the aid of a suitable function of the geodesic distance (formed for a suitable line element); the integral equation plays an essential role in the work, in particular, Fredholm's theory is locally applicable in $D$. Associated with $F$ there is constructed a sequence of "regular" approximating homogeneous boundary value problems, $(P_1)$, $(P_2)$, $\cdots$. When $F$ is self adjoint, theorems on existence of solutions, uniqueness properties and so forth are given for $\lambda$ nonreal; also, for $\lambda$ real, depending on sufficient "rarefication" of the totality of characteristic values formed for $(P_1)$, $(P_2)$, $\cdots$. An essential role is also played by an appropriate spectral theory based on $(P_1)$, $(P_2)$, $\cdots$. (Received November 28, 1940.)

### 51. H. S. Wall: *A theorem on real functions bounded in the unit circle.*

Let $E$ denote the class of functions $e(x)$ which are analytic for $|x|<1$, real when $x$ is real, and for which $M(e)=\text{l.u.b.}_{|x|<1}|e(x)|\leq 1$. Let $F$ denote the subclass of all functions $f(x)$ of $E$ which are of the form $f(x)=\int_0^1 d\phi(u)/(1+xu)$ where $\phi(u)$ is real, bounded and monotone nondecreasing on the interval $0\leqq u\leqq 1$. Then there is a one-to-one correspondence between the functions of $E$ and of $F$ such that if $e(x)\longleftrightarrow f(x)$, then $e(x)=[1-x-2f(z)]/[1-x+2xf(z)]$, $z=4x/(1-x)^2$, $|x|<1$. (Received November 7, 1940.)

### 52. J. L. Walsh and E. N. Nilson: *Approximation to an analytic function by functions analytic and bounded in a region.*

Let $R$ be a finite sum of disjoint regions containing a closed set $S$ in its interior, with the respective boundaries $C_1$ and $C_0$ composed of a finite number of analytic Jordan curves, each component region of $R$ containing at least one point of $S$, no point $R-S$ separated by $S$ from $C_1$. Define $u(z)$ as unity on $C_1$, zero on $C_0$, continuous in the extended plane, harmonic except on $C_1$ and $C_0$. If $v(z)$ is conjugate to $u(z)$ in $R-S$, then $u(z)=u(\infty)-(1/2\pi)\int_{C_1+C_0}\log|z-\zeta|\,dv(\zeta)$, for $z$ not on $C_1+C_0$. This equation is used to define poles and points of interpolation for sequences of rational functions which are used (Walsh, Proceedings of the National Academy of Sciences, vol. 24 (1938)) to prove: Let $f(z)$ be analytic on $S$ but not throughout $R$ (a single region); for $M>0$, let $f_M(z)$ be that function analytic and of modulus not greater than $M$ in $R$ for which [max $|f(z)-f_M(z)|$, $z$ on $S$] is least. Then, for $0\leqq\sigma<\rho$, $\limsup_{M\to\infty}$ [max $|f(z)-f_M(z)|$, $z$ in $\overline{R}_\sigma]^{1/\log M}=e^{(\sigma-\rho)/(1-\rho)}$, where $R_\sigma$ is the set on which $0\leqq u(z)<\sigma$ and $\rho$ is the largest number, $0<\rho<1$, such that $f(z)$ is analytic throughout $R_\rho$. (Received December 2, 1940.)

### 53. W. F. Whitmore: *Convergence theorems for functions of two complex variables.* II.

In continuation of the work of the first part of this paper (American Journal of Mathematics, vol. 62 (1940), pp. 687–696), results are obtained for convergence of a function of two complex variables in sector-domains $I^4=\sum_{|z_2|\leqq 1}I^2[\theta_2(z_2),\ \theta_1(z_2)]$ where $I^2(\theta_2,\ \theta_1)=E[\theta_2\geqq\arg z_1\geqq\theta_1]$ (superscript indices give the dimensionality of the domain in question). The limit function approached at the vertex surface $H^2=E[z_1=0$, $|z_2|\leqq 1]$ is also examined, and shown to be an analytic function of $z_2$. The principal tool is the theory of harmonic measure. These sextor-domains are useful as domains of comparison for arbitrary domains whose boundary hypersurfaces contain a segment of an analytic hypersurface. (Received October 28, 1940.)

### 54. D. V. Widder: *Necessary and sufficient conditions for the representation of a function in Lidstone series.*

A set of polynomials $\Lambda_n(x)$ is defined as follows: The first, $\Lambda_0(x)$, is $x$. For each positive integer $n$ the second derivative of $\Delta_n(x)$ is to be $\Lambda_{n-1}(x)$, and each polynomial except the first is to vanish at zero and at one. For example, $\Lambda_1(x)$ is $(x^3-x)/6$. In terms of these polynomials a Lidstone series for the functions $f(x)$ has the form $f(x)=\sum_{k=0}^{\infty}f^{(2k)}(1)\Lambda_k(x)+f^{(2k)}(0)\Lambda_k(1-x)$. Any partial sum of the series is a polynomial whose coefficients are determined so that it may coincide together with as many of its even derivatives as possible with $f(x)$ and its corresponding even derivatives at zero and one. These series, discovered by G. J. Lidstone, have been studied by H. Poritsky, J. M. Whittaker and I. J. Schoenberg. (References may be found in Schoenberg's paper, this Bulletin, vol. 42 (1936), pp. 284–288.) In the present paper the series are considered from the real point of view. A new proof of a theorem of the author is given (Proceedings of the National Academy, November, 1940) that a function $f(x)$ for which $(-1)^k f^{(2k)}(x) \geqq 0$ on an arbitrary interval is necessarily entire. By use of this result a necessary and sufficient condition that a function can be expanded in an absolutely convergent Lidstone series is obtained. (Received November 25, 1940.)

### 55. J. R. Woolson: *A Theory of projections in complex Banach spaces*. Preliminary report.

A projection $P$ is defined as a linear operator on a complex Banach space $B$ to $B$, such that $P^2 = P$. Using an inter-space product $[F, f]$, $f \in B$, $F \in (B)$ the set of complex valued linear functionals defined on $B$, it is possible to prove the usual theorems concerning projections by certain analogies with the Hilbert space inner-product. (Cf. M. H. Stone, *Linear Transformations in Hilbert Space and Their Applications to Analysis*, American Mathematical Society Colloquium Publications, vol. 15.) The theory is used to show that $(1/n)\sum_{i=1}^{\infty} A^i$ converges weakly to a projection and to characterize a linear operator in terms of the linear manifolds which it leaves invariant. (Received October 28, 1940.)

## APPLIED MATHEMATICS

### 56. Harry Bateman: *Aerodynamical effects of changes in the fundamental equations.*

Simple examples indicate that a change from the elliptic to the parabolic type may not be as drastic in its effects as a change from the elliptic to hyperbolic type. The effects of the various changes that have been made in the equations of viscous flow by Prandtl, Oseen, Kármán and others are reviewed by the author. Some remarks are made also on mixtures of fluids and changes of state. Some new equations are considered and reference is made to the work of Duhem, Schutz, Silberstein, Natanson, Bjerknes and Kozlowski. Some remarks are made on the equations of Burgers, Mattioli and others which differ from the classical equations and yet exhibit some of the phenomena of turbulence. (Received November 18, 1940.)

### 57. M. A. Biot: *Finite difference equations applied to aircraft engine vibrations.*

The vibration amplitudes of the crankshaft are shown to satisfy a finite difference equation of the second order with constant coefficients. The frequency equation derived from the end conditions is solved numerically by an artifice which shortens considerably the time required for the evaluation of the critical speeds. (Received November 18, 1940.)

### 58. J. Bjerknes: *Some uses of mathematics in meteorology.*

The atmosphere exhibits tide-like wave phenomena, but the lunar component of the atmospheric tide is only about one tenth of the solar component. If the atmospheric tide is gravitational, just as the ocean tides, the atmosphere must be able to act as a vibrating system with a proper period very close to 12 solar hours and thereby give the solar component of the tide a greater amplitude than the lunar component. That vibration problem is mathematically rather well defined but still unsolved in its general form. The atmospheric disturbances responsible for the day to day variations of weather are mainly aperiodic and disorderly but at times they are quasi-periodic and to some extent amenable to mathematical treatment. The nearest approximation to wave-like conditions are found in the upper atmosphere. The general westerly current which prevails there takes a sine curve shape under disturbed conditions and the sinuosities move along with the current but with less speed than the air. That wave phenomenon is mathematically less tractable than the tidal waves, mainly because of the difficult boundary conditions, but whatever little progress can be made with it would be a valuable contribution to the science of weather forecasting. (Received November 18, 1940.)

### 59. B. E. Gatewood: *Thermal stresses in regions bounded by epitrochords.*

Muschelisvili's method of solving the biharmonic equation when the first derivatives are known on the boundary of the region is used in solving the thermal stress problem for a long cylindrical body whose cross section is bounded by an epitrochord (see abstract 45-9-314.) (Received November 14, 1940.)

### 60. F. B. Hildebrand: *The approximate solution of singular integral equations arising in engineering practice.*

In this paper a method of numerical solution of integral equations given by Crout (Journal of Mathematics and Physics, M.I.T., vol. 19 (1940), pp. 34–92) is extended to the solution of certain integral equations wherein the unknown function, as well as the kernel, may be singular. Considering an equation of the first kind, $\phi(x) = \int_a^b \sigma(\xi)G(x, \xi)d\xi$, the unknown function $\sigma(x)$ is approximated by a linear combination $s(x)$ of suitably chosen singular functions, $s(x) = \sum_{j=1}^{r} \chi_j s_j(x)$, where the $\chi$'s are undetermined constants, plus a polynomial $y(x)$ of order $2n$, in the Lagrangean form $y(x)\sum_{i=-n}^{n} K_i(x)y_i$, where the $y$'s are the undetermined ordinates at $2n+1$ equally spaced points in $(a, b)$. The conditions $\phi(x_k) \approx \int_a^b \{s(\xi)+y(\xi)\} G(x_k, \xi)d\xi$, $k=1, 2, \cdots, m$, where $m \geqq 2n+r+1$, constitute $m$ linear equations in the $2n+r+1$ parameters, the parameters then being determined by a method of least squares. Several specific problems arising in static field theory and elasticity are solved numerically, the results being in good agreement with exact results obtained by other methods. The computation involved is organized by the use of matrices and reduced principally to operations for which the modern computing machine is well adapted. (Received November 25, 1940.)

### 61. Theodore von Kármán and W. R. Sears: *Solution of the problem of the airplane wing of finite span by expansion in eigen-functions.* Preliminary report.

The problem of the calculation of the lift distribution along the span of a wing according to the lifting-line theory of L. Prandtl can be reduced to the determination of a two-dimensional potential function which satisfies on the boundary of a circle a linear relation with variable coefficients between the values of the function and its normal derivative. In engineering practice this problem is usually solved by means of a development of the boundary values of the potential function in a trigonometric series. In the present paper the classical method of expansion in a series of the eigen-functions of the problem is carried out. It is shown that this method has several practical advantages; moreover, the eigen-functions and eigen-values required for the calculation are independent of the twist of the wing and each set of them is applicable for a certain family of planforms. They can be estimated by various approximate methods. (Received November 18, 1940.)

### 62. W. D. Rannie: *Tensor methods in the theory of turbulence.* Preliminary report.

Applications of tensor analysis to the statistical theory of isotropic turbulence as developed by T. von Kármán and L. Howarth, and later treated by H. P. Robertson, are reviewed. (Received November 18, 1940.)

### 63. Eric Reissner: *A new derivation of the equations for the deformation of elastic shells.*

The equations of the theory of small deformations of shells, first given by A. E. H. Love, are rederived in a simpler manner. The simplifications are accomplished by using (1) vector stress resultants and equilibrium conditions in vector form and (2) the three-dimensional system of orthogonal coordinates which goes with the lines of curvature on the middle surface of the shell and the strain components with respect to this system. The assumption that the normal to the undeformed middle surface is deformed into the normal to the deformed middle surface, satisfied by determining appropriate displacement components, is introduced into these strain components. (Received November 25, 1940.)

### 64. H. J. Stewart: *Steady state oscillations in an atmosphere on a rotation sphere.* Preliminary report.

If one plots the mean surface atmospheric pressure, averaged over a period of at least a week, one finds that in addition to the mean westerly flow of air, there exist large scale closed isobaric systems which change very slowly with time. Attempts to develop long range weather forecasting techniques have shown the positions of these systems to be of primary importance and a knowledge of the factors which control these systems is very useful as a guide in formulating forecasting methods. In the present paper certain steady state oscillations of the stratosphere are investigated and are shown to vary with the mean velocity in the same manner as the observed oscillations. (Received November 18, 1940.)

## GEOMETRY

### 65. P. O. Bell: *On differential geometry intrinsically connected with a surface element of projective arc length.*

In this paper a surface element of projective arc length is interpreted geometrically and used to obtain a new geometric interpretation for each of the following: a generalization of Bompiani's projective curvature, a generalization of Fubini's asymptotic curvature, a projective torsion introduced in this paper, conjugate tangents, the tangents of Darboux, and the tangents of Segre. The associate conjugate net of an arbitrary net $N_{\lambda_1\lambda_2}$ of a surface $S$ (introduced in this paper) is defined as the conjugate net whose tangents at a point $P$ of $S$ separate harmonically the tangents at $P$ of the net $N_{\lambda_1\lambda_2}$. The following characteristic property of this net is a typical result: Let arcs $PP_1$, $PP_2$ of equal projective length $s$ be measured, with respect to the form $ds=(2Rv')^{1/2}du$, from the point $P$ along the curves $C_{\lambda_1}$, $C_{\lambda_2}$, respectively, of the net $N_{\lambda_1\lambda_2}$. The tangent plane to $S$ at $P$ intersects the line joining $P_1P_2$ in a point $P_3$ which tends to a limit point $P_0$, distinct from $P$, as $s$ tends to zero. The tangent line joining $PP_0$ and its conjugate tangent envelop the conjugate associate of the net $N_{\lambda_1\lambda_2}$, as $P$ varies over $S$. (Received November 20, 1940.)

### 66. L. M. Blumenthal: *Betweenness in metric ptolemaic spaces.*

A metric ptolemaic space (*MP*-space) is a metric space for which the determinant $|(p_ip_j)^2|$, $(i, j=1, 2, 3, 4)$, is nonpositive for every quadruple of points $p_1$, $p_2$, $p_3$, $p_4$ of the space. Investigating such spaces with regard to the four-, three-, and two-triple properties shows: (1) each *MP*-space has the four-triple property; (2) an *MP*-space has the three-triple property unless it contains the vertices of a convex tripod (that is, four points with one of them between each two of the remaining three points); (3) if four points of an *MP*-space contain exactly two linear triples, they may be labelled $a$, $b$, $c$, $d$ so that either $abc$ and $abd$ or $acb$ and $adb$ hold. Thus an *MP*-space has the two-triple property unless it contains vertices of a fork or a bow. Besides the transitive property of the betweenness relation valid in all metric spaces, there is in *MP*-spaces the additional transitivity $abc$ and $bad \rightarrow cad$ and $abd$. These results are preliminary to a study of separable, complete, convex *MP*-spaces which contain for each three non-linear points at least one point equidistant from them. (Received November 23, 1940.)

### 67. L. M. Blumenthal and C. V. Robinson: *Helly theorems on the sphere.*

Theorems for families of convex subsets of the sphere are given corresponding to the following theorem due to E. Helly: If each $n+1$ members of a family of convex bodies of $E_n$ intersect, then all the members of the family have a common point. On the sphere the existence of a common point is implied by the intersection of each 3, 4, 5, or 6 sets of the family according to the generality of the family considered. (Received November 26, 1940.)

### 68. C. R. Cassity: *The double points of a pencil of cubics invariant under the quadratic transformation.*

The double points of a general pencil of cubics contained in the invariant web of cubics of the involutorial quadratic transformation lie at the invariant points of the transformation and by pairs on the four lines which contain the six base points of the pencil which are not base points of the web. (Received November 25, 1940.)

### 69. Nathaniel Coburn: *A note on conformal geometry.*

Two Riemannian spaces of $n$-dimensions $V_n$ and $'V_n$ are considered. If coordinate systems in $V_n(\xi^\lambda)$ and $'V_n(\xi^\lambda)$ can be chosen so that at corresponding points $P(\xi^\lambda)$, $'P(\xi^\lambda)$ the connections of the spaces are related by the conformal transformation, then are the spaces conformally related? The problem is shown to be equivalent to determining the number of independent solutions of a system of linear partial differential equations. By analyzing the integrability conditions of the system, it is shown that only one independent solution exists. Hence the spaces are necessarily conformal. The complete theorem is: If coordinate systems in $V_n(\xi^\lambda)$ and $'V_n(\xi^\lambda)$ exist so that at corresponding points $P(\xi^\lambda)$, $'P(\xi^\lambda)$ the connections of these spaces are related by the conformal transformation and if the principal directions of the metric tensor of $'V_n$ exist in $V_n$, then the spaces are conformal. This theorem is of interest in that it can be shown that the corresponding theorem is not valid in unitary spaces. (Received November 20, 1940.)

### 70. Nathaniel Coburn: *Unitary spaces with corresponding geodesics.*

In the first section of the paper, the equations of geodesic curves $X_1$, which depend on a real parameter ($t$) and which are imbedded in a unitary space of $n$-dimensions $K_n$, are derived from the calculus of variations (Euler equations). The principal result is: the equations of such geodesics differ from the equations of geodesics in Riemannian space in that the former contain the torsion affinor whereas the latter do not contain this affinor. In the second section, the discussion centers on the connections of two unitary spaces $K_n$ and $'K_n$ whose geodesics correspond. It is shown that: (1) if two unitary spaces, both with symmetric connections, have their geodesics in correspondence, then the connections are related by the projective transformations; (2) if $K_n$ has a connection with torsion and $'K_n$ has a symmetric connection, then their geodesics cannot correspond. The first result is obtained in the same manner as the similar result in Riemannian space. The problem of determining all connections of unitary spaces $K_n$, both with torsion, whose geodesics correspond is left open. (Received November 20, 1940.)

### 71. Richard Courant: *Critical points and unstable minimal surfaces.*

Morse and Tompkins, and independently Shiffman (Annals of Mathematics, (2), vol. 40 (1939), pp. 834–854), have shown that Morse's theory of critical points in function spaces can be applied to minimal surfaces spanning suitably smooth contours. The main difficulty to overcome is the proof of a deformation property of the Dirichlet functional in the space of harmonic vectors. For this purpose a thorough analysis of the explicit Douglas boundary functional or its equivalent is needed. The present note, in line with the author's previous work on the Douglas problem, attacks the question for polygonal contours in a wider space by intrinsic considerations. It solves the problem by reducing it to that of the stationary points of a function of a finite number of variables with continuous derivatives. (Received November 27, 1940.)

### 72. N. A. Court: *On the harmonic pole.*

If $U$, $V$ are two tetrahedrons (triangles) polar reciprocal for a quadric (conic), and $L$ is the pole of a plane $\lambda$ (line $l$) for the quadric (conic), the harmonic plane (line) of $L$ for $U$ and the harmonic pole of the plane $\lambda$ (line $l$) for $V$ are pole and polar plane (line) with respect to the quadric (conic). Various consequences of this proposition are considered, of which the following may be noted: The harmonic plane (line) of a point of a quadric (conic) for a tetrahedron (triangle) inscribed in the quadric (conic), and the harmonic pole, for the tangential tetrahedron (triangle), of the tangent plane (line) to the quadric (conic) at the point considered are polar for the quadric (conic). (Received November 23, 1940.)

### 73. N. A. Court: *On the skew cubic.*

Given a tetrahedron $T$ whose faces osculate a skew cubic $C_3$ the harmonic pole for $T$ of a variable osculating plane of $C_3$ lies on a fixed line $j$. Conversely, given $T$ and $j$, the harmonic plane for $T$ of a variable point of $j$ osculates a skew cubic. The transforms of $j$ in the three skew harmonic homologies having for axes the three pairs of opposite edges of $T$ are axes of the cubic, and so are the transforms of $j$ in the four homologies, of constant $-3$, of which the vertices and the respectively opposite faces of $T$ are the centers and planes of homology. Various properties connected with these axes of $C_3$ are considered. (Received November 23, 1940.)

### 74. S. B. Myers: *Complete Riemannian manifolds of positive mean curvature.*

The author has proved previously that if, on a complete $n$-dimensional Riemannian manifold $M$, the curvature at every point and with respect to every pair of directions is greater than a fixed positive constant, then $M$ is closed (compact) and so is its universal covering manifold. In the present paper the same conclusions are drawn from the weaker hypothesis that the mean curvature of $M$ at every point and with respect to every direction is greater than a fixed positive constant. In particular, a complete space of constant positive mean curvature is closed, and so is its universal covering manifold. Such spaces are important in the general theory of relativity. (Received November 25, 1940.)

## STATISTICS AND PROBABILITY

### 75. G. A. Baker: *Fundamental distributions of errors for agricultural field trials.*

Evidence from various sources is presented which shows that the fundamental error distribution for yield trials is represented by $[1/ab(t_1-t_0)]\int_0^a\int_0^b\int_{t_0}^{t_1}[1/\sigma(x, y, t)]$ $\exp -\frac{1}{2}\{[\xi-f(x, y, t)]^2/\sigma^2(x, y, t)\}\,dt\,dy\,dx$ where the integrals may be Stieltjes integrals. Under certain conditions the fundamental error distribution can be expressed as a Gram-Charlier series, but very rarely, if ever, as a normal distribution. For comparison with analysis of variance results based on the normal theory, the distribution of the ratio of independent estimates of the second moments of samples, if the fundamental distributions are Gram-Charlier series, are given. Similar considerations show that the distributions of the numbers attacked in field trials can rarely be represented by Poisson or binomial distributions as is usually assumed. (Received October 22, 1940.)

### 76. G. A. Baker: *Maximum likelihood estimation of the ratio of the components of nonhomogeneous populations.*

Let $f(x)=[1/(1+k)](f_1(x)+kf_2(x))$, $e\leqq x\leqq f$, $k>0$ and $k<\infty$, where $f_1(x)$ and $f_2(x)$ are probability functions. The problem is to find the maximum likelihood estimate of $k$, say $\bar{k}$. If $f_1(x)$ and $f_2(x)$ are rectangular with equal ranges that partially overlap, then the probability of a value of $\bar{k}=w/u$ (where $u$ is the number of individuals drawn from the nonoverlapped interval of $f_1(x)$, $w$ is the number of individuals drawn from the nonoverlapped interval of $f_2(x)$ and $v$ is the number of individuals drawn from the interval overlapped by $f_1(x)$ and $f_2(x)$) is $(n!/u!v!w!)(p_1)^u(p_2)^v(p_3)^w$ where the $p_i$'s are the probabilities of coming from the respective intervals. The cases for which $u=n$, $v=n$, $w=n$, $u=0$, $w=0$ are excluded because $\bar{k}$ is then indeterminate. Hence, the probability of a determinate value of $\bar{k}$ is $P=1+(p_2)^n-(p_2+p_3)^n-(p_1+p_2)^n$. The estimates of $k$ are biased. (Received October 22, 1940.)

### 77. G. F. McEwen: *Statistical problems of the range divided by the mean in samples of size n.*

Certain quantitative climatological studies are based upon the "precipitation ratio" or ratio to the mean annual rainfall of the difference between the maximum and minimum annual rainfall corresponding to the standard number of years. Available observations correspond to various values of the number of years $n$. Accordingly it is necessary to compute the precipitation ratio $I$ corresponding to a standard number of years from $I_n$ corresponding to $n$ years. Various assumptions regarding the frequency distribution of the precipitation lead to corresponding sets of theoretical factors for standardizing the precipitation ratio and certain of these agree well with observations. Results plotted on coordinate paper having a reciprocal scale for $I_n$ and a logarithmic scale for $n$ facilitate a graphical treatment of the observations since the graph approximates to a straight line. The standard error of $I_n$ is also computed corresponding to the two most important frequency distributions of the precipitation. In order to reduce the accidental fluctuations of $I_n$ it is sometimes desirable to use the difference between the averages of the $rn$ highest and the $rn$ lowest values, where $rn$ equals some appropriate value, say 3. Theoretical factors derived for correcting such values to obtain $I_n$ agree well with observations. (Received October 3, 1940.)

### 78. Jerzy Neyman: *A statistical problem in mass production and routine analyses.*

Let $x_{i1}, x_{i2}, \cdots, x_{in}$ denote the results of parallel analyses of some $i$th sample, $i=1, 2, \cdots, N$ (a sample of items of mass production). The $x$'s are considered as random variables following the law $p(x_{ij}) = (\sigma_i(2\pi)^{1/2})^{-1} \exp \{ (x_{ij}-\xi_i)^2/2\sigma^2_i \}$. Denote by $H$ the hypothesis that $\sigma_1=\sigma_2= \cdots =\sigma_N$, that is, that the precision of measuring $\xi_i$ is maximum, and let $x_i=(\sum_j x_{ij})/n$, $S_i^2=\sum_j(x_{ij}-x_i)^2$, $V=\sum_i S_i^2$, $U=\sum_i S_i^4$. Also let $E$ denote the observed point in the space $W$ of the $S_i^2$'s and $w(V)$ the part of the locus $V=$const. included in any region $w\subset W$. It is proved that (1) a necessary and sufficient condition for a test's probability of rejecting $H$ when true to be a fixed value $\alpha$ is that its critical region $w=\sum_v w(V)$, with the $w(V)$ arbitrary except that $P\{E\in w(V) \mid E\in W(V)\}=\alpha$ for any $V>0$. (2) If $H$ is not true and $h=\sigma^{-2}$ is a random variable with $p(h)=ch^{\alpha-1}e^{-\beta h}$, then the critical region $w_0$ determined by $U>U_\alpha V^2$, with $P\{U>U_\alpha V^2 | H, V\}=\alpha$, has the following useful properties: (a) if $H$ be true, then the probability of $w_0$ rejecting $H$ is equal to $\alpha$, (b) the first derivative of the power function of $w_0$, taken at the point of $H$ being true, is equal to zero, (c) the second derivative is maximum. ((b) is true for all regions satisfying (a).) (Received October 28, 1940.)

## THEORY OF NUMBERS

### 79. E. T. Bell: *Selective equations.*

The symbols ( )′, [ ]′ denote the least, greatest of the integers occurring within the symbols. Each such integer may be replaced by a symbol ( )′, [ ]′ referring to a new set of integers, and so on, a finite number of times. The most general system of equations consisting of equalities between single power products formed from ( )′, [ ]′ is solved non-tentatively and finitely, by passing to a unique dual of the system, in which ( ), the G.C.D., and [ ], the L.C.M., replace ( )′, [ ]′. The latter system is a simple multiplicative system, and hence is completely solvable non-tentatively and finitely. The problem solved arose in the detailed discussion of compound multiplicative systems. (Received October 26, 1940.)

### 80. Leonard Carlitz: *Finite differences and polynomials in a Galois field.*

This paper is concerned chiefly with the solution of equations such as $\sum f(M)M(t)=g(t)$ and $\sum f(M, u)M(t)=g(u, t)$, the summation extending over all polynomials $M=M(x)$ of degree less than $m$. (Received November 25, 1940.)

### 81. Douglas Derry: *On the boundary case of Minkowski's linear form theorem.* Preliminary report.

Group theoretic methods are used to study Minkowski's conjecture regarding the boundary case of his linear form theorem, with the restriction that the coefficients become integral upon multiplication by a fixed prime $p$. In this case the validity of the conjecture is established for $n$ forms provided $p>(n-3)/2$ and for all primes where $n\leqq 8$. (Received November 25, 1940.)

### 82. D. H. Lehmer : *A general summation formula for numerical functions.*

Let $h(n)$ be any single valued numerical function, and let $H(n)=h(1)+h(2)+\cdots+h(n)$ be its sum. The author obtains a formula for $H(n)$ in terms of two functions $f(n)$, $g(n)$ and their sum functions $F(n)$ and $G(n)$, such that $\sum f(\delta)g(n/\delta)=h(n)$, the sum extending over the divisors of $n$. The formula involves a parameter $R$ which can be chosen at will. For $R=1$ or $R=n$ one obtains a formula used in several special cases by Dirichlet, while for $R=[n^{1/2}]$, a general formula of Franel and Glaisher is obtained, special cases of which have been used by many writers. By choosing $R$ an appropriate function of $n$ depending on $h$, the number of terms in the formula may be minimized. Exact formulas for the number of square-free numbers equal to or less than $x$ and the excess of numbers equal to or less than $x$ having an even number of prime factors over those having an odd number of prime factors are given as examples. In these formulas the number of terms varies as the cube root of $x$. (Received October 22, 1940.)

### 83. Gordon Pall: *Simultaneous representation in a quadratic and linear form.*

The problem of solving in integers $x_i$ the pair of equations $a=\sum c_i x_i^2$, $b=\sum c_i x_i$, where the $c_i$, $a$, and $b$ are given integers, and $t=\sum c_i\neq 0$, is reduced for certain forms of $t$ to the representation of $ta-b^2$ in a specified form in one fewer variables. When this form is in a genus of one class the solution is complete. Twenty-nine such sets $(c_1, \cdots, c_4)$ with positive $c_i$ are found. (Received November 25, 1940.)

### 84. R. M. Robinson: *On the simultaneous approximation of two real numbers.*

For each positive integer $s$, the smallest $C(s)$ is determined, such that for every pair of real numbers $\xi_1$ and $\xi_2$, there exist integers $a_1$, $a_2$, $b$ with $0<b\leqq s$ and $|b\xi_k-a_k|\leqq C(s)$ $(k=1, 2)$. (Received September 28, 1940.)

### 85. J. B. Rosser: *Explicit bounds for some functions of prime numbers.*

In this paper the constants implicit in several "big $O$" theorems of the analytical theory of primes are evaluated and improved. A typical result is the following. Let $\pi(x)$ denote the number of primes less than or equal to $x$. Then for $55\leqq x$, $x/(\log x+2)<\pi(x)<x/(\log x-4)$. (Received November 23, 1940.)

### 86. H. A. Simmons: *Maximum numbers associated with a symmetric Diophantine equation in n reciprocals.*

In a previous paper (Duke Mathematical Journal, vol. 2 (1936), pp. 317–340) the writer excluded from consideration the equation $1/x_1+1/x_2+\cdots+1/x_n+\lambda/x_1x_2\cdots x_n=b/((c+1)b-1)$, in which $b$, $c$ are any positive integers, and $\lambda$ is an integer greater than 1. The present paper identifies a class of maximum numbers relative to this equation. Using the terms $E$-solution and Kellogg solution, and the symbol $\Sigma_{n,r}(x)$ in senses previously employed, it is proved that if $w$ is the Kellogg solution and if $X\neq w$ is any $E$-solution of the equation, then $\Sigma_{n,n-1}(w)>\Sigma_{n,n-1}(X)$ and $\Sigma_{n,n}(w)>\Sigma_{n,n}(X)$. On account of these two inequalities, it is possible to associate with this equation a class of infinitely many maximum numbers. It is also shown that $w_n>X_n$, which might be called the Kellogg property. (Received October 17, 1940.)

### 87. L. I. Wade: *Certain quantities transcendental over the field $GF(p^n, x)$.*

In a previous paper (abstract 46-1-129) the author proved certain quantities transcendental over the field $GF(p^n, x)$. The present paper includes certain additional transcendental quantities. For example, $\sum_{k=1}^{\infty}1/(x^{p^{nk}}-x)$ is transcendental, $\sum_{k=0}^{\infty}1/x^{q^k}$ $(q>1)$ is algebraic if $q$ is of the form $p^s$ and transcendental otherwise, and $\sum_{k=0}^{\infty}1/x^{k^q}$ $(q>1)$ is transcendental. (Received November 25, 1940.)

### 88. Max Zorn: *Idempotency of infinite cardinals.*

This paper contains a simple proof of the theorem that the product of two infinite numbers is equal to the maximal factor. The application of ordinal numbers is replaced by the use of the maximum principle. (Received October 28, 1940.)

## TOPOLOGY

### 89. Ben Dushnik and E. W. Miller: *On the dimension of a partial order.*

Let $A$ be any set. Let $K$ be any collection of linear orders, each defined on all of $A$. A partial order $P$ on $A$ is defined as follows: For any two elements $a_1$ and $a_2$ of $A$, $a_1<a_2$ (in $P$) if an only if $a_1<a_2$ in every linear order of the collection $K$. A partial order so obtained will be said to be realized by the linear orders of $K$. With the aid of a result due to Szpilrajn on the linear extensions of a partial order (Fundamenta Mathematicae, vol. 16 (1930), pp. 386–389), it is seen that if $P$ is any partial order on a set $A$, then there exists a collection $K$ of linear orders on $A$ which realize $P$. By the dimension of a partial order $P$ defined on a set $A$ is meant the smallest cardinal number $m$ such that $P$ is realized by $m$ linear orders on $A$. It is shown that if $n$ is any natural number, there exists a finite partial order of dimension $n$, and if $m$ is any transfinite cardinal, there exists a partial order of dimension $m$ defined on a set of power $m$. Reversible partial orders (see abstract 46-5-266) are those of dimension not greater than 2. (Received October 25, 1940.)

### 90. Samuel Eilenberg: *Imbedding of spaces into euclidean spaces.* Preliminary report.

Given a metric space $X$, let $P(X)$ be the subset of the cartesian product $X\times X$ consisting of all points $(x, y)\in X\times X$ such that $x\neq y$. If every $i$-dimensional compact Vietoris cycle mod 2 of $P(X)$ bounds in $P(X)$ for $i=1, 2, \cdots, n-1$, then $X$ is not imbeddable topologically into the euclidean $n$-space. Many other results of this type are obtained. Analogous theorems hold if the join $X\circ X$ is considered instead of the product $X\times X$. (Received November 25, 1940.)

### 91. Samuel Eilenberg: *Monotone families of manifolds.*

Given a closed $n$-manifold $M$ in the euclidean space $E^{n+1}$, $D(M)$ will denote the bounded component of $E^{n+1}-M$. A family $\{M_t\}$ of $n$-manifolds depending on a real parameter $t$ is called: (a) monotone if $D(M_{t_1}) \subset D(M_{t_2})$ for $t_1<t_2$, (b) strictly monotone if $M_{t_1} \subset D(M_{t_2})$ for $t_1<t_2$, (c) continuous if $\lim M_{t_n}=M_t$ whenever $\lim t_n=t$. If $\{M_t\}$ $(0<t\leqq 1)$ is monotone continuous and $\delta(M_t)\to 0$ as $t\to 0$, then $D(M_t)+M_t$ is contractible to a point and $M_t$ is a homology-sphere. If moreover $\{M_t\}$ is strictly monotone, then also $D(M_t)$ is contractible to a point. (Received November 25, 1940.)

### 92. O. G. Harrold: *On a class of continuous maps.*

In this note the class of continua having the property of Čech (that is, of admitting a map of finite sections into the interval) is identified as those Peano continua $M$ such that every dendrite $D$ in $M$ has the Čech property. For dendrites the problem was solved by Mazurkiewicz (Fundamenta Mathematicae, vol. 17, pp. 88–98). (Received November 23, 1940.)

### 93. F. B. Jones: *Topologically flat spaces.*

A nondegenerate continuous curve (in a complete Moore space) will be called topologically flat provided that (1) it contains no cut point, (2) the Jordan curve theorem holds true in it, and (3) it is locally remotely connected. Conditions are obtained under which a topologically flat continuous curve is a subset of a plane (complete separability being sufficient) and certain of these subsets are characterized by means of upper semicontinuous collections in the plane. (Received November 23, 1940.)

### 94. F. B. Jones: *Totally discontinuous linear functions whose graphs are connected.*

An example of a real function $f(x)$ of a real variable is constructed with the following properties: (1) $f(x+y)=f(x)+f(y)$; (2) $f(x)$ is defined for all real values of $x$ and is discontinuous for each value of $x$; (3) the graph of $y=f(x)$ is a connected subset $M$ of the number plane; (4) the set $M$ is dense in the plane but contains no continuum; (5) considered as a space, (a) $M$ is metric and linearly ordered, (b) $M$ is the sum of a countable number of arbitrarily small totally disconnected domains, and (c) $M$ contains a totally disconnected closed set which contains a domain. Other related problems are considered. (Received October 24, 1940.)

### 95. J. P. LaSalle: *Pseudo-normed linear sets over valued rings.* II.

A generalized concept of a linear function on a pseudo-normed linear set over a valued ring $T$ (a $P_1$-space) to a $P_1$-space $T'$ is introduced where the valued rings need not be the same, though a sort of generalized homogeneity is assumed. Bounded sets are defined for $P_1$-spaces, and it is shown that a linear function maps bounded sets into bounded sets. These generalized linear functions are shown to have others of the properties which linear functions possess in less general spaces. A differential with the usual properties can be defined for functions with arguments and values in $P_1$-spaces. When the $P_1$-spaces are in particular the "topological abelian groups" considered by Michal (*First order differentials of functions with arguments and values in topological abelian groups*, to be published in the Revista de Ciencas, University of Lima), the existence of the $M_1$-differential implies the existence of the differential considered in this paper. By replacing the valued ring by a valued division ring in which the valuation satisfies $M(\alpha\beta)=M(\alpha)M(\beta)$ (it is not required that the valuation be archimedean), additional theorems concerning linear functions, and so on, may be proved. In particular it can be shown that the differential, if it exists, can be calculated as a limit. (Received November 8, 1940.)

### 96. A. N. Milgram: *Extensible and inextensible decompositions.* Preliminary report.

If $A$ is a closed subset of a topological space $S$, and if $A$ is decomposed into a finite number of closed sets $A=A_1+A_2+\cdots+A_r$, then we shall call the decomposition extensible if $S=S_1+S_2+\cdots+S_r$ is a decomposition of $S$ into closed sets such that $S_i\supseteq A_i$ and the decomposition of $S$ has the same nerve as that of $A$. If the decomposition of $A$ is not extensible it will be called inextensible. (See Menger's *Dimensionstheorie*, pp. 312–313.) This paper is devoted to a study of extensible and inextensible decompositions of topological spaces. For example, it is shown that a compact space is of dimension not less than $n$ if and only if it contains a closed subset having an inextensible decomposition into $n+1$ closed sets, each $n$ of which intersect. Criteria to determine when a continuous mapping of a subset of a space can be extended to the whole space are given in terms of extensible decompositions. Further results will be reported in a later paper. (Received October 24, 1940.)

### 97. A. N. Milgram: *Extensions of decompositions.*

The system of sets $B_1, B_2, \cdots, B_s$ is said to govern the system $A_1, A_2, \cdots, A_s$, if $B_{i_1}\cdot B_{i_2}\cdot \cdots \cdot B_{i_k}=0$ implies $A_{i_1}\cdot A_{i_2}\cdot \cdots \cdot A_{i_k}=0$. For each absolute neighborhood retract $R$, there exists a decomposition into closed sets $R_1, R_2, \cdots, R_n$ which in a sense determines whether mappings of subspaces into $R$ can be extended to mappings of the whole space into $R$. Namely, if $f\in R^A$, where $A$ is a closed subset of the metric space $S$, then $f$ can be extended to a mapping $f^*\in R^S$ if and only if there exist closed sets $S_i\supseteq f^{-1}(R_i)$, where $i=1, 2, \cdots, n$, which cover $S$ and such that the system $S_1, S_2, \cdots, S_n$ is governed by the system $R_1, R_2, \cdots, R_n$. If $R$ is not an absolute neighborhood retract, the theorem is shown to be false by an example. A space $S$ is called connected between $n$ of its closed sets $A_1, A_2, \cdots, A_n$ (see Menger, *Dimensionstheorie*, pp. 312–313) if for each decomposition of $S$ into closed sets $S_1, S_2, \cdots, S_n$ such that $S_i\supseteq A_i$ we have $\prod S_i\neq 0$. It is shown that a separable metric space is of dimension not less than $n$ if and only if it is connected between some $n+1$ of its closed sets. (Received November 26, 1940.)

### 98. Harlan C. Miller: *Concerning certain types of end points of compact continua.*

H. M. Gehman made a study of various definitions of end points of bounded plane continua. The following definition was not included: The point $P$ of a compact continuum $M$ is said to be a terminal point of $M$ if every irreducible subcontinuum of $M$ which contains $P$ is irreducible from $P$ to some point. In this paper it is shown that this definition is not equivalent to any of those considered by Gehman, except for the case where $M$ is a continuous curve. Among others the following results are established: A compact nondegenerate continuum every subcontinuum of which is unicoherent and decomposable has two terminal points. A compact continuum every subcontinuum of which is unicoherent and decomposable is irreducible about the set consisting of all its terminal points. If $M$ is a compact hereditarily decomposable continuum and $K$ is a subset of the set of all its terminal points, then $M-K$ is connected. (Received November 25, 1940.)

### 99. Harlan C. Miller: *On the existence of a certain type of irreducible continuum.*

In this paper the following theorem is proved: if $M$ is a closed and compact subset of a line $L$ lying in a plane $E$, there exists a compact continuum $N$ lying in $E$ such that (1) one of the complementary domains of $L$ has no point in common with $N$, (2) the common part of $N$ and $L$ is $M$, and (3) if $x$ and $y$ are points of different components of $M$ then $N$ is irreducible from $x$ to $y$. (Received November 25, 1940.)

### 100. Harlan C. Miller and R. L. Swain: *Properties of two intersecting arcs.*

Let space satisfy Axioms 0–5 of R. L. Moore (American Mathematical Society Colloquium Publications, vol. 13). The following definition is due to R. L. Moore: The arc $AB$ is said to cross the arc $CF$ provided there exist intervals $A'B'$ and $C'F'$ of $AB$ and $CF$, respectively, and a simple closed curve $J$ containing $C'F'$ such that $A'$ and $B'$ belong to different complementary domains of $J$ and such that the arc segment $C'F'$ contains the common part of $J$ and $A'B'$. Some necessary and sufficient conditions are given for crossing, and the crossing relation is shown to be reciprocal. Two arcs may cross at a point or interval, may touch at a point or interval, or may cross in every neighborhood of a point or interval. Among other results obtained in this paper are (1) if two arcs cross, they cross in every neighborhood of some component of their common part, (2) if two arcs cross, but cross at no component of their common part, and $K$ is the set of all components $H$ of their common part such that they cross in every neighborhood of $H$, then $K$ is a perfect set of point sets. (Received November 25, 1940.)

### 101. S. B. Myers: *Compact groups whose elements are of finite order.*

Denote by $G$ a metrizable topological group all of whose elements are of finite order, and by $G_N$ a $G$ which satisfies the postulate that if a sequence of elements $g_i$ converges to the identity, then the order of $g_i$ becomes infinite. (This postulate is satisfied by any $G$ which is a group of periodic homeomorphisms of a manifold into itself.) In this note it is proved that a compact $G_N$ is 0-dimensional, and a compact abelian $G_N$ is finite. (Received November 25, 1940.)

### 102. J. W. Odle: *Non-separating and non-alternating transformations modulo a family of sets.*

In this paper the type of generalization of non-alternating and non-separating transformations introduced by E. P. Vance (Duke Mathematical Journal, vol. 6 (1940), pp. 66–79) is carried still further, and the most general possible transformations of this kind are shown to have the same characteristic properties as Vance's transformations. A systematic analysis of product and factor theorems involving monotone, non-separating, non-alternating, weakly non-separating, and weakly non-alternating transformations is made, and with the addition of the two new theorems in this paper, all possible theorems of this type are now known. The relationships between locally non-alternating and 0-regular transformations are studied, and it is proved that any 0-regular transformation is locally non-alternating. An example is given to show that the converse is not true in general. However, on certain special sets the transformations are equivalent. (Received October 17, 1940.)

### 103. N. E. Rutt: *Sets with a certain sort of derived set.*

Let $D$ be a set of objects. Suppose that corresponding to each subset $K_a$ of $D$ there is a second subset $K_a'$ of $D$ such that, when $K_a'$ is nonvacuous and at most a finite number of $K_a$ are not in $K_b$, then the product of $K_a'$ and $K_b'$ is nonvacuous. The sets $K_a'$ exhibit a few of the simpler aspects of derived sets and the nature of these is the subject of this paper. (Received November 18, 1940.)

### 104. G. E. Schweigert: *Border transformations.*

The nature of these transformations may be inferred from the fact that for $B=T(A)$ a perfect set, where $A$ is compact and metric, the transformation is interior if and only if $T$ is a (continuous) border transformation such that the inverse image of a point contains no open set. Questions as to what extent open or connected open sets are preserved are considered in detail. These considerations yield necessary and sufficient conditions that $T$ be border. On locally connected continua, theorems holding for $T$ interior are generalized and a factor theorem shows that a border $T$ is quasi-monotone. Some results showing the difference between border and interior transformations are also obtained. (Received November 25, 1940.)

### 105. R. H. Sorgenfrey: *Concerning triodic continua.*

It is the purpose of this paper to present several definitions of the term triodic, first defined by R. L. Moore in Fundamenta Mathematicae, vol. 13 (1929), p. 262, and to establish the following theorem: If the compact continuum $M$ contains three continua which have a point in common and such that no one of them is a subset of the sum of the other two, then $M$ contains a continuum $N$, three points $P$, $Q$, and $O$ not belonging to $N$, and continua $H$, $K$, and $L$ irreducible from $N$ to $P$, $Q$, and $O$, respectively, each two of the continua $H$, $K$, and $L$ having only $N$ in common. (Received November 23, 1940.)

### 106. R. L. Swain: *Distance axioms in Moore spaces.*

A Moore space satisfies Axiom 0 and Axiom 1 (condition (4) omitted) of R. L. Moore (American Mathematical Society Colloquium Publications, vol. 13); a weak Moore space satisfies Axiom 0 and Axiom 1′, where Axiom 1′ is Axiom 1 except that the condition $\mathbf{g}\subset(R-B)+A$ is replaced by $g\subset(R-B)+A$. In the following axioms, $X$, $Y$, $X_n$, $Y_n$ are points of a space $S$: (A) $d(XY)=d(YX)$ is a real number; $d(XY)=0$ if $X=Y$; $d(XY)>0$ if $X\neq Y$; (B) if $\lim d(YY_n)=0$, then $\liminf d(XY_n)\leqq d(XY)$; (C) if $\lim d(XX_n)=0$ and $\lim d(YY_n)=0$, then $\lim d(X_nY_n)=0$; (D) given $X$ and given a positive number $\epsilon$, there exists a positive number $\delta_{p\epsilon}$ such that if $\lim d(YY_n)=0$ and $d(XY_n)<\delta_{p\epsilon}$, then $d(XY)<\epsilon$. The point $X$ is said to be a limit point of a point set $M$ if, for each $n$, there exists a point $X_n$ of $M$ such that $\lim d(XX_n)=0$. In order that $S$ be a weak Moore space, it is necessary and sufficient that $S$ satisfy Axioms A, B, C. For $S$ to be a Moore space, it is necessary and sufficient that $S$ satisfy Axioms A, B, C, D. (Received November 25, 1940.)

### 107. R. L. Swain: *Linear metric space.*

A metric space $S$ is called linear if for each three points $X'$, $Y'$, $Z'$ of $S$ there exist three points $X$, $Y$, $Z$ such that $X+Y+Z=X'+Y'+Z'$ and $d(XZ)=d(XY)+d(YZ)$, where $d$ is a metric distance function for $S$. If $S$ does not consist of just four points, it is shown that there exists a reversibly continuous transformation which preserves order and distance and which maps $S$ into some subset of the $x$-axis. (Received November 25, 1940.)

### 108. P. M. Swingle: *An abstraction of types of connected sets.*

Let $U$ and $V$ be subsets of a set $M$ such that $M=U+V$ where not both $U$ and $V$ are $M$. Let $S(X)$ denote that set $X$ has property $S$, $U\ r\ V$ denote that $U$ has relation $r$ to $V$. A set $M$ is of type $(I, B, W)$ in $P$ and $r$ if and only if $M$ has property $P$ and, for every pair $U$ and $V$ with property $P$, $U\ r\ V$. Let $M$ be a set of type $(I, B, W)$. Then $M$ is an $SPQ$-set of type $B$ if and only if $U\ r\ V\equiv[S(U)\cdot S(V)$ has property $Q]$, $M$ is an $SPQ$-set of type $I$ if and only if $U\ r\ V\equiv[$either $S(U)$ or $S(V)$ has property $Q]$, and $M$ is an $SPQ$-set of type $W$ if and only if $U\ r\ V\equiv[$both $S(U)$ and $S(V)$ have property $Q]$. Certain general theorems are obtained concerning these sets and particular theorems obtained assigning various properties to $S$, $P$ and $Q$. One can take these such that types $B$, $W$ and $I$ are respectively biconnected, widely connected and indecomposable closed connected sets. (Received November 18, 1940.)

### 109. P. M. Swingle: *Indecomposable connexes.*

Let $M$ be a connected point set. Then $M$ is an indecomposable connex if and only if for every two connected subsets $H$, $K$ of $M$ such that $M=H+K$ either both $H$ and $M$ or both $K$ and $M$ have the same closure. And $M$ is an irreducible joining connex closure between $a$ and $b$ if and only if there exists a connected subset $N$ of $M$ such that $N+a+b$ is connected and for all such $N$'s the sets $N$ and $M$ have the same closure. Examples are given of these sets and among others the following theorem is proven: If $M$ is connected and the closure of $M$ is compact, then in order that $M$ be an indecomposable connex it is necessary and sufficient that there exist three points $x$, $y$, $z$ such that $M$ is an irreducible joining connex closure between each two of these points. (Received October 26, 1940.)

### 110. A. W. Tucker: *Barycentric mappings.*

To each cell $x$ of an augmentable complex $X$ let there correspond a vertex $v(x)$ of a simplicial complex $Y$ in such a way that $v(x), v(x'), \cdots, v(x^{(k)})$ are all vertices of some single simplex of $Y$ if $x<x'<\cdots<x^{(k)}$ ($<$ means "is a face of"). Then $Tx=v(x)TFx$ defines a chain mapping of $X$ into $Y$ satisfying the usual requirement that $FTx=TFx$. And if $Y$ is a closed subcomplex of $X$ such that each $x$ is joined in $X$ to the simplex formed from the vertices $v(x), v(x'), \cdots, v(x^{(k)})$ where $x<x'<\cdots<x^{(k)}$, then $Dx=v(x)\{x-DFx\}$ defines a chain deformation such that $DFx+FDx=x-Tx$. The correspondence $x\rightarrow v(x)$ may be thought of as a simplicial mapping of the barycentric subdivision of $X$ into $Y$—hence the name barycentric mapping. It has many convenient properties and a wide variety of uses, particularly in comparing one homology system with another. (Received October 26, 1940.)

### 111. C. W. Vickery: *A new proof of a theorem of Chittenden.*

It is shown that if $S$ is a space $(\mathcal{E})$ in which the écart is regular (terminology of Fréchet's *Les Espaces Abstraits*, p. 219), there exists an equicontinuous collection $G$ of real valued point functions defined over $S$ and satisfying the conditions of Theorem 1.6 of the author's paper *Spaces of uncountably many dimensions* (this Bulletin, vol. 45 (1939), p. 459). Thus $S$ is homeomorphic with a subset of a space $D^{\alpha}$ and hence is metric. This establishes a theorem of Chittenden (Transactions of this Society, vol. 18 (1917), p. 161). (Received October 12, 1940.)

### 112. R. L. Wilder: *Characterizations of those euclidean domains whose boundaries are $lc^k$.*

The central problem is to find positional properties which characterize those domains in the euclidean $n$-sphere whose boundaries are $lc^k$ $(k \geqq 0)$. The only solution previously given was for the case where $n=2$ and $k=0$ (R. L. Moore, Fundamenta Mathematicae, vol. 3 (1922), pp. 232–237). A complete solution is given in this paper, using the properties $S_i$ defined in an earlier paper (American Journal of Mathematics, vol. 61 (1939), pp. 823–832) and a weak property $S_i$ (equals $WS_i$) defined exactly as was $S_i$ except that carriers of chains need not be self-compact: In order that the boundary of a domain $D$ in the euclidean $n$-sphere should be $lc^k$, it is necessary and sufficient that $D$ have (1) property $WS_0^k$ and (2) property $S_{n-k-1}^{n-1}$ and property $S_{n-k-2}$ relative to bounding cycles. For $n=2$, $k=0$, this is equivalent to the result of Moore cited above, since $S_0$ implies $WS_0$. Weak uniform local $i$-connectedness (equals $i$-*wulc*) is also investigated, and among other results it is found that in order for a common boundary of two domains in the euclidean $n$-sphere to be $lc^k$ it is necessary and sufficient that the domains be $wulc_0^k$. (Received November 23, 1940.)

### 113. R. L. Wilder: *On the domains complementary to continua having certain avoidability properties.*

G. T. Whyburn showed (American Journal of Mathematics, vol. 61 (1939), pp. 733–749) that in the plane the boundary of a domain complementary to a compact semi-locally connected continuum is peanian. The present paper employs the property of almost local $i$-avoidability (equals $i$-*ala*; see American Journal of Mathematics, vol. 61 (1939), p. 832, footnote 18) which for $i=0=p^0$ is equivalent to semi-local connectedness. Among the results obtained are the following: Let $M$ be a subcontinuum of $S^n$ which is $ala_0^{n-2}$. Then (1) if only finitely many of the "small" $i$-cycles $(i \leqq n-2)$ of a complementary domain $D$ are linearly independent relative to homologies (*lirh*) in $D$, $F(D)$ is peanian; (2) if the complementary domains of $M$ form a null sequence and only finitely many "small" $i$-cycles $(0<i \leqq n-2)$ of $S^n-M$ are *lirh* in $S^n-M$, then $M$ is $lc^{n-2}$; (3) under condition (2) if $M$ is $i$-avoidable for $i \leqq n-2$, all but a finite number of complementary domain boundaries are g.c. $(n-1)$-m.'s. If $M$ is an $(n-2)$-*ala* Peano continuum in $S^n$, then the complementary domain boundaries are all peanian (since every Peano continuum is 0-*ala*, this is exactly the well known Torhorst theorem when $n=2$). (Received November 23, 1940.)

### 114. R. L. Wilder: *The duality between the S-properties of closed sets and their complements in the euclidean n-sphere.*

Let property $B_i$ denote property $S_i$ relative to bounding cycles (American Journal of Mathematics, vol. 61 (1939), pp. 823–832). The following duality is established: In order that a closed subset $M$ of the euclidean $n$-sphere $S^n$ should have property $B_i$, it is necessary and sufficient that $S^n-M$ have property $B_{n-i-2}$. Thus property $S_i$ of $S^n-M$ and finiteness of $p^{i+1}(S^n-M)$ are dual to property $S_{n-i-2}$ of $M$ and finiteness of $p^{n-i-1}(M)$, since for any set $K$ to have property $S_i$ is equivalent to having property $B_i$ and finite $p^i(K)$. The duality is a consequence of the following results: (1) If $M$ is a closed subset of $S^n$, then a necessary and sufficient condition that $S^n-M$ have property $B_i$ is that the closed subsets of $M$ be almost completely $(n-i-2)$-avoidable by bounding cycles of $M$; and (2) a necessary and sufficient condition for a compact metric space to have property $B_i$ is that the closed subsets of $M$ be almost completely $i$-avoidable by bounding cycles. (Received November 23, 1940.)

# NEW PUBLICATIONS

BLAND, J. R. See KELLS, L. M.

BOYD, E. N. A Diagnostic Study of Students' Difficulties in General Mathematics in First Year College Work. (Contributions to Education, no. 798.) New York, Columbia University Bureau of Publications, 1940. 152 pp. $1.85.

BRINK, R. W. Plane Trigonometry. Revised edition. New York and London, Appleton-Century, 1940. 12+226+110 pp. $2.00.

BROWN, J. T. The Elements of Analytical Geometry. Part 3. Conic Sections. London, Macmillan, 1940. 7+154+7 pp. 3s.

COOLIDGE, J. L. A History of Geometrical Methods. Oxford, Clarendon Press; London, Oxford University Press, 1940. 18+451 pp. $4.00.

CROSLAND, L. Upper School Algebra. Being an Abridged and Revised Edition of Hall and Knight's "Higher Algebra." London, Macmillan, 1940. 15+292 pp. 6s.

DAVIS, H. T. College Algebra. New York, Prentice-Hall, 1940. 13+423 pp. $2.50.

DINGLE, H. The Special Theory of Relativity. (Methuen's Monographs on Physical Subjects.) London, Methuen, 1940. 7+94 pp. 3s. 6d.

DOERFLING, R. Mathematik für Ingenieure und Techniker. Munich and Berlin, Oldenbourg, 1939. 533 pp. RM 9.60.

ECKERT, W. J. Punched Card Methods in Scientific Computation. New York, Columbia University, 1940. 9+136 pp. $2.00.

EISENHART, L. P. An Introduction to Differential Geometry with Use of the Tensor Calculus. Princeton, University Press; London, Humphrey Milford and Oxford University Press, 1940. 10+304 pp. $3.50.

ERTEL, H. Elemente der Operatorenrechnung mit geophysikalischen Anwendungen. Berlin, Springer, 1940. 6+133 pp. RM 7.50.

FRANKLIN, P. A Treatise on Advanced Calculus. New York, Wiley; London, Chapman and Hall, 1940. 14+595 pp. $6.00.

GREENWOOD, E. R. A Detailed Proof of the Chi-Square Test of Goodness to Fit. (The Harvard Phi Beta Kappa Prize Essay for 1939.) Cambridge, Harvard University Press; London, Oxford University Press, 1940. 13+61 pp. 8s.

HAUSSNER, R. Darstellende Geometrie. Part 1. Elemente; Ebenflächige Gebilde. 5th revised edition. (Sammlung Göschen, vol. 142.) Berlin, de Gruyter, 1940. 207 pp. RM 1.62.

—— See SCHUBERT H.

KELLS, L. M., KERN, W. F., and BLAND, J. R. Plane and Spherical Trigonometry. 2d edition. New York, McGraw-Hill, 1940. 15+401 pp. $2.00.

KERN, W. F. See KELLS, L. M.

KOMMERELL, K. Vorlesungen über analytische Geometrie des Raumes. Leipzig, Koehler, 1940. 7+388 pp. RM 20.

KOSTITZIN, V. A. Mathematical Biology. Translated from the French by T. H. Savory. London, Bombay and Sydney, Harrap, 1939. 238 pp. 7s. 6d.

KRULL, W. Elementare Algebra vom höheren Standpunkt. (Sammlung Göschen, no. 930.) Berlin, de Gruyter, 1939. 142 pp. RM 1.62.

LOTKA, A. J. Théorie Analytique des Associations Biologiques. Vol. 2. Analyse Démographique avec Application Particulière à l'Espèce Humaine. (Actualités Scientifiques et Industrielles, no. 780.) Paris, Hermann, 1940. 149 pp.

MACDUFFEE, C. C. An Introduction to Abstract Algebra. New York, Wiley; London, Chapman and Hall, 1940. 7+303 pp. $4.00.

Quine, W. V. Mathematical Logic. New York, Norton, 1940. 13+348 pp. $4.00.

Robinson, R. M. Stencils for Solving $x^2 \equiv a$ (mod $m$). Berkeley and Los Angeles, University of California Press, 1940. 14 pp.+272 Hollerith cards.

Rose, C. E. Matrix and Tensor Algebra for Engineers and Chemists. New York, Chemical Publishing Co., 1940. 8+143 pp. $4.00.

Savory, T. H. See Kostitzin, V. A.

Schubert, H. Vierstellige Tafeln und Gegentafeln für logarithmisches und trigonometrisches Rechnen in zwei Farben zusammengestellt. Revised by R. Haussner. (Sammlung Göschen, vol. 81.) Berlin, de Gruyter, 1940. 181 pp. RM 1.62.

Tropfke, J. Geschichte der Elementar-Mathematik in systematischer Darstellung mit besonderer Berücksichtigung der Fachwörter. Vol. 4. Ebene Geometrie. 3d edition, revised by K. Vogel. Berlin, de Gruyter, 1940. 316 pp. RM 13.

Vogel, K. See Tropfke, J.

Witting, A. Integralrechnung. 2d revised edition. (Sammlung Göschen, vol. 88.) Berlin, de Gruyter, 1940. 175 pp. RM 1.62.

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# RATIONAL METHODS IN MATRIX EQUATIONS

MARK H. INGRAHAM

I. **Introduction.**[2] I shall start with what I hope will not prove an overelaborate statement of the limitations of this paper in scope and treatment. I shall assume throughout a knowledge of the definitions of a field, a division algebra, of matrices and rational operations thereon. I shall also need to assume a knowledge of what is meant by the invariant factors and elementary divisors of a matrix. Little essential will be lost if the only field considered by the listeners is the rational number system and the only division algebra that of quaternions over the rational field.

Consider a field $\mathfrak{K}$ and a system of constant matrices $A_1, \cdots, A_l$, unknown matrices $X_1, \cdots, X_n$ and equations

$$\phi_i(A_1, \cdots, A_l, X_1, \cdots, X_n) = 0 \tag{1}$$

where the $\phi_i$'s are polynomials with coefficients in $\mathfrak{K}$. If the elements of $A_i$ are $a_{ijk}$ and of $X_i$ are $x_{ijk}$, (1) is equivalent to a system

$$\psi_s(\cdots, x_{ijk}, \cdots) = 0 \tag{2}$$

where the $\psi_s$ are polynomials with coefficients in $\mathfrak{K}$. (If $\mathfrak{K}$ is replaced by a division algebra $\mathfrak{d}$ over $\mathfrak{K}$, the number of equations in (2) is merely enlarged.) We have therefore "reduced" the equations (1) to those of (2). This process we shall technically designate without great exaggeration as the "worst possible algorithm," or, following modern style, W.P.A. This indicates that no "tour de force" which shows that ultimately a matrix problem can be solved in a finite time, but shows little else, is of interest. This is a topic in which the above simple proof of the existence of inelegant methods means that we need only pay attention to results that essentially use the matricial properties of matrices, only to results and methods having at least a minimum degree of elegance.

Partly as a consequence of the above, this lecture, as is often the case, is not so much a description of broad theories of the nature we desire, as a report on what special cases have been found to be seduci-

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[1] An address delivered before the Chicago meeting of the Society, April 14, 1939, by invitation of the Program Committee; presented in part April 15, 1939, under the title *An algorithm for the solution of the unilateral matrix equation.*

[2] The author wishes to thank H. C. Trimble, C. J. Everett, and J. H. Bell for help in preparing this paper. Their aid was made possible by the Research Committee of the University of Wisconsin.

ble. As a definition of the type of result I shall describe and the literary style in which this description will be given, I will lay down, for this lecture, four canons. The first and last of these I would not defend as completely valid dicta.

1. Irrational methods will be relegated to the background. In particular, we will not assume that we can reduce every polynomial into linear factors.

2. A theorem that is valid for the field of rational numbers is ipso facto of interest. This is particularly true if it extends to the case of all division algebras with finite basis over that field. (If a "field" had been defined to have characteristic zero and others to be "semi-fields," we would, I believe, have had a more wholesome set of values than at present.)

3. Proofs that are essentially algorithmic belong to the aristocracy.

4. The language of matrices, bases, vectors, linear spaces, polynomials, greatest common divisors, and so on, will be used instead of operators, modules, ideals, lattices, and so on. This is in deference to (a) the non-algebraists, (b) the frequency with which generalizations avoided are merely formal, (c) my personal taste.

The paper when presented as a symposium address included the discussion of the equation $TA=BT+C$ and a discussion of the unilateral matrix equation. Only the latter is included here since an extended form of the former is being published elsewhere. (Ingraham [8].)

II. **The unilateral equation.** We will now pass to the consideration of the unilateral (as to position of coefficients) equation

$$\sum A_iX^i = 0. \tag{3}$$

We will endeavor to show that much more can be said of this equation than has heretofore been shown. However, at best the general treatment is not simple so that it is desirable to use when available simpler theories for special types of equations, or for yielding special types of solutions.

What follows will be divided into three parts:

1. The case where all the $A_i$ are polynomials in a single matrix $A$ with scalar coefficients, and solutions $X$ are sought which are of the same type.

2. The case where (3) is of the type

$$\psi(X) = A \tag{4}$$

where the polynomial $\psi$ has scalar coefficients.

### 3. The general unilateral equation.

The solution of case 1 was first given by Szücs[3] and though his methods are not rational, the rewriting in rational form, as here done, is not difficult. This type of solution had been given, however, for (4) by Roth,[4] and later by a method similar to that of Szücs by Franklin.[5]

Consider

$$\phi(A, X) = 0 \tag{5}$$

and consider only such solutions $X$ as are polynomials in $A$. We will also assume that the field $\mathfrak{K}$ has characteristic zero. Let $X(\lambda)$ be a polynomial such that $X(A)$ is a solution of (5). Hence $\phi(A, X(A)) = 0$. A necessary and sufficient condition that this be true is that $\phi(\lambda, X(\lambda))$ be divisible by the first invariant factor of $A$, that is, the minimal polynomial $h$ such that $h(A) = 0$. Let $g^k$ be the highest power of an irreducible polynomial $g$ contained in $h$ as a factor. Hence if $\phi(A, X(A)) = 0$, $g^k$ must divide $\phi(\lambda, X(\lambda))$. Since $\mathfrak{K}$ has characteristic zero this is equivalent to $g$ dividing $\phi, \phi', \phi'', \cdots, \phi^{(k-1)}$. (In case the characteristic is not zero the condition of divisibility is still workable but not as neat.)

Let $X(\lambda) = X_0(\lambda) + X_1(\lambda)g + X_2(\lambda)g^2 + \cdots$ where the $X_i$ are reduced mod $g$.

Hence

$$\phi(\lambda, X_0) \equiv 0 \bmod g,$$

or in other words, $\phi(\lambda, X)$ has a zero in the field $\mathfrak{K}[\lambda]/\{g(\lambda)\}$ for every irreducible factor $g$ of $h$.

Consider

$$\begin{aligned} \phi'(\lambda, X) &= \phi_\lambda(\lambda, X) + \phi_X(\lambda, X)X' \\ &\equiv \phi_\lambda(\lambda, X_0) + \phi_X(\lambda, X_0)(X_0' + X_1 g') \\ &\equiv (X_1 g')\phi_X(\lambda, X_0) + \psi_1 \bmod g \end{aligned}$$

where $\psi_1$ is fixed by the determination of $X_0$ and, in general,

$$\phi^{(s)}(\lambda, X) \equiv s!X_s(g')^s\phi_X(\lambda, X_0) + \psi_s \bmod g$$

where $\psi_s$ is determined in terms of $X_0, \cdots, X_{s-1}$.

Note that since $g$ is irreducible, $g'$ is prime to $g$.

It is therefore necessary if $k > 1$ that either

$$\phi_X(\lambda, X_0) \not\equiv 0 \bmod g \tag{6}$$

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[3] Szücs [7].

[4] Roth [6].

[5] Franklin [1].

or that

$$\psi_i \equiv 0 \bmod g, \qquad i = 1, \cdots, k-1. \tag{7}$$

If for every irreducible factor $g$ these conditions are satisfied, then we have for each $g_i$ an $X_{0_i}$ such that $X$ is a solution of $\phi(A, X)=0$ if and only if

$$X \equiv X_{0_i} \bmod g_i^{k_i}.$$

That such an $X$ exists follows from the Chinese Remainder Theorem for polynomials. Hence

THEOREM 1. *If* $\mathfrak{K}$ *has characteristic* 0, *the equation* $\phi(A, X)=0$ *has a solution which is a polynomial in* $A$ *if and only if for every irreducible factor* $g$ *of the minimal polynomial of* $A$ *there exist in* $\mathfrak{K}[\lambda]$ *solutions of* $\phi(\lambda, X)\equiv 0 \bmod g$ *which, in case* $g$ *is a multiple factor of the minimal polynomial of* $A$, *satisfy* (6), *or* (7).

If the equation considered is of the form $\theta(X)=A$ where $\theta$ has scalar coefficients, then $\phi'(\lambda, X_0)=-1$ and (6) above becomes necessary so that we can state

THEOREM 2. *If* $\mathfrak{K}$ *has characteristic* 0, *the equation* $\theta(X)=A$ *has a solution which is a polynomial in* $A$ *if and only if* $\theta(X)\equiv\lambda \bmod g_i$ *has a solution in* $\mathfrak{K}[\lambda]$, *for every irreducible factor* $g_i$ *of the minimum function of* $A$ *which, in case* $g_i$ *is a multiple factor of the minimal function of* $A$, *does not satisfy* $\theta'(X)\equiv 0 \bmod g_i$.

As a corollary of this,

THEOREM 3. *If* $\mathfrak{K}$ *is algebraically closed and has characteristic* 0, *the equation* $X^n=A$ *has a solution* $X(A)$, *a polynomial in* $A$, *if and only if* 0 *is not a multiple root of the minimum function of* $A$.

It is not difficult to compute the number of such functions.

For certain cases the existence of solutions not polynomials in $A$ is easily established by example.

For instance

$$X^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has the solution

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

It is interesting, therefore, to study the general solutions of equations of the type discussed above. This is readily done for the equations $\theta(X)=A$, if $A$ is nonderogatory, that is, its minimum equation is its characteristic equation. Since $A$ is nonderogatory and every solution $X$ is commutative with $A$, $X$ is a polynomial in $A$. In this case our problem is completely solved. In case $\phi$ has scalar coefficients, the general solution of

$$\phi(X) = A$$

is determined as follows.[6]

If we find a matrix $Y$ such that $\phi(Y)$ is similar to $A$, then a nonsingular matrix $S$ may be found such that $A=S\phi(Y)S^{-1}=\phi(SYS^{-1})$ and hence $SYS^{-1}$ is effective as $X$. Consider the invariant factors of $A-\lambda I$, $h_1, h_2, \cdots, h_k$. Then $\phi(Y)$ is similar to $A$ if and only if the nullity of $h_i(\phi(Y))$ equals the nullity of $h_i(A)$. Hence the invariant factors of $Y$ must be divisors of $h_1(\phi(\lambda))$. Under this restriction and the fact that the nullity of any polynomial $\phi$ in $Y$ is determined as the sum of the degrees of the greatest common divisors of the invariant factors of $Y$, with the polynomial $\phi$, diophantine equations may be written, any solution of which determines the invariant factors of a solution $X$, and hence determines $X$ to within a transformation by a matrix commutative with $A$. All solutions are the transforms by matrices commutative with $A$ of a certain finite dissimilar system of solutions $X_1, X_2, \cdots, X_k$. Those $X_i$ which are polynomials in $A$ are those solutions which are dissimilar to all other solutions. For example if

$$X^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then

$$X = \begin{pmatrix} 0 & 0 & 0 \\ m & 0 & k \\ 1/k & 0 & 0 \end{pmatrix}$$

with $k\neq 0$.

This work generalizes with only moderate difficulty to the case where the scalars involved belong to a division algebra provided we ask the question correctly.[7] In the case where the scalars are in a field and $\phi=\sum\lambda^i a_i$ then $\phi(X)=A$ is equivalent to $\phi(X)\xi=\sum X^i\xi a_i$

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[6] Ingraham [2].

[7] Ingraham [3].

$=A\xi$ for every vector $\xi$. This is not the case if the $a_i$ belong to a noncommutative division algebra $\mathfrak{D}$. The correct question which can be answered is not to find a solution of $\phi(X)=A$, but given a set $[a_i]$ in $\mathfrak{D}$ to find a matrix $X$ such that $\sum X^i\xi a_i=A\xi$ for every $\xi$. This problem turns out to be equivalent to a system of equations

$$\phi_i(X) = A_i$$

where the coefficients of $\phi_i$ are in the centrum of the division algebra $\mathfrak{D}$.

Let us now consider the general unilateral matrix equation

$$\sum A_iX^i = 0, \tag{8}$$

where the $A_i$ are $n\times n$ matrices of constants.

Fundamental to the consideration of (8) is the fact that if $X$ is any constant matrix then $A(\lambda)=\sum A_i\lambda^i$ can be expressed in one and only one way as

$$(\sum B_i\lambda^{i-1})(\lambda I - X) + B_0$$

and in fact

$$B_0 = \sum A_iX^i.$$

Hence

THEOREM 4. *$X$ is a solution of $\sum A_iX^i=0$ if and only if $\lambda I-X$ is a right factor of the matrix $A(\lambda)=\sum A_i\lambda^i$.*

Theorem 5, which is a corollary of a theorem due to Phillips,[8] yields us an algorithm better than the W.P.A. This theorem is as follows:

THEOREM 5. *If $X$ is a solution of $\sum A_iX^i=0$, then the determinant of the matrix $\lambda I-X$ divides the determinant of the matrix $\sum A_i\lambda^i$.*

This is, of course, a corollary of Theorem 4. The proof originally given was far more complicated than this proof which is due to C. C. MacDuffee. Phillips' theorem yields a necessary condition on the invariant factors of any solution. If $Y$ has such invariant factors and $X=PYP^{-1}$ is a solution, then $\sum A_iPY^i=0$. The problem is then reduced to finding all $P$'s satisfying this equation for canonical $Y$'s with possible invariant factors.

Far stronger conditions can be secured from Theorem 6, which is a consequence of the following lemma.

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[8] Phillips [5].

LEMMA. *If $A$, $B$, $C$, are $\lambda$-matrices such that $A=BC$, then the invariant factors of $B$ (or $C$) divide the corresponding invariant factors of $A$.*

Thus

THEOREM 6. *If $X$ is a solution of $\sum A_iX^i=0$, then the invariant factors of the matrix $\lambda I-X$ are divisors of the corresponding invariant factors of the matrix $\sum A_i\lambda^i$.*

This eliminates many cases not strained out by the sieve of Phillips' theorem.

That these conditions or any conditions in terms of the invariant factors of $A(\lambda)$ are insufficient to guarantee the existence of solutions with a given set of invariant factors for $\lambda I-X$ is shown by the two equations

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}X^2+\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}=0$$

and

$$X+\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}=0.$$

The $\lambda$-matrices for these are respectively

$$\begin{pmatrix}\lambda^2 & 0\\ 0 & 1\end{pmatrix},\quad \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix},$$

each of which has invariant factors $\lambda^2$, 1; but the first equation has no solution and the second has the solution

$$\begin{pmatrix}0 & -1\\ 0 & 0\end{pmatrix}.$$

We now pass to the description of an algorithm believed to be new for the solution of (8).

We shall call a matrix with elements polynomials in $\lambda$ unimodular if its determinant is a scalar not zero, or what is equivalent, if it is nonsingular in the field of rational functions of $\lambda$, and if its reciprocal has elements polynomials in $\lambda$.

We say that $A$ is in triangular form if all the elements below the main diagonal are zero. We say it is in canonical triangular form if it is in triangular form and all the elements above the main diagonal are of lower degree than the elements in the same column on the main diagonal and if when a zero occurs on the main diagonal the whole row in which it occurs is zero. We also specify that the leading coefficients of the polynomials along the main diagonal be 1.

If $U$ is unimodular, then $\lambda I - X$ is a right factor of $A$ if and only if it is a right factor of $UA$. Moreover, a unimodular matrix $U$ may be found such that $UA$ is in canonical triangular form, this form being completely determined by[9] $A$. From now on we will assume that $A$ is in such form.

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

where the $a_{ij}$'s are polynomials in $\lambda$.

If $A = H(\lambda I - X)$ and if $U$ is the unimodular matrix such that $U(\lambda I - X)$ is in canonical triangular form, then

$$A = HU^{-1}U(\lambda I - X)$$

and $T = U(\lambda I - X)$ is a right factor of $A$.

Our problem is therefore reduced to finding the triangular factors of $A$ which are the canonical triangular forms of matrices of type $\lambda I - X$ where $X$ is independent of $\lambda$.

We study then the problem of finding canonical triangular factors $T$ of the matrix $A$ and then of selecting those which are the canonical triangular forms of matrices of type $\lambda I - X$ for some $X$.

If $A = ST$ where the matrices $A$ and $T$ are in triangular form, $S$ is in triangular form, and

$$(9) \qquad a_{i,i+j} = \sum_{k=0}^{j} s_{i,i+k} t_{i+k,i+j}, \qquad i = 1, \cdots, n;\ j = 0, \cdots, n - i.$$

For the case $j = 0$ these relations reduce to

$$a_{ii} = s_{ii} t_{ii}$$

and hence to the fact that the diagonal elements of $T$ are factors of the corresponding diagonal elements of $A$.

Equations (9) give linear congruences each conditioning (not necessarily uniquely) the possible $t_{ij}$ in terms of previously determined elements of $T$ below $t_{ij}$ occurring in the $j$th column and elements of $s$ to the left of $s_{ij}$ in the $i$th row—these may be considered therefore as recursive formulae for the $t_{ij}$. In these congruences the coefficient of $t_{ij}$ is $s_{ii}$ and the modulus is $t_{jj}$, so that, in case $\mathfrak{K}$ has infinitely many elements, an infinitude of solutions exists if and only if $s_{ii}$ and $t_{jj}$ are not relatively prime and at least one solution exists.

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[9] MacDuffee [4].

It should be noted that if $T$ is the triangular form of $M$, the product of the first $i$ diagonal terms is the greatest common divisor of the determinants of the $i$th order minors of the first $i$ columns of $M$. Hence if $T$ is of the form $U(\lambda I-X)$ where $U$ is unimodular, the sum of the degrees of the first $i$ terms of the principal diagonal of $T$ must be less than or equal to $i$ and the sum of the degrees of all the diagonal terms is $n$, the order of $T$. I believe in most equations written at random these last conditions would preclude a solution.

We turn now to determining $X$ such that a $T$ satisfying the above conditions is the triangular canonical form for some $(\lambda I-X)$. It can be shown (but the proof is omitted) that $X$ is unique if it exists.

If $T=U^{-1}(\lambda I-X)$ where $U$ is unimodular, then

$$U = (\lambda I - X)T^{-1}. \tag{10}$$

If we choose a matrix $X$, $(\lambda I-X)T^{-1}$ will have elements rational functions of $\lambda$ but not, in general, polynomials.

The problem therefore reduces to seeking $X$ such that $(\lambda I-X)T^{-1}$ is a matrix with elements polynomials in $\lambda$.

Let $T^{-1}=F/d$ where $F$ is in triangular form and $d$ is the determinant of $T$. The $f_{ij}$ will be polynomials.

If (10) is fulfilled

$$u_{ij} = (\lambda f_{ij} - \sum x_{ik}f_{kj})/d;$$

hence $u_{ij}$ is a polynomial if and only if

$$\sum x_{ik}f_{kj} \equiv \lambda f_{ij} \bmod d.$$

Since the $f_{ij}$ are of degree less than $d$, these linear congruences are easily reduced, by equating coefficients, to $n$ systems of linear equations for the $x_{ik}$ ($i$ fixed) all of the systems having the same matrix of coefficients, the rank of which must (in light of the uniqueness theorem) be $n$ if a solution of all the systems exists.

Since this paper was delivered at Chicago Mr. J. H. Bell has furnished me with the following note on the conditions upon $T$ for a matrix $X$ to exist such that $T$ is a triangular form of $\lambda I-X$.

The existence of $X$ depends directly upon the coefficients of the elements of $T$.

We may obtain an $n\times n$ matrix $D$ whose elements are scalars by the following steps.

1. Augment each element of $T$ by the proper powers of $\lambda$ with zero coefficients so that terms of the same degree as in the corresponding diagonal element appear.

2. Break up each column into separate columns, each one of which involves only monomials of the same degree in $\lambda$, for example,

$$\begin{pmatrix} a\lambda + b \\ c\lambda + d \end{pmatrix} \rightarrow \begin{pmatrix} a\lambda, & b \\ c\lambda, & d \end{pmatrix}.$$

3. Delete the columns which do not involve $\lambda$.

4. Set $\lambda = 1$ obtaining $D$.

A necessary and sufficient condition that there exist a matrix $X$ such that $T = U^{-1}(\lambda I - X)$ is that $|D| \neq 0$.

If $T = \sum_r^0 T_i \lambda^i$ ($T_i^{n \times n}$ involving scalars only), then the above condition is equivalent to saying that $n$ is the rank of

$$M = (T_r, T_{r-1}, \cdots, T_2, T_1).$$

$X$ may be obtained quite readily from $D$. We first find $B = (b_{ik})$ as follows. If $t_{jj} \neq 1$, $(b_{j1} \cdots b_{jn}) = (0 \cdots 0, -1, 0 \cdots 0)D^{-1}$ where the $-1$ occurs in the position corresponding to the column of $D$ containing $t_{1,jj}$ (where $t_{kj} = \sum_s^0 t_{i,kj} \lambda^i$). If $t_{ll} = 1$, $(b_{l1} \cdots b_{ln}) = \eta_l D^{-1}$ where $\eta_l$ is a $1 \times n$ vector obtained by taking the $l$th row of the matrix obtained from the matrix in step 2 by deleting the columns involving the leading term of $t_{kk}$, for every $k$, and setting $\lambda = 1$.

Then

$$X = BT_0.$$

## BIBLIOGRAPHY

1. P. Franklin, *Algebraic matric equations*, Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 10 (1932), pp. 289–314.

2. M. H. Ingraham, *On the rational solution of the matrix equation* $P(X) = A$, Journal of Mathematics and Physics, vol. 13 (1934), pp. 46–50.

3. M. H. Ingraham, *On certain equations in matrices whose elements belong to a division algebra*, this Bulletin, vol. 44 (1938), pp. 117–124.

4. C. C. MacDuffee, *Matrices with elements in a principal ideal ring*, this Bulletin, vol. 39 (1933), pp. 564–584.

5. H. B. Phillips, *Functions of matrices*, American Journal of Mathematics, vol. 41 (1919), pp. 266–278.

6. W. E. Roth, *A solution of the matric equation* $P(X) = A$, Transactions of this Society, vol. 30 (1928), pp. 579–596.

7. A. Szücs, *Sur les équations définissant une matrice en fonction algébrique d'une autre*, Acta Litterarum ac Scientiarum, Szeged, vol. 7 (1934–1935), pp. 48–50.

8. M. H. Ingraham and H. C. Trimble, *On the matric equation* $TA = BT + C$, American Journal of Mathematics, vol. 63, pp. 9–27.

UNIVERSITY OF WISCONSIN

# MAXIMUM OF CERTAIN FUNDAMENTAL LAGRANGE INTERPOLATION POLYNOMIALS[1]

M. S. WEBSTER

This note extends some of the results obtained in a previous paper[2] which we shall designate as I. The notations are the same.

We are concerned with the polynomials

$$l_k^{(n)}(x) \equiv \frac{\phi_n(x)}{\phi_n'(x_k)(x-x_k)}, \qquad k = 1, 2, \cdots, n,$$

where $\phi_n(x) \equiv (x-x_1)(x-x_2)\cdots(x-x_n)$ is the Jacobi polynomial of degree $n$ which satisfies the differential equation $(1-x^2)\phi_n''(x) + [\alpha-\beta-(\alpha+\beta)x]\phi_n'(x)+n(n+\alpha+\beta-1)\phi_n(x)=0$. The parameters $\alpha$, $\beta$ are positive and $n$ is a positive integer. It is known that $-1<x_n<x_{n-1}< \cdots <x_1<1$. Throughout the paper, $x$ is always restricted to the interval $-1 \leqq x \leqq 1$.

It was shown in I, for example, that, if $\alpha=\beta=\frac{3}{2}$, $\max |l_k^{(n)}(x)| <2$ and $l_1^{(n)}(1)\to 2$ as $n\to\infty$.

Now we use[3]

$$\phi_n(1) = \frac{2^n\Gamma(n+\beta)\Gamma(n+\alpha+\beta-1)}{\Gamma(\beta)\Gamma(2n+\alpha+\beta-1)}$$

and the asymptotic expressions[4]

$$\phi_n(\cos\theta) = \frac{2^n\Gamma(n+1)\Gamma(n+\alpha+\beta-1)}{(\pi n)^{1/2}\Gamma(2n+\alpha+\beta-1)}\left(\sin\frac{\theta}{2}\right)^{1/2-\beta}\left(\cos\frac{\theta}{2}\right)^{1/2-\alpha}$$
$$\cdot\left\{\cos\,[N\theta-(2\beta-1)\pi/4]+(n\sin\theta)^{-1}O(1)\right\},$$

$$\phi_n(\cos\theta) = \frac{2^n\Gamma(n+1)\Gamma(n+\alpha+\beta-1)}{\Gamma(2n+\alpha+\beta-1)}\left(\sin\frac{\theta}{2}\right)^{1-\beta}\left(\cos\frac{\theta}{2}\right)^{1-\alpha}$$
$$\cdot\left\{\frac{\Gamma(n+\beta)}{\Gamma(n+1)}\left(\frac{\theta}{\sin\theta}\right)^{1/2}\frac{J_{\beta-1}(N\theta)}{N^{\beta-1}}+\theta^{1/2}O(n^{-3/2})\right\},$$

where $N=n+(\alpha+\beta-1)/2$, $cn^{-1}\leqq\theta\leqq\pi-\epsilon$, $c$, $\epsilon$ positive constants and where $J_m(x)$ is Bessel's function of order $m$. Since $\phi_n'(x;\ \alpha,\ \beta) = n\phi_{n-1}(x;\alpha+1,\beta+1)$, these yield immediately the following results:

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[1] Presented to the Society, April 13, 1940.

[2] M. Webster, *Note on certain Lagrange interpolation polynomials*, this Bulletin, vol. 45 (1939), pp. 870–873.

[3] C. Winston, *On mechanical quadratures formulae involving the classical orthogonal polynomials*, Annals of Mathematics, (2), vol. 35 (1934), pp. 658–677.

[4] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, 1939, pp. 191–192, 121, 123.

LEMMA. *For* $x_k$ *such that* $-1+\epsilon \leqq x_k \leqq 1-\epsilon$ *and* $|x-x_k| \geqq \epsilon' > 0$, max $|l_k^{(n)}(x)| \to 0$ as $n\to\infty$ *even if* $x\to\pm 1$ $(\alpha,\ \beta<\frac{3}{2};\ \epsilon,\ \epsilon'>0)$.

THEOREM 1. *For* $x_k$ *such that* $-1+\epsilon \leqq x_k \leqq 1-\epsilon$ *and* $|x-x_k| \geqq \epsilon' > 0$, max $|l_k^{(n)}(x)| = O(n^\gamma)$ *as* $n\to\infty$ *where* max $(\alpha,\beta)=\gamma>\frac{3}{2};\ \epsilon,\epsilon'>0$. *The exponent* $\gamma$ *cannot be decreased.*

The method used in the proof of Theorem 5 in I really gives the following slightly stronger result:

THEOREM 2. *If* $-1+\epsilon\leqq x_k\leqq 1-\epsilon,\ -1+\epsilon'\leqq x\leqq 1-\epsilon'$, max $|l_k^{(n)}(x)| \to 1$ *as* $n\to\infty$ $(\epsilon,\ \epsilon'>0)$.

Combining Theorem 2 and the lemma, we obtain the following:

THEOREM 3. *For* $x_k$ *such that* $-1+\epsilon\leqq x_k\leqq 1-\epsilon$, max $|l_k^{(n)}(x)|\to 1$ *as* $n\to\infty$ $(\alpha,\ \beta<\frac{3}{2},\ \epsilon>0)$.

This result is a considerable improvement over Theorem 5 in I. Moreover, if the hypothesis $-1+\epsilon\leqq x_k\leqq 1-\epsilon$ is removed, the theorem is not true as Erdös and Grünwald[5] showed in case $\alpha=\beta=\frac{1}{2}$. In view of Theorems 1, 4, 5, 6, the restriction $\alpha,\ \beta<\frac{3}{2}$ is also necessary. In particular, this theorem holds for the case of Tschebycheff $(\alpha=\beta=\frac{1}{2})$ and Legendre $(\alpha=\beta=1)$ polynomials.

THEOREM 4. *If* $\alpha=\beta=\frac{3}{2}$ *and* $x_k\to t$ *as* $n\to\infty$, *then* max $|l_k^{(n)}(x)|\to 1+|t|$ *as* $n\to\infty$ $(-1\leqq t\leqq 1)$. *This is also an upper bound if* $|x_k|<|t|$ *at least for large values of* $n$.

PROOF. It was shown in I that $l_k^{(n)}(1)=1+x_k$ and for $x_{k+1}\leqq x\leqq x_{k-1}$, max $|l_k^{(n)}(x)|<1.87$. Since (I) max $|l(_k^{(n)}x)|$ is attained either between $x_{k+1}$ and $x_{k-1}$ or at $x=\pm 1$, the theorem is valid for $t=1$ and, by symmetry, for $t=-1$.

If $|t|<1$, the preceding paragraph and Theorem 2 complete the proof. In fact, max $|l_k^{(n)}(x)|=1+|x_k|$ at least for large $n$.

The next two theorems are obtained in a similar manner.

THEOREM 5. *If* $\alpha=\frac{1}{2},\ \beta=\frac{3}{2}$ *and* $x_k\to t$ as $n\to\infty$, *then* max $|l_k^{(n)}(x)| \to 4/\pi$ *if* $t=-1$, 1 if $-1<t\leqq-\frac{1}{2}$, $(2(1+t))^{1/2}$ *if* $-\frac{1}{2}\leqq t\leqq 1$.

THEOREM 6. *If* $\alpha=\frac{3}{2},\ \beta=\frac{1}{2}$ *and* $x_k\to t$ *as* $n\to\infty$, *then* max $|l_k^{(n)}(x)| \to [2(1-t)]^{1/2}$ *if* $-1\leqq t\leqq\frac{1}{2}$, 1 *if* $\frac{1}{2}\leqq t<1$, $4/\pi$ *if* $t=1$.

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[5] Erdös and Grünwald, *Note on an elementary problem of interpolation*, this Bulletin, vol. 44 (1938), pp. 515–518.

The max $|l_1^{(n)}(x)|$ is attained at $x=\pm 1$ since[4] (I) $\theta_{k+1}-\theta_k \leqq 2\pi/(2n+\alpha+\beta-1)$ provided $\frac{1}{2}\leqq\alpha, \beta\leqq\frac{3}{2}$ and $x_k\equiv\cos\theta_k$. Using the second asymptotic formula and the fact[4] that $n\theta_k\to j_k$ as $n\to\infty$ where $j_k$ is the $k$th positive zero of $J_{\beta-1}(x)$, we find that

$$|l_k^{(n)}(1)|\to(\tfrac{1}{2}j_k)^{\beta-2}|\Gamma(\beta)J_\beta(j_k)|^{-1} \qquad \text{as n} \to \infty,\ k \text{ constant},$$

$l_1^{(n)}(-1)\to 0$ which proves the theorem:

THEOREM 7. Max $|l_1^{(n)}(x)|\to(\frac{1}{2}j_1)^{\beta-2}|\Gamma(\beta)J_\beta(j_1)|^{-1}$ *as* $n\to\infty$ (*where* $\frac{1}{2}\leqq\alpha, \beta\leqq\frac{3}{2}$, $j_1$ *is first positive zero of* $J_{\beta-1}(x)$).

A similar result holds for $l_n^{(n)}(x)$ if $\beta$ is replaced by $\alpha$.

For Legendre polynomials ($\alpha=\beta=1$) this limit is approximately 1.602. For $\alpha=\beta=\frac{1}{2}$ and $\alpha=\beta=\frac{3}{2}$ the limit of Theorem 7 is also an upper bound for max $|l_1^{(n)}(x)|$ and max $|l_k^{(n)}(x)|$. Whether this is true, in general, remains unanswered.

PURDUE UNIVERSITY

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# AN INVARIANCE THEOREM FOR SUBSETS OF $S^n$ [1]

SAMUEL EILENBERG

The purpose of this paper is to establish the following.

INVARIANCE THEOREM. *Let $A$ and $B$ be two homeomorphic subsets of the n-sphere $S^n$. If the number of components of $S^n-A$ is finite, then it is equal to the number of components of $S^n-B$.*

In the case when $A$ and $B$ are closed this theorem is a very well known consequence of Alexander's duality theorem and its generalizations. In our case we also derive our result as a consequence of a duality theorem. However, the duality is established only for the dimension $n-1$.

Given a metric space $X$ we shall say that $\Gamma^k$ is a $k$-cycle in $X$ if there is a compact subset $A$ of $X$ such that $\Gamma^k$ is a $k$-dimensional convergent (Vietoris) cycle in $A$ with coefficients modulo 2. We shall write $\Gamma^k\sim 0$ if $\Gamma^k\sim 0$ holds in some compact subset of $X$. The homology group of $X$ obtained this way will be denoted by $\mathfrak{H}^k(X)$; the corresponding connectivity number, by $p^k(X)$. The number $p^k(X)$ can be either finite or $\infty$.

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[1] Presented to the Society, December 28, 1939.

DUALITY THEOREM. *Let* $A \subset S^n$ *and let* $z_0, z_1, \cdots, z_m$ *belong to* $m+1$ *different quasi-components*[2] *of* $S^n - A$. *There are* $m$ *linearly independent (modulo* 2) $(n-1)$*-cycles*

$$(1) \qquad \Gamma_1^{n-1}, \cdots, \Gamma_m^{n-1}$$

*of* $A$ *such that*

$$(2) \qquad v(\Gamma_i^{n-1}, \overset{0}{\gamma_j}) = \delta_{ij}, \qquad i, j = 1, \cdots, m,$$

*where* $\gamma_j^0$ *is the* 0*-cycle* $z_0 + z_j$ *(consisting of the two points* $z_0, z_j$ *each of them with coefficient* 1) *and* $v(\Gamma, \gamma)$ *is the linking number.*

*In case* $S^n - A$ *has only* $m+1$ *quasi-components, the cycles* (1) *form a basis for* $\mathcal{H}^{n-1}(A)$.

PROOF. In case $A$ is closed the theorem turns out to be a particular case of the generalized Alexander duality theorem.[3] We shall prove our theorem for arbitrary sets $A$ using the theorem for closed sets.

Since $z_0, z_1, \cdots, z_m$ belong to $m+1$ different quasi-components of $S^n - A$ there is a decomposition $S^n - A = A_0 + A_1 + \cdots + A_m$ such that $z_i \in A_i$ and $A_i \bar{A}_j + \bar{A}_i A_j = 0$ for $i \neq j$, $i, j = 0, 1, \cdots, m$. Let $B_0, B_1, \cdots, B_m$ be open disjoint sets such that $A_i \subset B_i$ for $i = 0, 1, \cdots, m$ and let $B = S^n - (B_0 + B_1 + \cdots + B_m)$. Clearly $B$ is a closed subset of $A$ and $z_0, z_1, \cdots, z_m$ belong to $m+1$ different quasi-components (equals components) of $S^n - B$.

Applying the duality theorem to the closed set $B$ we obtain the cycles (1) satisfying (2). In order to prove that they determine linearly independent elements modulo 2 of $\mathcal{H}^{n-1}(A)$ consider a cycle $\Gamma^{n-1} = a_1 \Gamma_1^{n-1} + \cdots + a_m \Gamma_m^{n-1}$ where $a_j = 0, 1$. It follows from (2) that $v(\Gamma^{n-1}, \gamma_j^0) = a_j$. Therefore $\Gamma^{n-1} \sim 0$ in $A$ implies $a_1 = \cdots = a_m = 0$.

Suppose now that $S^n - A$ consists of exactly $m+1$ quasi-components. It follows that the sets $A_0, A_1, \cdots, A_m$ are connected.

Let $\Gamma^{n-1}$ be an $(n-1)$-cycle of $A$ contained in some closed set $D \subset A$. Let $E_i$ be the component of $S^n - (B + D)$ containing $A_i$ $(i = 0, 1, \cdots, m)$ and let $E = S^n - (E_0 + E_1 + \cdots + E_m)$. It follows that (1°) $E$ is a closed subset of $A$, (2°) $S^n - E$ consists of exactly $m+1$ quasi-components (equals components), (3°) the points

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[2] Two points $x_1, x_2 \in X$ belong to the same quasi-component of $X$ if there is no decomposition $X = A_1 + A_2$ such that $x_1 \in A_1$, $x_2 \in A_2$ and $A_1 \bar{A}_2 + \bar{A}_1 A_2 = 0$. If the number of quasi-components of $X$ is finite then every quasi-component is a component.

[3] Alexander, J. W., Transactions of this Society, vol. 23 (1922), pp. 333–349; Frankl, F., Sitzungsberichte der Wiener Akademie der Wissenschaften, Mathematisch-Naturwissenschaftliche Klasse, 2A, vol. 136 (1927), pp. 689–699; Alexandroff, P., Annals of Mathematics, (2), vol. 30 (1928), p. 163.

$z_0, z_1, \cdots, z_m$ belong to different quasi-components of $S^n-E$, (4°) the cycles (1) and $\Gamma^{n-1}$ are contained in $E$. According to the duality theorem for closed sets the cycles (1) form a basis for $\mathcal{H}^{n-1}(E)$. This implies the existence of $a_1, a_2, \cdots, a_m$ $(a_i=0, 1)$ such that

$$\Gamma^{n-1} \sim a_1\Gamma_1^{n-1} + \cdots + a_m\Gamma_m^{n-1} \text{ in } E.$$

This proves the theorem since $E\subset A$.

Given a metric space $X$ let the number $b_0(X)$ be defined as follows:

$$\begin{aligned} b_0(X) &= 0 \quad \text{if } X = 0, \\ b_0(X) &= m \quad \text{if } X \neq 0 \text{ and } X \text{ has exactly } m+1 \text{ components}, \\ b_0(X) &= \infty \quad \text{if } X \text{ has an infinity of components.} \end{aligned}$$

Clearly the value of $b_0(X)$ remains unchanged if we replace in its definition components by quasi-components. The duality theorem implies therefore the following:

(I) *For every subset $A$ of $S^n$ we have*

$$p^{n-1}(A) = b_0(S^n - A).$$

(II) *For every two homeomorphic subsets $A$ and $B$ of $S^n$ we have*

$$b_0(S^n - A) = b_0(S^n - B).$$

The invariance theorem stated in the introduction follows directly from (II).

If $X$ consists of an infinity of components, then instead of taking $b_0(X)=\infty$ we could define $b_0(X)$ to be the cardinal number corresponding to the class of all components of $X$. Similarly $p^k(X)$ could be redefined as a cardinal number. But with these new definitions (I) and (II) are no longer true.[4] In fact, let $A$ be a subset of $S^1$ such that $S^1-A$ is closed and enumerably infinite, and let $B$ be a subset of $S^1$ such that $S^1-B$ is perfect and non-dense. It is clear that $A$ and $B$ are homeomorphic, that $b_0(S^1-A)=p^0(A)=p^0(B)=\aleph_0$, and that $b_0(S^1-B)=2^{\aleph_0}$.

UNIVERSITY OF MICHIGAN

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[4] That (II) is no longer true was first pointed out to me by Dr. L. Zippin.

# MEASURABILITY AND MODULARITY IN THE THEORY OF LATTICES[1]

M. F. SMILEY

In a previous note[2] a notion of measurability (with respect to a function $\mu$) of elements of an arbitrary lattice was introduced. Our purpose there was to study closure properties of the subset of measurable elements. To do this it was convenient to assume that the lattice was modular. Results of V. Glivenko,[3] and of L. R. Wilcox and the author[4] indicate that the idea of measurability and that of modularity are intimately related. The purpose of this note is to exhibit a further relationship which does not depend on metric properties of the function $\mu$.

In a lattice $L$ we call the ordered pair $(a, b)$ of elements of $L$ *modular* and write $(a, b)M$ in case $(a_1+a)b=a_1+ab$ for every $a_1 \leqq b$. This relation has been studied by L. R. Wilcox.[5] If $\mu(a)$ is a real valued function defined on $L$ we say that an element $c \in L$ is $\mu$-measurable in case

$$\mu(c) + \mu(b) = \mu(c + b) + \mu(cb)$$

for every $b \in L$. The symbol $L(\mu)$ will denote the totality of elements of $L$ which are $\mu$-measurable. We call $\mu$ *proper* in case $a \leqq b$ with $\mu(a)=\mu(b)$ implies $a=b$.

Before we discuss the general case, let us consider a lattice $L_0$ of finite dimension (that is, satisfying both chain conditions) in which every principal chain joining two elements $a, b \in L_0$ has the same length $n(a, b)$. Examples of such lattices are well known.[6] Let 0 denote the least element of $L_0$ and define $\mu_0(a) \equiv n(0, a)$. We note that $\mu_0$ is proper. These considerations lead to the following theorem.

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[1] Presented to the Society, February 24, 1940.

[2] *A note on measure functions in a lattice*, this Bulletin, vol. 46 (1940), pp. 239–241. This concept specializes to that of permutability with all subgroups (that is, the *quasi-normality* of O. Ore, *Structures and group theory* I, Duke Mathematical Journal, vol. 3 (1937), p. 162) if we set $\mu(s) = \log o(s)$ in the lattice of subgroups of a finite group, and to that of *measurability* in the sense of Carathéodory with respect to an *outer measure* function.

[3] *Contributions à l'étude des systèmes de choses normées*, American Journal of Mathematics, vol. 59 (1937), pp. 933–934.

[4] *Metric lattices*, Annals of Mathematics, (2), vol. 40 (1939), p. 313.

[5] *Modularity in the theory of lattices*, Annals of Mathematics, (2), vol. 40 (1939), p. 491 ff.

[6] The "exchange" lattices discussed by Saunders Mac Lane (*A lattice formulation for transcendence degrees and p-bases*, Duke Mathematical Journl, vol. 4 (1938), pp. 455–468) as well as the "semi-modular" lattices of finite dimension of Wilcox (op. cit., pp. 502–505) have these properties. We make no use, however, of the metric properties of these systems.

THEOREM 1. *If $c \in L_0$ then $c_1 \in L_0(\mu_0)$ for every $c_1 \leqq c$ if and only if $a \leqq c$ implies that $(b, a)M$ for every $b \in L_0$ and furthermore that $x = ax + b$ for every $x \in L_0$ for which $b \leqq x \leqq a + b$.*

PROOF. To show the sufficiency of the condition consider elements $b, c_1 \in L_0$ for which $c_1 \leqq c$. It is easily seen that $c_1 \in L_0(\mu_0)$ provided that $n(c_1 b, c_1) = n(b, c_1 + b)$. We shall prove in fact that $c_1/c_1 b \simeq c_1 + b/b$. As usual, for $x_1 \in c_1/c_1 b$, that is, $c_1 b \leqq x_1 \leqq c_1$, define $f(x_1) \equiv x_1 + b$. Clearly $f(x_1) \in c_1 + b/b$. Also $f(c_1/c_1 b) = c_1 + b/b$; for if $b \leqq x \leqq c_1 + b$, by hypothesis $f(c_1 x) = x$. Suppose $f(x_1) = f(x_2)$, that is, $x_1 + b = x_2 + b$. Clearly $(x_1 + b)c_1 = (x_2 + b)c_1$. By hypothesis $(b, c_1)M$, and it follows that $x_1 + bc_1 = x_2 + bc_1$, $x_1 = x_2$. Thus $f(x_1)$ establishes a one-to-one correspondence between $c_1/c_1 b$ and $c_1 + b/b$. This correspondence preserves the ordering relation and hence $c_1/c_1 b \simeq c_1 + b/b$. The proof of the sufficiency of the condition is complete. The proof of the necessity is included in the proof of the following theorem.

THEOREM 2. *Suppose that $\mu(a)$ is a proper real valued function defined on a lattice $L$. Consider an element $c \in L$. If $c_1 \in L(\mu)$ for every $c_1 \leqq c$, then $(b, a)M$ for every $a \leqq c$ and every $b \in L$, and furthermore $x = ax + b$ for every $x \in L$ such that $b \leqq x \leqq a + b$.*

PROOF. Consider elements $a, b \in L$ with $a \leqq c$. To show that $(b, a)M$, it suffices, since $\mu$ is proper, to prove that

$$\mu((a_1 + b)a) = \mu(a_1 + ab)$$

for every $a_1 \leqq a$. By hypothesis $a_1, a \in L(\mu)$ and consequently

$$\begin{aligned} \mu((a_1 + b)a) &= \mu(a_1 + b) + \mu(a) - \mu(a + b) \\ &= \mu(a_1 + b) + \mu(ab) - \mu(b) \\ &= \mu(a_1) + \mu(b) - \mu(a_1 b) + \mu(ab) - \mu(b) \\ &= \mu(a_1) + \mu(ab) - \mu(a_1 b) = \mu(a_1 + ab). \end{aligned}$$

Thus we have $(b, a)M$. Now consider an element $x \in L$ for which $b \leqq x \leqq a + b$. Clearly $ax \leqq a \leqq c$ and hence $a, ax \in L(\mu)$. It follows that

$$\begin{aligned} \mu(ax + b) &= \mu(ax) + \mu(b) - \mu(ab) \\ &= \mu(x) + \mu(a) - \mu(a + x) + \mu(b) - \mu(ab) \\ &= \mu(x) + \mu(a) + \mu(b) - \mu(a + b) - \mu(ab) \\ &= \mu(x). \end{aligned}$$

Since $\mu$ is proper we have $x = ax + b$ and this completes the proof.

COROLLARY. *If $p \in L$ is a point, then $p \in L(\mu)$ implies that $b+p$ is prime over $b$ for every $b \in L$ such that $bp = 0$.*

PROOF. Consider a point $p \in L(\mu)$ and an element $b \in L$ for which $bp = 0$. We always have $(b, p)M$ and thus the first part of our conclusion is vacuous.[7] The second part yields $x = b + px$ for every $x \in L$ for which $b \leqq x \leqq b + p$. Since $px = 0$ or $px = p$, it follows immediately that $b + p$ is prime over $b$.

REMARK. Consider the example of a lattice of six elements 0, $p$, $q$, $b$, $c$, 1, with $p < c$, $q < b$, 0 the least element, and 1 the greatest element. Our corollary yields the fact that no proper real valued function $\mu$ can be defined over this lattice for which $p$ or $q$ is $\mu$-measurable.

LEHIGH UNIVERSITY

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[7] L. R. Wilcox, op. cit., p. 491.

# A STIELTJES INTEGRAL EQUATION[1]

F. G. DRESSEL

**Introduction.** By making use of the notion of derivative taken with respect to a function of bounded variation, solutions of the Young-Stieltjes integral equations of the following type

$$F(x) = M(x) + \lambda \int_0^1 H(x, y)dF(y) \tag{1}$$

are obtained in this paper. Since the integration by parts formula is not valid for Young-Stieltjes integrals, equation (1) cannot be immediately changed to the equation

$$f(x) = m(x) + \lambda \int_0^1 f(y)dK(x, y). \tag{2}$$

Fischer[2] in his treatment of equation (2) imposed three conditions on $K(x, y)$ to obtain solutions. In the second part of this paper, we show that one of these conditions is sufficient to insure the existence of a bounded monotone increasing function $g(y)$ such that

$$| K(x, y_2) - K(x, y_1) | \leqq | g(y_2) - g(y_1) |. \tag{3}$$

It has been shown[3] that if condition (3) holds, then equation (2) can be changed into an ordinary Fredholm equation. Thus it appears that only one of the three conditions imposed on $K(x, y)$ by Fischer is needed to insure the solution of (2).

Before handling equation (1), let us note that if $g(y_1) < g(y_2)$ for $y_1 < y_2$, we may apply the transformation[4]

$$\beta(s) = \limsup E_y[s \geqq g(y)], \qquad g(0) \leqq s \leqq g(1), \tag{4}$$

and send the integral equation

$$v(x) = w(x) + \lambda \int_0^1 G(x, y)v(y)dg(y) \tag{5}$$

into the Fredholm equation

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[1] Presented to the Society, February 24, 1940.

[2] C. A. Fischer, *The Fredholm theory of the Stieltjes equation*, Annals of Mathematics, (2), vol. 25 (1923–1924), pp. 124–158.

All functions used in this paper are assumed to be measurable Borel.

[3] F. G. Dressel, *A note on Fredholm-Stieltjes integral equations*, this Bulletin, vol. 44 (1938), pp. 434–437.

[4] $E_y[s \geqq g(y)]$ designates the set of values of $y$ such that $s \geqq g(y)$.

$$v(\beta(t)) = w(\beta(t)) + \lambda \int_\alpha^\delta G(\beta(t), \beta(s))v(\beta(s))ds, \tag{6}$$

where $\alpha = g(0)$, $\delta = g(1)$, and $t$ is any of the possible solutions of

$$x = \beta(t), \qquad g(0) \leqq t \leqq g(1).$$

Now in $L^2$ the Fredholm theory applies to (6) if

$$\int_0^1 \int_0^1 |G(x, y)|^2 dg(x)dg(y) = \int_\alpha^\delta \int_\alpha^\delta |G(\beta(t), \beta(s))|^2 dtds \tag{7}$$

exists.[5] Also if $h(t) \subset L^2$, we have by Schwartz's inequality

$$\left| \int_0^1 G(x, \beta(t))h(t)dt \right|^2 \leqq \int_0^1 |G(x, \beta(t))|^2 dt \int_0^1 |h(t)|^2 dt,$$

and from this we see $\int_0^1 G(x, \beta(t))h(t)dt$ is integrable with respect to $g(x)$ if (7) holds. The above statements lead to the following lemma:

LEMMA 1. *If* $g(x_1) < g(x_2)$ *for* $x_1 < x_2$ *with* $g(x)$ *bounded, and if under the transformation* (4) *the functions* $w(x)$, $G(x, y) \subset L^2$, *and also if* (7) *holds, then for all values of* $\lambda$, *except perhaps the characteristic values*[5] *of the integral equation* (6), *the equation* (5) *has a solution integrable with respect to* $g(x)$.

**Stieltjes integral equations.** Consider now the equation (1) where the given functions $M(x)$ and $H(x, y)$ are assumed to be absolutely continuous[6] $g(x)$. Here $g(x)$ is a bounded increasing function, continuous from the right except perhaps at $x = 0$. Let $m(x)$ and $h(x, y)$ be the derived numbers of $M(x)$ and $H(x, y)$ taken with respect to $g(x)$, and form the equation

$$v(x) = m(x) + \lambda \int_0^1 h(x, y)v(y)dg(y). \tag{8}$$

We see from our lemma that equation (8) will have a solution under rather weak restrictions on $m(x)$ and $h(x, y)$. Let us now assume (8) has a solution, say $\bar{v}(x)$, which is integrable $g(x)$, and define the function $Q(y)$ as

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[5] For list of references and terminology see E. Hille and J. D. Tamarkin, *On the characteristic values of linear integral equations*, Proceedings of the National Academy of Sciences, vol. 14 (1928), pp. 911–914.

[6] For terminology see A. J. Maria, *Generalized derivatives*, Annals of Mathematics, (2), vol. 28 (1926–1927), pp. 419–432. Let $f(x)$ be a function of bounded variation and denote by $\bar{f}(e)$ the completely additive function of sets defined by $f(x)$. (The facts here are that for any interval $\alpha(x_1 \leqq t \leqq x_2)$ the relation $\bar{f}(\alpha) = f(x_2+0) - f(x_1-0)$ holds.) The function $f(x)$ is said to be absolutely continuous with respect to the non-decreasing function $g(x)$ if whenever $E$ is a set for which $\bar{f}(E)$ is defined and $\bar{g}(E) = 0$, then $\bar{f}(E) = 0$.

$$\begin{aligned} Q(y) &= \int_0^y \bar{v}(x)dg(x), && y \neq 0, \\ &= 0, && y = 0. \end{aligned}$$

We are now able to write the solution of (1) as

$$F(x) = M(x) + \lambda \int_0^1 H(x, y)dQ(y). \tag{9}$$

To see that the above $F(x)$ furnishes a solution of (1), substitute the right side of (9) in both sides of equation (1). After cancelling $M(x)$ we have

$$\begin{aligned} \lambda \int_0^1 H(x, y)dQ(y) &= \lambda \int_0^1 H(x, y)dM(y) \\ &\quad + \lambda^2 \int_0^1 H(x, y)d_y \int_0^1 H(y, t)dQ(t). \end{aligned} \tag{10}$$

Since $\bar{v}(x)$ is a solution of (8), on integration of that equation with respect to $g(x)$, there results

$$\begin{aligned} Q(x) &= M(x+0) - M(0) + \lambda \int_0^1 [H(x+0, t) - H(0, t)]\bar{v}(t)dg(t),^{7} && x \neq 0, \\ &= M(x+0) - M(0) + \lambda \int_0^1 [H(x+0, t) - H(0, t)]dQ(t), && x \neq 0, \\ &= 0, && x = 0. \end{aligned}$$

If the above is substituted in the left side of (10), it is easily seen to be equal to the right side, hence (9) satisfies (1).

From the above and our lemma we have the following theorem:

THEOREM 1. *Let $g(x)$ be continuous from the right in* $0 < x \leq 1$, *and satisfy the conditions of Lemma* 1; *let $M(x)$ and $H(x, y)$ in equation* (1) *be absolutely continuous $g(x)$ and have $m(x)$ and $h(x, y)$ as derived numbers $(g(x))$. Then if $m(x)$, $h(x, y)$ satisfy the conditions placed on $w(x)$, $G(x, y)$ respectively in Lemma* 1, *equation* (9) *furnishes a solution of* (1) *for all values of* $\lambda$ *except those that are characteristic values of the integral equation* (8).

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[7] F. G. Dressel, loc. cit., p. 435.

COROLLARY. *If*

$$|H(x_2, y) - H(x_1, y)| \leqq |g(x_2) - g(x_1)|,$$
$$|M(x_2) - M(x_1)| \leqq |g(x_2) - g(x_1)|$$

*where* $g(x)$ *meets the requirements of Theorem* 1, *then* (9) *is the solution of* (1) *for all values of* $\lambda$ *that are not characteristic values of the integral equation* (8).

**The Fischer theory.** In the treatment of equation (2), Fischer imposed the following condition on the kernel $K(x, y) = K(x, y+0)$:

$$V_2K(x, y) < N.$$

Here $N$ is a constant and $V_2K$ has the following meaning:

$$V_2K(x, y) = \text{l.u.b.} \sum_{i=1}^{n} | K(x_i, y_i) - K(x_i, y_{i-1}) |,$$

it being understood that $0 = y_0 < y_1 < \cdots < y_n = 1$, and $\{x_i\}$ are any $x$'s in $0 \leqq x \leqq 1$. We shall soon show that the above condition implies that inequality (3) holds.

Define the set $E$ on the interval $0 \leqq y \leqq 1$ to be given by

$$y = m/2^n, \qquad m = 0, 1, 2, \cdots, m \leqq 2^n, \; n = 1, 2, \cdots,$$

and let $g(x, y)$ be a function with the properties

(a) $\qquad g(x, y) = g(y, x) \geqq 0, \qquad g(x, y) \leqq g(x, z) + g(z, y)$

and if $0 \leqq y_0 < y_1 < \cdots < y_n \leqq 1$ then

(b) $$\sum_{i=1}^{n} g(y_i, y_{i-1}) < N \qquad \text{for all } n.$$

Let $y_j \subset E$ and define $A_n(y_j)$ as

$$A_n(y_j) = \sum_{i=1}^{m} g\left(\frac{i}{2^n}, \frac{i-1}{2^n}\right), \qquad 0 < \frac{m}{2^n} \leqq y_j < \frac{m+1}{2^n},$$
$$= 0, \qquad\qquad y_j < \frac{1}{2^n}.$$

Due to the conditions on $g(x, y)$ we have

$$A_n(y_j) \leqq A_{n+1}(y_j), \qquad A_n(y_j) < N;$$

hence $\lim_{n\to\infty} A_n(y_j)$ exists: let us call it $A(y_j)$. Again let $y_1, y_2 \subset E$ with $y_1 < y_2$; then if $y_1 = r/2^k$, $y_2 = m/2^k$,

$$A_k(y_2) - A_k(y_1) = {\sum_{i=r+1}^{m}}' g\left(\frac{i}{2^k}, \frac{i-1}{2^k}\right) \geqq g(y_2, y_1);$$

hence

$$A_k(y_2) - A_k(y_1) \geqq g(y_2, y_1) \geqq 0.$$

Thus $A(y)$ as defined on $E$ is a bounded monotone increasing function; hence we may extend the definition of $A(y)$ on $CE$ so as to define a monotone increasing function of bounded variation[8] on $0 \leqq y \leqq 1$. We make $A(y)$ unique by taking $A(y+0) = A(y)$ if $y \subset CE$.

Turn now to Fischer's condition $V_2K(x, y)$ and define

$$g(y_2, y_1) = \underset{0 \leqq x \leqq 1}{\text{l.u.b.}} \; | K(x, y_2) - K(x, y_1) |.$$

This $g(x, y)$ is easily seen to satisfy conditions (a) and (b); hence the following theorem is valid.

THEOREM 2. *If for the function* $K(x, y) = K(x, y+0)$ *the condition* $V_2K(x, y) < N$ *holds, then there exists a bounded monotone increasing function* $A(y)$ *such that*

$$| K(x, y_2) - K(x, y_1) | \leqq | A(y_2) - A(y_1) |.$$

COROLLARY. *The condition* $V_2K(x, y) < N$ *imposed on* $K(x, y) = K(x, y+0)$ *by Fischer is sufficient to change equation* (2) *into an ordinary Fredholm equation.*[9]

The existence of the following limit is essential in Fischer's treatment of equation (2):

$$\lim_{\delta \to 0} \sum_{i=1}^{n} [K(x_i, x_i) - K(x_i, x_{i-1})], \qquad 0 = x_0 < x_1 < \cdots < x_n = 1,$$

where $\delta = \max (x_i - x_{i-1})$. The Fischer theory will therefore not apply to the kernel

$$\begin{aligned} K_1(x, y) &= 1, \qquad x = \tfrac{1}{2}, \qquad \tfrac{1}{2} \leqq y \leqq 1, \\ &= 0, \qquad \text{elsewhere.} \end{aligned}$$

However, the integral equation (2) with the above kernel is easily treated since $V_2K_1(x, y) < 2$.

BROWN UNIVERSITY AND
DUKE UNIVERSITY

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[8] G. C. Evans, *Logarithmic Potential*, American Mathematical Society Colloquium Publications, vol. 6, 1927, pp. 5–6.

[9] F. G. Dressel, loc. cit., p. 436.

# FOURIER COEFFICIENTS OF BOUNDED FUNCTIONS[1]

BERNARD FRIEDMAN

The results of this paper are divided into two parts. First: inequalities for the Fourier coefficients of any bounded function; second: an approximation theorem for the Fourier development of an arbitrary bounded function.

Inequalities for Fourier coefficients have been discussed in a paper by Professor Szász.[2] However, his work deals mainly with linear inequalities for complex coefficients. The inequalities to be investigated in this paper are not linear. Nevertheless, they are the best possible, for this reason: given any set of numbers which makes the inequality an equality, there exists a bounded function which has these numbers as its Fourier coefficients.

A simple illustration will clarify this. Let $f(x)$ be a bounded measurable function in $(-\pi, \pi)$ such that $|f(x)| \leqq 1$. The Fourier coefficients of $f(x)$ are given by the formulae

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx,$$
$$n = 1, 2, 3, \cdots .$$

Then, it is clear that

$$| a_n | \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} | f(x) | | \cos nx | \, dx \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} | \cos nx | \, dx,$$
$$| b_n | \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} | f(x) | | \sin nx | \, dx \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} | \sin nx | \, dx$$

since $|f(x)| \leqq 1$.

Since the cosine is negative in the intervals $(-\pi, -\pi/2)$ and $(\pi/2, \pi)$ and positive in the remaining interval $(-\pi/2, \pi/2)$,

$$\int_{-\pi}^{\pi} | \cos x | \, dx = \int_{-\pi}^{-\pi/2} (-\cos x) dx + \int_{-\pi/2}^{\pi/2} \cos x \, dx$$
$$+ \int_{\pi/2}^{\pi} (-\cos x) dx = 4.$$

Therefore, $|a_1| \leqq 4/\pi$. However, if $f_1(x) = -1, \ -\pi < x < -\pi/2$; $f_1(x)$

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[1] Presented to the Society, September 8, 1939.

[2] American Journal of Mathematics, vol. 61 (1939).

$=1,\ -\pi/2<x<\pi/2;\ f_1(x)=-1,\ \pi/2<x<\pi$; then

$$\int_{-\pi}^{\pi} f_1(x) \cos x\, dx = \int_{-\pi}^{-\pi/2} (-\cos x)dx + \int_{-\pi/2}^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} (-\cos x)dx$$

and the bound $4/\pi$ will actually be attained for the function $f_1(x)$.

Likewise, $\int_{-\pi}^{\pi}|\cos nx|\,dx$ and $\int_{-\pi}^{\pi}|\sin nx|\,dx$ both equal four, so that $|a_n|$ and $|b_n| \leqq 4/\pi$; these bounds will be attained for the functions signum cos $nx$ and signum sin $nx$ respectively.

But these bounds for the Fourier coefficients cannot be attained simultaneously. If one of the coefficients is too large, the others must be small. For example: $b_1$ and $b_2$ must satisfy the following inequality:

$$\left(\frac{\pi}{4} b_1\right)^2 + \frac{\pi}{4} | b_2 | \leqq 1$$

so that if $b_1=4/\pi$ then $b_2=0$ while if $b_2=4/\pi$ then $b_1=0$. Conversely, any numbers $b_1$, $b_2$ satisfying this relation will be the Fourier coefficients of some bounded function.

Similar relations hold between the other Fourier coefficients, but they are much more complicated:

$$a_n^2 + b_n^2 \leqq \left(\frac{4}{\pi}\right)^2,$$

$$\left(\frac{\pi}{4} b_1 + 1\right)^3 - 3\left(\frac{\pi}{4} b_1 + 1\right) - 1 \leqq 3\left(\frac{\pi}{4} b_3\right) \leqq \left(\frac{\pi}{4} b_1 - 1\right)^3 - 3\left(\frac{\pi}{4} b_1 - 1\right) + 1.$$

After the inequalities for the Fourier coefficients are obtained, they are used to derive an approximation theorem. But instead of being an approximation to the function itself, it will be an approximation to the Fourier development of the function. More precisely: take a bounded measurable function $f(x)$ whose absolute value is less than one; let its Fourier development be $a_0/2+\sum(a_n \cos nx+b_n \sin nx)$. Then, for any $N$ there exists a function $g_N(x)$ which takes the values 1 and $-1$ only, such that the first $N$ Fourier coefficients of $g_N(x)$ are the same as the first $N$ Fourier coefficients of $f(x)$.

In general, there will be $N+1$ discontinuities. If $f(x)$ is even or odd, $g_N(x)$ will be even or odd. For example, let $f(x)=(\pi-x)/\pi$,

$0<x<\pi; f(-x)=-f(x)$. Then its Fourier series will be

$$\frac{2}{\pi}\sum_{1}^{\infty}\frac{\sin nx}{n},$$

$$\begin{aligned} g_1(x) &= +1, \quad 0<x<\frac{2\pi}{3}; & g_2(x) &= +1, \quad 0<x<\frac{\pi}{2};\\ g_1(x) &= -1, \quad \frac{2\pi}{3}<x<\pi; & g_2(x) &= -1, \quad \frac{\pi}{2}<x<\frac{2\pi}{3};\\ & & g_2(x) &= +1, \quad \frac{2\pi}{3}<x<\pi. \end{aligned}$$

Incidentally, one of the inequalities leads to the following result: If $F(x)$ is a bounded measurable function in absolute value less than $\frac{1}{2}$,

$$\int_0^{\pi} F(x)\cos x\,dx \leqq \cos\left(\int_0^{\pi} F(x)dx\right).$$

The method used in proving the above inequalities is as follows: Let $f(x)$ be a bounded measurable function in $(-\pi,\pi)$ such that $|f(x)|\leqq 1$. Let the Fourier development of $f(x)$ be

$$\frac{a_0}{2}+\sum (a_n\cos nx + b_n\sin nx).$$

Consider the expression

$$f_n(\mu, a) \equiv \mu_0 a_0 + \mu_1 b_1 + \mu_2 a_1 + \cdots + \mu_{2n-1} b_n + \mu_{2n} a_n \tag{1}$$

where $\mu_0, \mu_1, \cdots, \mu_{2n}$ are arbitrary real parameters. The functions $f_n(\mu, a)$ can be expressed in terms of the function $f(x)$ by using the usual formulae for Fourier coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

From these formulae and (1), it is clear that

$$\begin{aligned} f_n(\mu, a) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)[\mu_0 + \mu_1\sin x + \mu_2\cos x + \cdots \\ &\qquad + \mu_{2n-1}\sin nx + \mu_{2n}\cos nx]dx \\ &\equiv \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)C_n(\mu, x)dx, \end{aligned}$$

say.

Assuming the $\mu$'s are fixed, the maximum of $f_n(\mu, a)$ will be attained when the integrand $f(x)C_n(\mu, x)$ is as large as possible. But

$$|f(x)C_n(\mu, x)| \leqq |C_n(\mu, x)|$$

since $|f(x)| \leqq 1$. However, if $f(x)$ is taken as signum $C_n(\mu, x)$; that is, $f(x)=1$ when $C_n(\mu, x)>0$ and $f(x)=-1$ when $C_n(\mu, x)<0$, then

$$|f(x)C_n(\mu, x)| = |C_n(\mu, x)|.$$

The following result is now obvious:

$$f_n(\mu, a) \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} |C_n(\mu, x)|\, dx \equiv D_n(\mu), \tag{2}$$

and the equality holds only when

$$f(x) = \text{signum } C_n(\mu, x) \equiv g_n(\mu, x).$$

For example, if $n=1$ and $\mu_1=0$, formula (2) states that

$$\mu_0 a_0 + \mu_2 a_1 \leqq \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(x)C_1(\mu, x)dx$$

where

$$f_1(x) = \text{signum } (\mu_0 + \mu_2 \cos x).$$

Assume $\mu_0+\mu_2 \cos x=0$ has the real roots $\pm\alpha_1$; then $g_1(\mu, x)$ will be defined as follows:

$$\begin{array}{lll}
g_1(\mu, x) = +1, & -\alpha_1 < x < \alpha_1, & \\
g_1(\mu, x) = -1, & \begin{cases} -\pi < x < -\alpha_1, \\ \alpha_1 < x < \pi, \end{cases} & \text{if } \mu_0 > 0; \\
g_1(\mu, x) = -1, & -\alpha_1 < x < \alpha_1, & \\
g_1(\mu, x) = +1, & \begin{cases} -\pi < x < -\alpha_1, \\ \alpha_1 < x < \pi, \end{cases} & \text{if } \mu_0 < 0;
\end{array}$$

and

$$\begin{aligned}
\int_{-\pi}^{\pi} |C_1(\mu, x)|\, dx &= \pm\frac{1}{\pi}\int_{-\pi}^{\pi} g_1(\mu, x)C_1(\mu, x)dx \\
&= \pm\frac{1}{\pi}\left[\int_{-\pi}^{-\alpha_1} - C_1(\mu, x)dx + \int_{-\alpha_1}^{\alpha_1} C_1(\mu, x)dx \right. \\
&\qquad \left. + \int_{\alpha_1}^{\pi} - C_1(\mu, x)dx\right]
\end{aligned}$$

$$= \pm \frac{1}{\pi}\,[\mu_0(4\alpha - 2\pi) + \mu_2 4 \sin \alpha] = \pm \frac{4}{\pi}\left[\mu_0\left(\alpha - \frac{\pi}{2}\right) + \mu_2 \sin \alpha\right].$$

If $\mu_0$ is positive, the upper sign holds; if $\mu_0$ is negative, the lower sign is used.

But $C_1(\mu, \alpha_1) = 0$; that is, $\mu_0 + \mu_2 \cos \alpha_1 = 0$ or

$$\cos \alpha = -\mu_0/\mu_2,$$

$$\alpha = \arccos(-\mu_0/\mu_2), \qquad \alpha - \pi/2 = \arcsin \mu_0/\mu_2,$$

$$\sin \alpha = (1 - \mu_0^2/\mu_2^2)^{1/2},$$

so that

$$D_1(\mu) = \pm \frac{4}{\pi}\,[\mu_0 \arcsin(\mu_0/\mu_2) + \mu_2(1 - \mu_0^2/\mu_2^2)^{1/2}].$$

Therefore

$$\mu_0 a_0 + \mu_2 a_1 \leqq \pm \frac{4}{\pi}\,[\mu_0 \arcsin(\mu_0/\mu_2) + \mu_2(1 - \mu_0^2/\mu_2^2)^{1/2}] \tag{3}$$

where $a_0$ and $a_1$ are the Fourier coefficients of any bounded function. It is easy to see that the equality sign holds only when $a_0$ and $a_1$ equal the Fourier coefficients of $g_1(\mu, x)$.

Similar results hold for any $n$. Asume $C_n(\mu, x) = 0$ has $2n$ real roots $\alpha_i$ where $-\pi < \alpha_1 \leqq \alpha_2 \leqq \alpha_3 \leqq \cdots \leqq \alpha_{2n} \leqq \pi$. Call $-\pi = \alpha_0$ and $\pi = \alpha_{2n+1}$.

Assume $C_n(\mu, x)$ is positive between $\alpha_0$ and $\alpha_1$. Then $g_n(\mu, x)$ will be defined as follows: $g_n(\mu, x) = (-1)^i$, $\alpha_i < x < \alpha_{i+1}$, $i = 0, 1, \cdots, 2n+1$. If $C_n(\mu, x)$ is negative, then $g_n(\mu, x) = (-1)^{i+1}$, $\alpha_i < x < \alpha_{i+1}$. Therefore,

$$D_n(\mu) = \frac{1}{\pi}\int_{-\pi}^{\pi} g_n(\mu, x) C_n(\mu, x)\,dx = \pm \frac{1}{\pi} \sum_{0}^{2n+1} \int_{\alpha_i}^{\alpha_{i+1}} (-1)^i C_n(\mu, x)\,dx$$

so that from (2)

$$\mu_0 a_0 + \mu_1 b_1 + \mu_2 a_1 + \cdots + \mu_{2n-1} b_n + \mu_{2n} a_n \leqq D_n(\mu). \tag{4}$$

Let

$$a_i = x_{2i}, \qquad b_i = x_{2i-1}. \tag{5}$$

Consider the $2n+1$ dimensional space of points whose coordinates are the $x_j$ $(j = 0, 1, 2, \cdots, 2n+1)$. The inequality (4) can be written as

$$\sum_{j=0}^{2n+1} \mu_j x_j \leqq D_n(\mu).$$

This implies that, for any bounded measurable function, the point $P$ whose coordinates are given by (5) must lie on one side of the hyperplane whose equation is

$$\sum_{0}^{2n+1} \mu_j x_j = D_n(\mu). \tag{6}$$

The point $P$ will lie on the plane if (4) becomes an equality. This will occur when and only when $f(x)$ has its first $2n+1$ Fourier coefficients equal to those of $g_n(\mu, x)$.

Since the $\mu_j$ were arbitrary, the same result holds for any set $\mu_0, \mu_1, \cdots, \mu_{2n+1}$. Therefore, the point $P$, whose coordinates are $(x_j)$, $j=0, 1, \cdots, 2n+1$, must lie on one side of every plane in the family of planes defined by the equation (6).

The previous example will clarify this. Equation (3) implies that if, for any $f(x)$, $a_0$ is plotted as the $x$ coordinate and $a_1$ as the $y$ coordinate of a point, then such points will lie on the origin side of the straight line whose equation is

$$\mu_0 x + \mu_2 y = D_1(\mu).$$

The point will lie on the straight line if and only if $f(x)$ has its Fourier coefficients $a_0$ and $a_1$ equal to those of $g_1(\mu, x)$.

Since the $\mu$'s were arbitrary, the same result holds for any set $\mu_0, \mu_2$. Therefore, the point $P$, whose coordinates are $(a_0, a_1)$ must lie on one side of every line in the family of straight lines defined by the equation

$$\mu_0 x + \mu_2 y = D_1(\mu).$$

This implies that the point $P$ will lie inside the region enveloped by these straight lines.

The usual procedure in finding the equation of this envelope is, first, to differentiate partially with respect to the parameter, and second, to eliminate the parameters between the resulting equation and the original equation. The partial differentiation results in the following two equations:

$$x = \frac{\partial D}{\partial \mu_0} = \pm (4/\pi) \arc\sin (\mu_0/\mu_2), \quad y = \frac{\partial D}{\partial \mu_2} = \pm (4/\pi)(1 - \mu_0^2/\mu_2^2)^{1/2}.$$

After the elimination of the parameters, the following equation is obtained for the envelope:

$$y = (4/\pi) \cos (\pi x/4).$$

This implies that

$$a_1 \leqq (4/\pi) \cos (\pi a_0/4).$$

Equality will hold, as before, if $f(x)$ has the same first two Fourier coefficients as any function $g_1(\mu, x)$.

In the general case, the equation of the envelope is obtained by the same method. When formula (6) is differentiated partially with respect to the parameters, the following equations result:

$$x_j = \frac{\partial D_n(\mu)}{\partial \mu_j}, \qquad j = 0, 1, \cdots, 2n+1.$$

But

$$\frac{\partial D_n(\mu)}{\partial \mu_j} = \pm \frac{1}{\pi} \frac{\partial}{\partial \mu_j}\left[ \sum_{i=0}^{2n+1} \int_{\alpha_i}^{\alpha_{i+1}} (-1)^i C_n(\mu, x) dx \right]$$

where the $\alpha_i$, the roots of $C_n(\mu, x) = 0$, also depend on the $\mu_j$. Now

$$\frac{\partial}{\partial \mu_j} \int_{\alpha_i}^{\alpha_{i+1}} (-1)^i C_n(\mu, x) dx = \int_{\alpha_i}^{\alpha_{i+1}} (-1)^i \frac{\partial C_n(\mu, x)}{\partial \mu_j} dx$$
$$+ (-1)^i C_n(\mu, \alpha_{i+1}) \frac{\partial \alpha_{i+1}}{\partial \mu_j} - (-1)^i C_n(\mu, \alpha_i) \frac{\partial \alpha_i}{\partial u_j}$$

by the standard rule for differentiating under the integral sign.

However $C_n(\mu, \alpha_{i+1}) = C_n(\mu, \alpha_i) = 0$ and

$$\frac{\partial C_n(\mu, x)}{\partial \mu_j} = \sin (j+1)x/2 \text{ if } j \text{ is odd,}$$
$$= \cos (jx/2) \text{ if } j \text{ is even,}$$

so that

$$\text{(7)} \quad \begin{aligned} x_j &= \pm \frac{1}{\pi} \sum_{i=0}^{2n+1} \int_{\alpha_i}^{\alpha_{i+1}} (-1)^i \sin (j+1)x/2 dx \text{ if } j \text{ is odd,} \\ &= \pm \frac{1}{\pi} \sum_{i=0}^{2n+1} \int_{\alpha_i}^{\alpha_{i+1}} (-1)^i \cos (jx/2) dx \text{ if } j \text{ is even,} \end{aligned}$$

where the signs are either all positive or all negative.

Instead of eliminating the parameters $\mu$, it is simpler to eliminate the parameters $\alpha$, thus obtaining the equation of the envelope $\phi_n(x_j) = 0$ which implies the inequality

$$\text{(8)} \qquad \phi_n(a, b) \leqq 0.$$

Since $\phi_n(x_j)$ is obtained as the envelope of planes, it is obvious that the region bounded by it is the convex hull of a family of half-spaces. Again, the inequality will hold if and only if $f(x)$ has its first $2n+1$ Fourier coefficients $(a_0, a_1, b_1, \cdots, a_n, b_n)$ equal to the corresponding Fourier coefficients of any one of the functions $g_n(\mu, x)$.

A few illustrations will clarify this method. Let $n=2$ and $\mu_0=\mu_2=\mu_4=0$. Then the $\alpha_i$ will be the roots of the equation

$$\mu_1 \sin x + \mu_3 \sin 2x = 0,$$

and from formula (7)

$$x_1 = \pm \frac{1}{\pi} \sum \int_{\alpha_i}^{\alpha_{i+1}} (-1)^i \sin x \, dx = \pm (-4/\pi \cos \alpha_3),$$

since $\alpha_1 = -\alpha_3$, and

$$x_3 = \pm \frac{1}{\pi} \sum \int_{\alpha_i}^{\alpha_{i+1}} (-1)^i \sin 2x \, dx = \pm (2/\pi)(1 - \cos 2\alpha_3).$$

Therefore $(\pi x_1/4)^2 + \pi x_3/4 = 1$ or $(\pi x_1/4)^2 - \pi x_3/4 = 1$ according as the plus or minus signs are used. But since $x_1$ was the coordinate for $b_1$ and $x_3$ for $b_2$, the following inequality results:

$$(\pi b_1/4)^2 + \pi \left| b_3 \right| /4 \leqq 1.$$

From the result proved above: that the Fourier coefficients of $f(x)$ satisfy certain inequalities or, stating it geometrically, lie in certain convex regions, the approximation theorem follows. The theorem can be stated as follows:

Let $f(x)$ be a real, bounded, measurable function in absolute value less than one. Let its Fourier development be

$$\frac{a_0}{2} + \sum (a_n \cos nx + b_n \sin nx).$$

Then for every $N$, there exists a step-function $g_N(x)$ which takes on the values of $+1$ and $-1$ only, such that the first $N$ Fourier coefficients of $g_N(x)$ are the same as the first $N$ Fourier coefficients of $f(x)$, that is,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g_N(x) \cos nx \, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g_N(x) \sin nx \, dx,$$

$$n = 0, 1, \cdots, N.$$

Again, let $a_i = x'_{2i}$, $b_i = x'_{2i-1}$. Then, as before, the point $P'$ whose coordinates are $(x'_0, x'_1, \cdots, x'_{2N})$ must satisfy the inequality (8):

$\phi_N(x_j') \leqq 0$, or, geometrically, the point $P'$ must lie inside the convex region bounded by the hypersurface whose equation is

$$\phi_N(x_0, x_1, x_2, \cdots, x_{2N}') = 0.$$

Suppose now that $P'$ lies on the hypersurface, that is, its coordinates satisfy the equation

$$\phi_N(x_0', x_1', x_2', \cdots, x_{2N}') = 0.$$

Since $\phi_N(x_j) = 0$ was obtained as the envelope of the family of hyperplanes represented by equation (6), $\sum \mu_j x_j = D_N(\mu)$, there will be a plane of the family tangent to the surface at $P'$. But since $P'$ lies on this plane, whose equation can be written

$$\mu_0' x_0 + \mu_1' x_1 + \mu_2' x_2 + \cdots + \mu_{2N}' x_{2N} = D_N(\mu'),$$

its coordinates must equal those of the function $g_N(\mu', x)$. Therefore, $g_N(\mu_0', \cdots, \mu_{2N}', x)$ is the function described in the theorem.

If $P'$ lies inside the hypersurface, consider the points in $2N+3$ dimensional space whose coordinates are $(x_0', x_1', x_2', \cdots, x_{2N}', x_{2N+1}, x_{2N+2})$ where $x_{2N+1}$ is an arbitrary variable. They will lie on the straight line through $(x_0', x_1', x_2', \cdots, x_{2N}', 0, 0)$ perpendicular to the hyperplane $x_{2N+1} = x_{2N+2} = 0$.

Since $\phi_{N+1}(x_j) = 0$ is a closed convex surface for all $n$, the above straight line will intersect it in some point whose coordinates are $(x_0', x_1', x_2', \cdots, x_{2N}', x_{2N+1}', x_{2N+2}')$. Just as before, there will exist a step-function $g_{N+1}(x)$ whose Fourier development is $a_0'/2 + \sum (a_n' \cos nx + b_n' \sin nx)$ and such that $a_i' = x_{2i}'$, $b_i' = x_{2i-1}'$; $i = 0, 1, 2, \cdots, N+1$.

This function satisfies the conditions of the theorem for

$$x_{2i}' = a_i = a_i', \qquad x_{2i-1}' = b_i' = b_i.$$

University of Wisconsin

# THE GENERALIZATION OF A LEMMA OF M. S. KAKEYA

J. GERONIMUS

We shall prove the following:

LEMMA. *It is always possible to find the unique polynomial*

$$\phi^*(z) = \sum_{k=0}^{2s} \gamma_k^* z^k$$

*of degree* $2s$ *possessing the following properties:*

I. $$\phi^*(z) = c i^2(z)\tau(z)\tau^*(z), \qquad c = \text{const.},$$

*the polynomial* $i(z)$ *of degree* $\sigma \leqq s$ *having all roots in the domain* $|z| > 1$:

$$i(z) = \prod_{i=1}^{\sigma} (z - a_i), \qquad |a_i| > 1, \quad i = 1, 2, \cdots, \sigma,$$

*and the polynomial* $\tau(z)$ *being of degree* $\nu = s - \sigma$:

$$\tau(z) = \prod_{i=1}^{\nu} (z - \alpha_i), \qquad \tau^*(z) = z^\nu \bar{\tau}\left(\frac{1}{z}\right) = \prod_{i=1}^{\nu} (1 - z\bar{\alpha}_i).$$

II. *It is subject to the conditions*

$$\omega_i(\phi^*) = \sum_{k=0}^{2s} \gamma_k^* c_k^{(i)} = d_i, \qquad i = 0, 1, \cdots, s,$$

*the given linear functionals* $\omega_i$ *being such that every polynomial* $\phi(z)$ *of degree* $n \geqq 2s$ *for which*

$$\omega_i(\phi) = \sum_{k=0}^{2s} \gamma_k c_k^{(i)} = 0, \quad (i = 0, 1, \cdots, s), \qquad \phi(z) = \sum_{k=0}^{n} \gamma_k z^k,$$

*has* $s+1$ *roots at least in the domain* $|z| < 1$.

In the particular case when

$$\omega_i(\phi) = \phi^{(i)}(z_k), \qquad |z_k| < 1,$$

this lemma has been proved by M. S. Kakeya [1];[1] without being aware of his result we have proved this lemma in the case[2]

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[1] Numbers in brackets refer to the bibliography at the end.

[2] In [1] and [2] one may find the application of this lemma to some extremal problems.

$$\omega_i(\phi) = \frac{1}{i!}\left(\frac{d^i\phi}{dz^i}\right)_{z=0}, \qquad i = 0, 1, \cdots, s.$$

In order to prove this lemma in the most general case we consider the following extremal problem:

PROBLEM. *To find the minimum of the integral*

$$L(b) = \int_0^{2\pi} | t(z) |^2 b(\theta)d\theta, \qquad z = e^{i\theta}, \tag{1}$$

*$t(z)$ being the given polynomial of degree $s$ with $t(0)\neq 0$ and $b(\theta)$ being a trigonometric polynomial of order $n \geqq 2s$:*

$$b(\theta) = R\left\{z^n\bar{\phi}\left(\frac{1}{z}\right)\right\} = R\sum_{k=0}^{n}\bar{\gamma}_k e^{i(n-k)\theta}, \qquad z = e^{i\theta},$$

*subject to the conditions*[3]

$$\omega_i(b) = \omega_i(\phi) = \sum_{k=0}^{2s}\gamma_k c_k^{(i)} = d_i, \quad i = 0, 1, \cdots, s. \tag{2}$$

The fundamental property of our functionals $\omega_i$ yields at once that every trigonometric polynomial $b(\theta)$ subject to the conditions

$$\omega_i(b) = 0, \qquad i = 0, 1, \cdots, s,$$

has in $(0, 2\pi)$ no more than $2(n-s-1)$ changes of sign. It is clear that there exists a solution of our problem. Further, the necessary conditions for an extremum are

$$\operatorname{sgn} b^*(\theta) \mid t(z) \mid^2 = R\sum_{k=n-2s}^{\infty} A_k z^k, \qquad z = e^{i\theta},$$

whence we find at once that the Fourier expansion of sgn $b^*(\theta)$ is of the form

$$\operatorname{sgn} b^*(\theta) = R\sum_{k=n-s}^{\infty} B_k z^k, \qquad z = e^{i\theta}.$$

We have shown in [2] that every trigonometric polynomial with this property must be of the form

$$b^*(\theta) = R\left\{\bar{c}z^{n-2s+\nu}q^2(z)\right\}\tau(z)\bar{\tau}(1/z), \qquad z = e^{i\theta},$$

$q(z)$ being a polynomial of degree $\sigma \leqq s$ all of whose roots lie in the domain $|z| < 1$, and $\tau(z)$ being a polynomial of degree $\nu = s - \sigma$.

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[3] The functionals $\omega_i$ are the same as above.

The polynomial $b^*(\theta)$ for which the minimum is attained is unique. If there were two such polynomials, $b_1^*(\theta)$ and $b_2^*(\theta)$, then we would have

$$L(b_1^*) \leqq L\left(\frac{b_1^* + b_2^*}{2}\right) \leqq L(b_1^*);$$

then $b_1^*(\theta)$ and $b_2^*(\theta)$ would change sign at the same points, that is, the polynomial

$$b_1^*(\theta) - b_2^*(\theta) = R\{z^{n-2s+\nu}q^2(z)\}\{\bar{c}_1|\tau_1(z)|^2 - \bar{c}_2|\tau_2(z)|^2\}, \qquad z = e^{i\theta},$$

would have at least $2(n-\nu)$ changes of sign in $(0, 2\pi)$; but since

$$\omega_i(b_1^* - b_2^*) = 0, \qquad i = 0, 1, \cdots, s,$$

the polynomial $b_1^*(\theta) - b_2^*(\theta)$ cannot have more than $2(n-s-1)$ changes of sign in $(0, 2\pi)$; this contradiction proves the unicity of the polynomial solving our problem. *Thus we find that there exists the unique polynomial* $b^*(\theta)$ *minimizing* (1) *under conditions* (2) *and it must be of the form*

$$\begin{aligned} b^*(\theta) &= R\{\bar{c}z^{n-2s+\nu}q^2(z)\tau(z)\bar{\tau}(1/z)\} \\ &= R\{\bar{\gamma}_0^*z^n + \bar{\gamma}_1^*z^{n-1} + \cdots + \bar{\gamma}_{2s}^*z^{n-2s}\}, \qquad z = e^{i\theta}. \end{aligned}$$

Since the real parts of two polynomials coincide on the unit circle, these polynomials are identical, that is,

$$\bar{c}z^{n-2s}q^2(z)\tau(z)\tau^*(z) = \bar{\gamma}_0^*z^n + \bar{\gamma}_1^*z^{n-1} + \cdots + \bar{\gamma}_{2s}^*z^{n-2s},$$

whence we find finally

$$\phi^*(z) = \gamma_0^* + \gamma_1^*z + \cdots + \gamma_{2s}^*z^{2s} = ci^2(z)\tau(z)\tau^*(z),$$

where

$$i(z) = q^*(z) = z^\sigma\bar{q}(1/z).$$

Thus we have found the polynomial $\phi^*(z)$ satisfying all the conditions of our lemma.

## BIBLIOGRAPHY

1. S. Kakeya, *Maximum modulus of some expressions of limited analytic functions*, Transactions of this Society, vol. 22 (1921), pp. 489–504.

2. J. Geronimus, *On a problem of F. Riesz and on the generalized problem of Tchebycheff-Korkine-Zolotareff*, Bulletin de l'Académie des Sciences de l'URSS., vol. 3 (1939), pp. 279–288.

TECHNOLOGICAL INSTITUTE,
KHARKOW, U.S.S.R.

# NOTE ON CERTAIN ORTHOGONAL POLYNOMIALS[1]

DUNHAM JACKSON

1. **Introduction.** A well known theorem[2] states that if $K_n(x, t) = \sum_0^n p_k(x)p_k(t)$ is the kernel associated with a system of orthonormal polynomials on an interval $(a, b)$ with weight function $\rho(t)$, and if $x_0$ is a real number not belonging to the open interval $(a, b)$, the function $K_n(x, t)$ is orthogonal to every polynomial of lower degree with respect to $|t-x_0|\rho(t)$ as weight function. This result is to be extended below to orthogonal trigonometric sums, and more generally to other orthogonal polynomials in two real variables on an algebraic curve.[3] The fact that a single polynomial of the $n$th degree in the original formulation is replaced by two or more sums or polynomials of like degree in the generalized orthogonal systems imparts to the extension some features of novelty. Certain other generalizations are briefly mentioned also.

2. **Trigonometric sums.** Let $U_0(x)$, $U_1(x)$, $V_1(x)$, $\cdots$ be orthonormal trigonometric sums for weight $\rho(x)$, and let

$$K_n(x, s) = U_0(x)U_0(s) + U_1(x)U_1(s) + \cdots + U_n(x)U_n(s) + V_1(x)V_1(s) + \cdots + V_n(x)V_n(s).$$

For definiteness it may be assumed that $U_k(x)$ contains no term in $\sin kx$, while $V_k(x)$ contains $\sin kx$ with a nonvanishing coefficient; the function $K_n(x, s)$ would be unchanged if $U_k$, $V_k$ were replaced by any equivalent pair of sums of the $k$th order. If $T_n(x)$ is any trigonometric sum of the $n$th order,

$$\int_{-\pi}^{\pi} \rho(s)K_n(x, s)T_n(s)ds = T_n(x).$$

*Suppose* $\rho(x) \equiv 0$ *throughout an interval* $(\alpha, \beta)$. Let $x_1$, $x_2$ be any two distinct points of $(\alpha, \beta)$. Let $\tau_{n-1}(x)$ be an arbitrary trigonometric sum of order $n-1$ at most, and let $\sin \frac{1}{2}(x-x_1) \sin \frac{1}{2}(x-x_2)\tau_{n-1}(x) = T_n(x)$; the product

$$\sin \tfrac{1}{2}(x - x_1) \sin \tfrac{1}{2}(x - x_2) = \tfrac{1}{2} \cos \tfrac{1}{2}(x_1 - x_2) - \tfrac{1}{2} \cos [x - \tfrac{1}{2}(x_1 + x_2)]$$

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[1] Presented to the Society, December 29, 1939.

[2] See, e.g., G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, New York, 1939, p. 39.

[3] See D. Jackson, *Orthogonal polynomials on plane curves*, Duke Mathematical Journal, vol. 3 (1937), pp. 228–236.

is a trigonometric sum of the first order. Then

$$
(1)\qquad \int_{-\pi}^{\pi} \rho(s)K_n(x, s) \sin \tfrac{1}{2}(s - x_1) \sin \tfrac{1}{2}(s - x_2)\tau_{n-1}(s)ds
= \sin \tfrac{1}{2}(x - x_1) \sin \tfrac{1}{2}(x - x_2)\tau_{n-1}(x).
$$

The right-hand member vanishes for $x=x_1$ and for $x=x_2$; that is to say, $K_n(x_1, s)$ *and* $K_n(x_2, s)$ *as trigonometric sums of the nth order in* $s$ *are orthogonal to every sum* $\tau_{n-1}(s)$ *of lower order, with respect to* $\sin \frac{1}{2}(s-x_1) \sin \frac{1}{2}(s-x_2)\rho(s)$ *as weight function.* This weight function is of constant sign, since $\rho(s)$ vanishes throughout the interval $(x_1, x_2)$.

*The sums* $K_n(x_1, s)$ *and* $K_n(x_2, s)$ *are linearly independent.* An identity $K_n(x_2, s)=CK_n(x_1, s)$ would require in particular that $U_0(x_2) =CU_0(x_1)$, $U_1(x_2)=CU_1(x_1)$, $V_1(x_2)=CV_1(x_1)$, that is, that $C=1$, $\cos x_2=\cos x_1$, $\sin x_2=\sin x_1$.

In the limiting case $x_2=x_1$,

$$
\int_{-\pi}^{\pi} \rho(s)K_n(x, s) \sin^2 \tfrac{1}{2}(s - x_1)\tau_{n-1}(s)ds = \sin^2 \tfrac{1}{2}(x - x_1)\tau_{n-1}(x)
$$

and by differentiation with respect to $x$ under the sign of integration

$$
\int_{-\pi}^{\pi} \rho(s)K_{nx}(x, s) \sin^2 \tfrac{1}{2}(s - x_1)\tau_{n-1}(s)ds
= \sin \tfrac{1}{2}(x - x_1) \cos \tfrac{1}{2}(x - x_1)\tau_{n-1}(x) + \sin^2 \tfrac{1}{2}(x - x_1)\tau'_{n-1}(x),
$$

where $K_{nx}(x, s)$ means $(\partial/\partial x)K_n(x, s)$. Both right-hand members vanish for $x=x_1$; $K_n(x_1, s)$ and $K_{nx}(x_1, s)$ as trigonometric sums of the $n$th order in $s$ are orthogonal to every sum of lower order for weight $\sin^2 \frac{1}{2}(s-x_1)\rho(s)$. Here no assumption with regard to vanishing of $\rho$ is required. It is obvious that $K_n$ and $K_{nx}$ are linearly independent, for one contains a constant term and the other does not.

Vanishing of $\rho$ is again unnecessary if $x_1$ and $x_2$ are allowed to be complex. Let $x_1=\gamma+i\delta$, $x_2=\gamma-i\delta$, with $\gamma$ and $\delta$ real,

$$
\begin{aligned}
\sin \tfrac{1}{2}(x - x_1) \sin \tfrac{1}{2}(x - x_2) &= \tfrac{1}{2} \cos i\delta - \tfrac{1}{2} \cos (x - \gamma) \\
&= \tfrac{1}{2} \cosh \delta - \tfrac{1}{2} \cos (x - \gamma).
\end{aligned}
$$

Since each member of (1) represents a trigonometric sum in $x$, and they are identical for real values of $x$, they are equal also when $x$ is complex. For $x=x_1$ the right-hand member is zero, and the real and pure imaginary parts of the expression on the left must vanish separately. The same result is obtained by setting $x=x_2$. The real and pure imaginary parts of $K_n(x_1, s)$ for real $s$ (or of its conjugate $K_n(x_2, s)$) are orthogonal to every trigonometric sum of lower order in $s$ with respect to the non-negative weight function $[\frac{1}{2}\cosh\delta-\frac{1}{2}\cos(s-\gamma)]\rho(s)$. These real and pure imaginary components, being linearly independent combinations of the linearly independent functions $K_n(x_1, s)$ and $K_n(x_2, s)$, are themselves linearly independent trigonometric sums of the $n$th order in $s$.

3. **Polynomials on an algebraic curve.** Orthogonal trigonometric sums can be regarded as a special instance of orthogonal polynomials in two real variables on an algebraic curve, the curve in this case being a circle (or, without essentially greater generality, an ellipse). Considerations analogous to those outlined above apply to the more general problem. As an illustration the reasoning will be formulated for a curve of the third degree. It will be immediately apparent that the same type of argument can be applied to curves of higher degree, if exceptional cases are avoided, or, on the other hand, to an alternative treatment of orthogonal polynomials on a circle, or to orthogonal polynomials on hyperbolic arcs. (Orthogonal polynomials on a parabola are essentially orthogonal polynomials in a single variable.)

Let $C$ be a curve of the third degree. Let $\rho(x, y)$ be a non-negative weight function defined on $C$, and, for simplicity, vanishing identically except on an arc or arcs of finite extent. Let $p_{01}(x, y)$, $p_{11}(x, y)$, $p_{12}(x, y)$, and, for $n\geq 2$, $p_{n1}(x, y)$, $p_{n2}(x, y)$, $p_{n3}(x, y)$ denote the members of a corresponding system of orthonormal polynomials on $C$, each polynomial being of total degree indicated by its first subscript. Let

$$K_n(x, y, u, v) = \sum_{k\leq n} p_{kj}(x, y)p_{kj}(u, v),$$

the summation being extended over all the relevant pairs of subscripts. As in the special case of trigonometric sums, the function $K_n$ is completely determinate when the points $(x, y)$ and $(u, v)$ are on the curve $C$ in their respective planes, although the individual polynomials of any specified degree in the system admit orthogonal transformation among themselves.

Let $ax+by+c=0$ be the equation of a line intersecting $C$ in three distinct points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, and so situated that the domain of orthogonality of the polynomials is all on one side of it, that is, so that $ax+by+c$ is of constant sign where $\rho\neq 0$. The intersections may be real, or two of them may be conjugate complex; it is understood that the coefficients $a$, $b$, $c$, as well as those in the equation of $C$, are real. It is to be understood further, if $C$ is degenerate, that the domain of orthogonality includes a portion of positive measure (with respect to arc length as variable of integration) on each of the loci defined by the vanishing of an irreducible factor in the equation of $C$.

If $P_n(x, y)$ is any polynomial of the $n$th or lower degree, then for $(x, y)$ on $C$

$$\int_C \rho(u, v)K_n(x, y, u, v)P_n(u, v)ds = P_n(x, y),$$

where $s$ denotes arc length on the curve $C$ in the $(u, v)$ plane. While the identity holds in general only on the curve, it is valid at all points of the curve, not merely where $\rho \neq 0$. If $\pi_{n-1}(x, y)$ is an arbitrary polynomial of degree $n-1$ at most, and $P_n(x, y) = (ax+by+c)\pi_{n-1}(x, y)$, the identity states that

$$\begin{aligned}(2) \qquad \int_C \rho(u, v)K_n(x, y, u, v)(au + bv + c)\pi_{n-1}(u, v)ds& \\ = (ax + by + c)\pi_{n-1}(x, y).&\end{aligned}$$

Since the points $(x_\nu, y_\nu)$, $\nu = 1, 2, 3$, lie on the curve and at the same time on the line $ax+by+c=0$, the relation holds with the right-hand member equal to zero if any one of these pairs of values is substituted for $(x, y)$. The functions $K_n(x_\nu, y_\nu, u, v)$ as polynomials in $u$ and $v$ are orthogonal to every polynomial of lower degree for weight $(au+bv+c)\rho(u, v)$.

The three polynomials $K_n(x_\nu, y_\nu, u, v)$ are linearly independent on $C$ if $n \geqq 2$. For, as the polynomials $p_{kj}(u, v)$ are linearly independent on the curve, a relation of linear dependence of the form $\sum_\nu C_\nu K_n(x_\nu, y_\nu, u, v) = 0$ would imply corresponding relations of linear dependence among the quantities $p_{kj}(x_\nu, y_\nu)$ as coefficients in the $K$'s. In particular, the first six of the equations $\sum_\nu C_\nu p_{kj}(x_\nu, y_\nu)$ would require that

$$\begin{aligned} C_1 + C_2 + C_3 &= 0, \\ C_1x_1 + C_2x_2 + C_3x_3 &= 0, \\ C_1y_1 + C_2y_2 + C_3y_3 &= 0, \\ C_1x_1^2 + C_2x_2^2 + C_3x_3^2 &= 0, \\ C_1x_1y_1 + C_2x_2y_2 + C_3x_3y_3 &= 0, \\ C_1y_1^2 + C_2y_2^2 + C_3y_3^2 &= 0. \end{aligned}$$

The first, second, and fourth of these equations are possible with $C$'s not all zero only if two of the $x$'s are equal, and the first, third, and sixth only if two of the $y$'s are equal. But this is inconsistent with the hypothesis that $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ are three distinct points on a straight line. Similarly, though more simply, any two of the polynomials $K_1(x_\nu, y_\nu, u, v)$ are linearly independent on $C$. The expressions $K_n(x_\nu, y_\nu, u, v)$ give the complete system of orthogonal polynomials corresponding to the weight function $(ax+by+c)\rho(x, y)$. If two of the points $(x_\nu, y_\nu)$ are conjugate complex, real polynomials for the orthogonal system can be obtained by taking the real and pure imaginary parts of the corresponding $K$'s.

Similar reasoning is clearly effective for a curve of the $N$th degree with a straight line meeting it in $N$ distinct points. For a treatment of certain cases of coincident points of intersection the discussion will be limited to the curve of the third degree.

Suppose now that the line $ax+by+c=0$, instead of meeting the curve in three distinct points, is tangent to it at an ordinary point $(x_1, y_1)$, and intersects it at a point $(x_2, y_2)$ distinct from $(x_1, y_1)$. It is still assumed that the part of the curve on which $\rho\neq 0$ is all on one side of the line, that is, that $ax+by+c$ is of constant sign there. At least one of the coefficients $a$, $b$ is different from zero; without loss of generality it may be supposed that $b\neq 0$, that is, that the line is not parallel to the $y$-axis. In the neighborhood of the point $(x_1, y_1)$, $y$ is a single-valued function of $x$ on the curve, with a derivative whose value reduces at $(x_1, y_1)$ to $(-a/b)$, the slope of the tangent line. In this neighborhood on $C$, by differentiation of (2),

$$\begin{aligned}\int_C \rho(u, v)K_{nx}(x, y, u, v)(au + bv + c)\pi_{n-1}(u, v)ds \\ = \frac{d}{dx}\left[(ax + by + c)\pi_{n-1}(x, y)\right] \\ = (ax + by + c)\frac{d}{dx}\pi_{n-1}(x, y) + (a + by')\pi_{n-1}(x, y),\end{aligned}$$

where $K_{nx}(x, y, u, v)$ means the total derivative of $K_n$ with respect to $x$, equal to $\partial K_n/\partial x+y'\partial K_n/\partial y$. At $(x_1, y_1)$ both $ax+by+c$ and $a+by'$ vanish. The function $K_{nx}(x_1, y_1, u, v)$, which is still a polynomial as regards its dependence on $u$ and $v$, is orthogonal to $\pi_{n-1}(u, v)$ for weight $(au+bv+c)\rho(u, v)$.

There have been obtained then for $n\geqq 2$ three polynomials $K_n(x_1, y_1, u, v)$, $K_{nx}(x_1, y_1, u, v)$, $K_n(x_2, y_2, u, v)$, each possessing the property of orthogonality. These are once more linearly independent on the curve. Without effect on the form of $K_n$, and so without any impairment of generality, it may be assumed that the first six polynomials in the orthogonal system are specifically those obtained by applying the Schmidt process to the monomials $1, x, y, x^2, xy, y^2$, taken in that order. Then $p_{11}(x, y)$ does not involve $y$, and $p_{21}(x, y)$ involves neither $xy$ nor $y^2$. Suppose there were an identity of the form

$$C_1K_n(x_1, y_1, u, v) + C_2K_{nx}(x_1, y_1, u, v) + C_3K_n(x_2, y_2, u, v) = 0$$

with $C$'s not all zero. Then

$$C_1p_{kj}(x_1, y_1) + C_2p'_{kj}(x_1, y_1) + C_3p_{kj}(x_2, y_2) = 0$$

for each pair of indices $(k, j)$ in which $k \leqq n$, the accent indicating total differentiation with respect to $x$. The equations for the first four pairs of indices may be simplified to the form

$$\begin{aligned} C_1 + C_3 &= 0, \\ C_1x_1 + C_2 + C_3x_2 &= 0, \\ C_1y_1 - C_2a/b + C_3y_2 &= 0, \\ C_1x_1^2 + 2C_2x_1 + C_3x_2^2 &= 0. \end{aligned}$$

The first, second, and fourth of these equations require that $x_1 = x_2$, which is impossible under the hypothesis that $(x_1, y_1)$ and $(x_2, y_2)$ are distinct and $b \neq 0$. It is immediately apparent that any two of the three $K$'s are linearly independent for $n = 1$. So the $K$'s give the complete orthogonal system for weight $(ax+by+c)\rho(x, y)$.

Finally, let the line meet the curve at a double point $(x_1, y_1)$ at which there are distinct tangents with finite slopes, and at another point $(x_2, y_2)$. It is assumed, as always, that $ax+by+c$ is of constant sign where $\rho$ does not vanish. Let $K_1(u, v)$ and $K_2(u, v)$ be the total derivatives $K_{nx}(x_1, y_1, u, v)$ calculated respectively for the two branches of the curve through $(x_1, y_1)$; let the corresponding slopes be $\lambda_1$ and $\lambda_2$. Then

$$\int_C \rho(u, v)K_\nu(u, v)(au + bv + c)\pi_{n-1}(u, v)ds$$
$$= (a + b\lambda_\nu)\pi_{n-1}(x_1, y_1), \qquad \nu = 1, 2.$$

It is seen that $(a+b\lambda_2)K_1(u, v) - (a+b\lambda_1)K_2(u, v)$ is orthogonal to $\pi_{n-1}(u, v)$ for the composite weight function. This polynomial and the two polynomials $K_n(x_1, y_1, u, v)$, $K_n(x_2, y_2, u, v)$ are found to be linearly independent on $C$ for $n \geqq 2$, and once more a complete orthogonal system is obtained.

These illustrations will be allowed to suffice for the case of multiple intersections.

4. **Other cases.** The corresponding theory for a non-algebraic curve is complicated by the fact that the number of polynomials of the $n$th degree in the orthogonal system increases with $n$, but is compensatingly simplified by the observation that the representation corresponding to (2) holds for *all* values of $x$ and $y$, so that (except for the assumption that the domain of orthogonality is all on one side of the line) the points where the straight line meets the curve are no longer a matter of special concern. If $(x_\nu, y_\nu)$ are any $n+1$ distinct points on the line, independent polynomials of the $n$th degree orthogonal to every polynomial of lower degree with respect to the composite weight function are given by $K_n(x_\nu, y_\nu, u, v)$.

A similar conclusion holds for orthogonality on a two-dimensional region.

THE UNIVERSITY OF MINNESOTA

## NOTE ON AN INEQUALITY OF STEINER[1]

T. RADÓ AND P. REICHELDERFER

Let $Q$ denote the unit square $0 \leqq x, y \leqq 1$. If $f(x, y)$ be any function defined and continuous on $Q$, the relation $z=f(x, y)$ yields a continuous surface defined over $Q$. The Lebesgue area[2] of this surface will be denoted by $L(f)$. Let $z=f_1(x, y)$, $z=f_2(x, y)$ be two continuous surfaces defined over $Q$; then clearly $z=[f_1(x, y)+f_2(x, y)]/2$ is a continuous surface defined over $Q$. The *inequality of Steiner*[3] states that $L([f_1+f_2]/2) \leqq [L(f_1)+L(f_2)]/2$. McShane[4] obtained interesting and important results concerning the situation where the sign of equality holds in this relation. In this note we improve his results and, in a sense, give them a final form.

In order to emphasize and to clarify what is significant and interesting in the results of McShane and in our improvements thereon, we remind our reader of a few facts concerning the Lebesgue area.[5] Given a continuous surface $z=f(x, y)$ defined over $Q$; if $L(f)$ is finite then the partial derivatives $f_x$ and $f_y$ exist almost everywhere in $Q$, the integral $\iint_Q [1+f_x^2+f_y^2]^{1/2} dxdy$ exists, and the relation

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[1] Presented to the Society, April 13, 1940.

[2] See S. Saks, *Theory of the Integral*, Warsaw and Lwów, 1937, chap. 5, for the facts used in this paper concerning the Lebesgue area.

[3] E. J. McShane, *On a certain inequality of Steiner*, Annals of Mathematics, (2), vol. 33 (1932), pp. 125–138.

[4] Loc. cit.[3]

[5] Cf.[2]

$\iint_Q[1+f_x^2+f_y^2]^{1/2}dxdy \leqq L(f)$ is valid. The sign of equality holds in this last relation if and only if $f(x, y)$ is absolutely continuous in the sense of Tonelli. Next, if a sequence of continuous functions $f_n(x, y)$ converges uniformly on $Q$ to $f(x, y)$, then it is always true that $\liminf_{n\to\infty} L(f_n) \geqq L(f)$, but the sign of equality holds in this last relation only if the sequence $f_n(x, y)$ is chosen with extreme care.

Suppose $z=f_1(x, y)$, $z=f_2(x, y)$ are two continuous surfaces defined over $Q$ such that $L([f_1+f_2]/2) = [L(f_1)+L(f_2)]/2$. If we assume, for the moment, that both $f_1(x, y)$ and $f_2(x, y)$ are absolutely continuous in the sense of Tonelli, then so is $[f_1(x, y)+f_2(x, y)]/2$, and the preceding equality is equivalent to

$$\iint_Q\left\{\frac{[1+f_{1x}^2+f_{1y}^2]^{1/2}+[1+f_{2x}^2+f_{2y}^2]^{1/2}}{2} - \left[1+\left(\frac{f_{1x}+f_{2x}}{2}\right)^2+\left(\frac{f_{1y}+f_{2y}}{2}\right)^2\right]^{1/2}\right\}dxdy = 0. \tag{1}$$

Using the inequality of Schwartz one sees that the integrand in (1) is never negative; thus the integrand is equal to zero almost everywhere in $Q$, which implies that $f_{1x}=f_{2x}$, $f_{1y}=f_{2y}$ almost everywhere in $Q$—that is, $[(f_{1x}-f_{2x})^2+(f_{1y}-f_{2y})^2]^{1/2}=0$ almost everywhere in $Q$.[6]

Now suppose we discard the assumptions that $f_1$ and $f_2$ are absolutely continuous in the sense of Tonelli; then, of course, the preceding reasoning yields no result. Still McShane proves that $[(f_{1x}-f_{2x})^2+(f_{1y}-f_{2y})^2]^{1/2}=0$ almost everywhere in $Q$. Indeed, he proves a sharper result to the effect that, when $[L(f_1)+L(f_2)]/2 - L([f_1+f_2]/2)$ is small, then $[(f_{1x}-f_{2x})^2+(f_{1y}-f_{2y})^2]^{1/2}$ is small *in measure*, so to speak—quite exactly, he establishes a contrapositive statement:[7] *If the functions $f_1(x, y)$, $f_2(x, y)$ are defined and continuous on a square $Q$ and there exist positive numbers $G$, $\epsilon$, $\delta$ such that*

(1) $L(f_1) \leqq G$, $L(f_2) \leqq G$;

(2) $[(f_{1x}-f_{2x})^2+(f_{1y}-f_{2y})^2]^{1/2} \geqq \epsilon$ *on a set of measure at least equal to* $\delta$;

*then*

$$L([f_1+f_2]/2) \leqq [L(f_1)+L(f_2)]/2 - \mu(G, \epsilon, \delta),$$

*where $\mu$ is a positive number whose value depends only on $G$, $\epsilon$, and $\delta$.*

We improve this result by replacing smallness with respect to

---

[6] From this it follows (loc. cit.,[3] Theorem I) that $f_1-f_2$ is constant on $Q$, but we shall not be concerned with this fact.

[7] Loc. cit.,[3] Theorem IV.

*measure* by smallness with respect to *exponent*. First, we show that, *if $f_1(x, y)$ and $f_2(x, y)$ are two continuous functions defined on the square $Q$ for which $L(f_1)$ and $L(f_2)$ are both finite, then*

$$\left\{\iint_Q [(f_{1x} - f_{2x})^2 + (f_{1y} - f_{2y})^2]^{1/4} dxdy\right\}^4 \leqq [L(f_1) + L(f_2)]^3\left[\frac{L(f_1) + L(f_2)}{2} - L\left(\frac{f_1 + f_2}{2}\right)\right]. \tag{2}$$

Since the Hölder inequality reveals

$$\left\{\iint_Q [(f_{1x} - f_{2x})^2 + (f_{1y} - f_{2y})^2]^{\lambda/2} dxdy\right\}^{2/\lambda}$$

to be a non-decreasing function of $\lambda$, it follows that, for every exponent $\lambda$ satisfying $0<\lambda\leqq 1/2$, it is true that

$$\left\{\iint_Q [(f_{1x} - f_{2x})^2 + (f_{1y} - f_{2y})^2]^{\lambda/2} dxdy\right\}^{2/\lambda} \leqq [L(f_1) + L(f_2)]^3\left[\frac{L(f_1) + L(f_2)}{2} - L\left(\frac{f_1 + f_2}{2}\right)\right]. \tag{3}$$

Next, we prove that, *under the same assumptions, it follows that*

$$\left\{\iint_Q [(f_{1x} - f_{2x})^2 + (f_{1y} - f_{2y})^2]^{\lambda/2} dxdy\right\}^{2/\lambda} \leqq [L(f_1) + L(f_2)]^{(1+\lambda)/\lambda}\left[\frac{L(f_1) + L(f_2)}{2} - L\left(\frac{f_1 + f_2}{2}\right)\right]^{(1-\lambda)/\lambda} \tag{4}$$

*for every exponent $\lambda$ satisfying* $1/2\leqq\lambda\leqq 1$. The reader will observe that inequality (4) reduces to (2) for $\lambda=1/2$.

Let, now, $f_n(x, y)$ be a sequence of continuous functions converging uniformly on $Q$ to $f(x, y)$; assume that $L(f_n)$ and $L(f)$ are finite, and that $L(f_n)$ converges to $L(f)$. Under these hypotheses McShane proves that $[(f_{nx}-f_x)^2+(f_{ny}-f_y)^2]^{1/2}$ converges to zero *in measure*.[8] We improve this result by showing that *under these hypotheses it is true that*

$$\lim_{n\to\infty}\iint_Q [(f_{nx} - f_x)^2 + (f_{ny} - f_y)^2]^{\lambda/2} dxdy = 0 \tag{5}$$

*for every exponent $\lambda$ satisfying* $0<\lambda<1$. Since McShane has shown that the relation (5) is not generally true for the exponent $\lambda = 1$,[9] our result is the best obtainable under the given hypotheses.

---

[8] Loc. cit.,[3] Theorem V.

In fact, we shall give two proofs for relation (5). One proof will reveal this result as an immediate consequence of inequalities (3) and (4); the other will depend upon a general lemma concerning the Lebesgue integral.

We proceed presently to the proof of inequality (2). Given two continuous functions $f_1(x, y)$, $f_2(x, y)$ defined on $Q$ for which $L(f_1)$, $L(f_2)$ are both finite; then $f_{1x}, f_{1y}; f_{2x}, f_{2y}$ exist almost everywhere in $Q$. Set

$$
\begin{aligned}
u &= [(f_{1x} - f_{2x})^2 + (f_{1y} - f_{2y})^2]^{1/2}; & v &= 1 + f_{1x}f_{2x} + f_{1y}f_{2y};\\
w_1 &= [1 + f_{1x}^2 + f_{1y}^2]^{1/2}; & w_2 &= [1 + f_{2x}^2 + f_{2y}^2]^{1/2};\\
w &= [1 + ([f_{1x} + f_{2x}]/2)^2 + ([f_{1y} + f_{2y}]/2)^2]^{1/2}; & &\\
\alpha &= (w_1 + w_2)/2 - w; & \beta &= (w_1 + w_2)/2 + w;\\
\gamma &= w_1w_2 - v; & \delta &= w_1w_2 + v.
\end{aligned}
$$

The reader will easily verify the following identities:

$$
\alpha + \beta = w_1 + w_2, \qquad \alpha \cdot \beta = \gamma/2, \tag{6}
$$

$$
\gamma + \delta = 2w_1w_2, \qquad \gamma \cdot \delta = u^2 + (f_{1x}f_{2y} - f_{2x}f_{1y})^2. \tag{7}
$$

Now (7) implies that $\gamma \geqq 0$, $\delta \geqq 0$, whence it follows that $\alpha \cdot \beta \geqq 0$, and that

$$
\delta \leqq 2w_1w_2 \leqq (w_1 + w_2)^2/2.
$$

In view of these relations (6) implies that $\alpha \geqq 0, \beta \geqq 0$; hence $\beta \leqq w_1 + w_2$. So the reader sees that

$$
u^2 \leqq \gamma\delta = 2\alpha\beta\delta \leqq (w_1 + w_2)^3\{(w_1 + w_2)/2 - w\}. \tag{8}
$$

As we observed above, it is true that

$$
\iint_Q w_i\,dxdy \leqq L(f_i), \qquad i = 1, 2. \tag{9}
$$

A reasoning of McShane[10] shows that

$$
\iint_Q \{(w_1 + w_2)/2 - w\}\,dxdy \leqq [L(f_1) + L(f_2)]/2 - L([f_1 + f_2]/2). \tag{10}
$$

Using the Hölder inequality the reader will speedily verify that inequality (2) follows from (8), (9), and (10).

---

[9] Loc. cit.,[3] Theorem VI.

[10] Loc. cit.,[3] p. 129.

We proceed next to establish inequality (4). It follows by the triangle inequality that $u \leqq w_1 + w_2$; hence we have from (9),

$$(11) \qquad \int\!\!\int_Q [(f_{1x} - f_{2x})^2 + (f_{1y} - f_{2y})^2]^{1/2} dxdy \leqq L(f_1) + L(f_2).$$

Suppose that $\lambda_2$ is a positive number and $F(x, y)$ is a non-negative function such that $F^{\lambda_2}$ is summable on $Q$. If $\lambda_1$ be any positive number less than $\lambda_2$, it follows by the Hölder inequality that $F^{\lambda_1}$ is also summable on $Q$. Now every number $\lambda$ satisfying $\lambda_1 \leqq \lambda \leqq \lambda_2$ may be written in the form $\lambda = \alpha_1\lambda_1 + \alpha_2\lambda_2$, where $0 \leqq \alpha_1, \alpha_2 \leqq 1$, $\alpha_1 + \alpha_2 = 1$. From the Hölder inequality we have

$$(12) \qquad \int\!\!\int_Q F^{\lambda} dxdy \leqq \left[\int\!\!\int_Q F^{\lambda_1} dxdy\right]^{\alpha_1} \left[\int\!\!\int_Q F^{\lambda_2} dxdy\right]^{\alpha_2}.$$

The reader will verify that (4) follows from (2), (11), and (12) with $F = u$, $\lambda_1 = 1/2$, $\lambda_2 = 1$.

Now let $f_n(x, y)$ be a sequence of continuous functions converging uniformly on $Q$ to $f(x, y)$; assume that $L(f_n)$ and $L(f)$ are finite, and that $L(f_n)$ converges to $L(f)$. We first show that

$$(13) \qquad \lim_{n \to \infty} \left[\frac{L(f_n) + L(f)}{2} - L\left(\frac{f_n + f}{2}\right)\right] = 0.$$

Since $(f_n + f)/2$ converges uniformly to $f$ on $Q$, it follows that

$$(14) \qquad \liminf_{n \to \infty} L\left(\frac{f_n + f}{2}\right) \geqq L(f).$$

But $\lim_{n \to \infty} [L(f_n) + L(f)]/2 = L(f)$, and by the inequality of Steiner it follows that $L([f_n + f]/2) \leqq [L(f_n) + L(f)]/2$. Thus

$$(15) \qquad \limsup_{n \to \infty} L\left(\frac{f_n + f}{2}\right) \leqq L(f).$$

From (14) and (15), the assertion in (13) follows.

Now the inequality (3) shows that, for $0 < \lambda \leqq 1/2$, it is true that

$$(16) \qquad \begin{aligned} &\int\!\!\int_Q [(f_{nx} - f_x)^2 + (f_{ny} - f_y)^2]^{\lambda/2} dxdy \\ &\qquad \leqq [L(f_n) + L(f)]^{3\lambda/2} \left[\frac{L(f_n) + L(f)}{2} - L\left(\frac{f_n + f}{2}\right)\right]^{\lambda/2}, \end{aligned}$$

while inequality (4) shows that, for $1/2 \leqq \lambda \leqq 1$, it is true that

$$\begin{aligned}(17)\qquad &\iint_Q [(f_{nx}-f_x)^2+(f_{ny}-f_y)^2]^{\lambda/2}dxdy\\ &\leqq [L(f_n)+L(f)]^{(1+\lambda)/2}\left[\frac{L(f_n)+L(f)}{2}-L\left(\frac{f_n+f}{2}\right)\right]^{(1-\lambda)/2}.\end{aligned}$$

From (13), (16), (17) the relation (5) clearly follows.

In order to give a second proof of relation (5) we use the following lemma.[11] *Let $F_n(x, y)$ be a sequence of non-negative functions defined on $Q$ for which there exists a positive number $\lambda_0$ such that* (1) *the functions $F_n$ converge to zero in measure*; (2) *the functions $F_n^{\lambda_0}$ are summable on $Q$*; (3) *the integrals $\iint_Q F_n^{\lambda_0}dxdy$ are uniformly bounded by a positive number $M$. Then, for every positive number $\lambda$ less than $\lambda_0$, it is true that* $\lim_{n\to\infty}\iint_Q F_n^{\lambda}dxdy=0$.

Proof. Given $\epsilon>0$, since $F_n$ converges to zero in measure there exists a positive integer $n(\epsilon)$ such that, for every $n>n(\epsilon)$ the set $E_n(\epsilon)$ of points $(x, y)$ in $Q$ for which $F_n(x, y)\geqq\epsilon^{1/\lambda}$ is of measure less than $\epsilon^{\lambda_0/(\lambda_0-\lambda)}\cdot M^{-\lambda/(\lambda_0-\lambda)}$. Now obviously

$$(18)\qquad \iint_Q F_n^{\lambda}dxdy=\iint_{Q-E_n(\epsilon)}F_n^{\lambda}dxdy+\iint_{E_n(\epsilon)}F_n^{\lambda}dxdy.$$

On $Q-E_n(\epsilon)$ it is clear that $F_n^{\lambda}(x, y)\leqq\epsilon$; hence

$$(19)\qquad \iint_{Q-E_n(\epsilon)}F_n^{\lambda}dxdy\leqq\epsilon.$$

Using the Hölder inequality, we have, for $n>n(\epsilon)$,

$$\begin{aligned}(20)\qquad &\iint_{E_n(\epsilon)}F_n^{\lambda}dxdy\\ &\leqq\left[\iint_{E_n(\epsilon)}F_n^{\lambda_0}dxdy\right]^{\lambda/\lambda_0}\left[\iint_{E_n(\epsilon)}dxdy\right]^{(\lambda_0-\lambda)/\lambda_0}<\epsilon.\end{aligned}$$

Since $\epsilon$ is arbitrary, the lemma follows immediately from (18), (19), (20).

Now let $f_n(x, y)$ be a sequence of continuous functions converging uniformly on $Q$ to a function $f(x, y)$; assume that $L(f_n)$ and $L(f)$ are finite, and that $\lim_{n\to\infty}L(f_n)=L(f)$. Consider the sequence of functions $F_n(x, y)=[(f_{nx}-f_x)^2+(f_{ny}-f_y)^2]^{1/2}$. In view of the results of McShane stated above, it is clear that this sequence of functions satisfies the three conditions of the preceding lemma with $\lambda_0=1$. Thus, for every exponent $\lambda$ satisfying $0<\lambda<1$ it is true that

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[11] This lemma is an immediate corollary of certain important results of F. Riesz, *Untersuchungen über Systeme integrierbarer Funktionen*, Mathematische Annalen, vol. 69 (1910), pp. 449–497. We give a direct proof for the convenience of the reader.

$$\lim_{n\to\infty}\iint_Q [(f_{nx}-f_x)^2+(f_{ny}-f_y)^2]^{\lambda/2}dxdy = 0.$$

So we have a second proof of our result (5).

THE OHIO STATE UNIVERSITY

# AN ADDITIONAL CRITERION FOR THE FIRST CASE OF FERMAT'S LAST THEOREM[1]

BARKLEY ROSSER

In an earlier paper[2] it was shown that if $p$ is an odd prime and

$$a^p + b^p + c^p = 0$$

has a solution in integers prime to $p$, then

$$m^{p-1} \equiv 1 \pmod{p^2}$$

for each prime $m \leqq 41$. In this paper the result is extended to $m \leqq 43$.

We will use the notations and conventions of I throughout, and a reference to a numbered equation will refer to the equation of that number in I. With $p$ assumed to be an odd prime such that (1) has a solution in integers prime to $p$, we assume that a $t$ exists such that the values of (2) satisfy (4), (5), and (6) with $m=43$. Put $g(x)=f(x)f(-x)$ and

$$h(x) = (x^{42} - 1)/(x^6 - 1).$$

Then $g(x)$ divides $h(x)$, and $g(x)$ can be completely factored modulo $p$.

*Case* 1. Assume that a root of $g(x)$ is a root of

$$h(x)/(x^{12} + x^{10} + x^8 + x^6 + x^4 + x^2 + 1).$$

Then this root belongs to either the exponent 21 or the exponent 42 modulo $p$. Hence $p \equiv 1 \pmod{42}$. So there is an $\omega$ such that

$$\omega^2 + \omega + 1 \equiv 0.$$

Then $g(x)$, $g(\omega x)$, and $g(\omega^2 x)$ all divide $h(x)$. Moreover, the only cases in which two of $g(x)$, $g(\omega x)$, and $g(\omega^2 x)$ have a common factor are

I. $a^6+1\equiv 0,$

II. $a^6+a^3+3a^2+3a+1\equiv 0,$

III. $a^6-a^3-3a^2-3a-1\equiv 0,$

or cases derived from these by replacing $a$ by one of the other roots of $f(x)$. So if we show that $h(x)$ has no factor in common with any of $x^6+1$, $x^6+x^3+3x^2+3x+1$, or $x^6-x^3-3x^2-3x-1$, then we can conclude that $g(x)g(\omega x)g(\omega^2 x)$ must divide $h(x)$.

Clearly $h(x)$ has no factor in common with $x^6+1$.

---

[1] Presented to the Society, April 27, 1940.

[2] *A new lower bound for the exponent in the first case of Fermat's last theorem*, this Bulletin, vol. 46 (1940), pp. 299–304. This paper will be referred to as I.

Suppose $h(x)$ has a factor in common with $x^6+x^3+3x^2+3x+1$. This latter has the factors $x^2+x+1$ and $x^4-x^3+2x+1$. The first has no factor in common with $h(x)$, since it divides $x^6-1$, which has no factor in common with $h(x)$. To test the second, we try it successively with each of the four factors of $h(x)$, getting the eliminants

$$13\cdot 19^2\cdot 127\cdot 163^2,\quad 5\cdot 36913,\quad 2\cdot 127,\quad 5\cdot 7.$$

Suppose $h(x)$ has a factor in common with $x^6-x^3-3x^2-3x-1$. This latter has the factors $x^2-x-1$ and $x^4+x^3+2x^2+2x+1$. The first has no factor in common with $h(x)$ by Lemma 3 of I. Trying the second factor successively with each of the four factors of $h(x)$, we get the eliminants

$$7^3\cdot 43,\quad 2^2\cdot 7\cdot 13\cdot 43,\quad 7,\quad 43.$$

So $g(x)g(\omega x)g(\omega^2 x)$ must divide $h(x)$. Since both are of degree 36, they must be equal. Putting $b=c+5$ and equating coefficients, we get

$$A+1 = 2c^3+3c^2-24c+13 \equiv 1,$$
$$B+1 = c^6+12c^5+42c^4+18c^3-9c^2-222c+173 \equiv 1,$$
$$C+1 = -2c^6+12c^5+171c^4+132c^3-666c^2+132c+201 \equiv 1.$$

Dividing $16B$ and $8C$ by $A$, we get the remainder

$$43D = 43(99c^2+192c-116) \equiv 0$$

from each. Then

$$2cE = 29A+3D = 2c(29c^2+192c-60) \equiv 0.$$

As $c\equiv 0$ would give $A\equiv 12\equiv 0$, we have

$$28cF = 15D-29E = 28c(23c-96) \equiv 0,$$
$$29cG = 8E-5F = 29c(8c-49) \equiv 0,$$
$$8F-23G = 359 \equiv 0.$$

*Case* 2. Assume that no root of $g(x)$ is a root of

$$h(x)/(x^{12}+x^{10}+x^8+x^6+x^4+x^2+1).$$

Then, since $g(x)$ divides $h(x)$ and is of degree 12,

$$g(x) \equiv x^{12}+x^{10}+x^8+x^6+x^4+x^2+1.$$

So $2c+1\equiv 1$ and $c^2+5\equiv 1$.

PRINCETON UNIVERSITY

# SOME NEW PROPERTIES OF TRANSFINITE ORDINALS[1]

SEYMOUR SHERMAN

1. **Introduction.** Sierpiński[2] has stated the following theorem on the Cantor normal form:

*Every ordinal* $A > 0$ *can be represented in the form*

$$A = \sum_{i=0}^{n-1} \omega^{\alpha_i} a_i,$$

*where*

$$\omega > n,\ a_0,\ a_1,\ a_2,\ \cdots,\ a_{n-1} > 0,$$
$$A \geqq \alpha_0 > \alpha_1 > \alpha_2 > \cdots > \alpha_{n-1} \geqq 0.$$

Henceforth in this paper, when an ordinal is written in summation form, it will be assumed that the summation satisfies the above-stated requirements.

Elsewhere Sierpiński[3] has proved:

*The set of all "divisors on the left" of an ordinal number is closed.*[4]

One of the purposes of this paper is to analyze further, by means of the Cantor normal form, the left and right factors of transfinite ordinals. In addition the Cantor normal form will be used to prove that for ordinals $A$, $B$, and $C$

$$(A + B)C \leqq AC + BC.$$

We can also specify the necessary and sufficient condition that the equality hold. F. Siecza[5] has used the Cantor normal form to discuss factorization of transfinite ordinals, but his problem was different from the one considered here.

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[1] Presented to the Society, December 27, 1939. The author is indebted to Professor W. A. Hurwitz for his indispensable advice in the preparation of this paper.

[2] W. Sierpiński, *Leçons sur les Nombres Transfinis*, Paris, 1928, p. 202; hereafter referred to as Sierpiński (I). The theorem was first stated for ordinals of the first and second ordinal class by G. Cantor, *Beiträge zur Begründung der transfiniten Mengenlehre*, Mathematische Annalen, vol. 49 (1897), p. 237, and for all ordinals by G. Hessenberg, *Grundbegriffe der Mengenlehre*, Abhandlungen der Fries'schen Schule, New Series 1.4, Göttingen, 1906, p. 587.

[3] W. Sierpiński, *A property of ordinal numbers*, Bulletin of the Calcutta Mathematical Society, vol. 20 (1930), pp. 21–22; hereafter referred to as Sierpiński (II).

[4] A set of ordinal numbers is said to be closed if it contains the least upper bound of each subset.

[5] F. Siecza, *Sur l'unicité de la décomposition de nombres ordinaux en facteurs irreductibles*, Fundamenta Mathematicae, vol. 5 (1924), pp. 172–175.

## 2. Left and right factors.

We shall now consider left factors.

2.1. THEOREM. *A necessary and sufficient condition that $x$ be a left factor of $A=\sum_{i=0}^{n-1}\omega^{\alpha_i}a_i$ is that either $0<x<\omega^{\alpha_{n-1}}$ or $x=\omega^{\alpha_j}p_j+\sum_{i=j+1}^{n-1}\omega^{\alpha_i}a_i$, where $p_j$ is a factor of $a_j$ and $0\leqq j\leqq n-1$.*

If $0<x<\omega^{\alpha_{n-1}}$, let $x=\sum_{i=0}^{m-1}\omega^{\beta_i}b_i$. Thus $\beta_0<\alpha_{n-1}$ and $-\beta_0+\alpha_j$, $0\leqq j\leqq n-1$, exists.[6] We have

$$\left\{\sum_{i=0}^{m-1}\omega^{\beta_i}b_i\right\}\left\{\sum_{i=0}^{n-1}\omega^{-\beta_0+\alpha_i}\right\}=\sum_{i=0}^{n-1}\omega^{\alpha_i}a_i,$$

and hence $x$ is a left factor of $A$.

If $x=\omega^{\alpha_j}p_j+\sum_{i=j+1}^{n-1}\omega^{\alpha_i}a_i$, where $a_j=p_j\lambda_j$, then

$$\left\{\omega^{\alpha_j}p_j+\sum_{i=j+1}^{n-1}\omega^{\alpha_i}a_i\right\}\left\{\sum_{i=0}^{j-1}(\omega^{-\alpha_j+\alpha_i})a_i+\lambda_j\right\}$$
$$=\sum_{i=0}^{j-1}(\omega^{\alpha_j-\alpha_j+\alpha_i})a_i+\omega^{\alpha_j}p_j\lambda_j+\sum_{i=j+1}^{n-1}\omega^{\alpha_i}a_i$$
$$=\sum_{i=0}^{j-1}\omega^{\alpha_i}a_i+\omega^{\alpha_j}a_j+\sum_{i=j+1}^{n-1}\omega^{\alpha_i}a_i=\sum_{i=0}^{n-1}\omega^{\alpha_i}a_i$$

and $x$ is a left factor of $A$.

Thus we have proved the sufficiency of the condition.

Suppose that $x=\sum_{i=0}^{m-1}\omega^{\beta_i}b_i$ is a left factor of $A$ and that $x\geqq\omega^{\alpha_{n-1}}$. Then

$$A=\left(\sum_{i=0}^{m-1}\omega^{\beta_i}b_i\right)\left(\sum_{i=0}^{l-1}\omega^{\gamma_i}c_i\right).$$

If $\gamma_{l-1}=0$ then

$$\sum_{i=0}^{n-1}\omega^{\alpha_i}a_i=\left(\sum_{i=0}^{m-1}\omega^{\beta_i}b_i\right)\left(\sum_{i=0}^{l-1}\omega^{\gamma_i}c_i\right)$$
$$=\sum_{i=0}^{l-2}(\omega^{\beta_0+\gamma_i})c_i+\omega^{\beta_0}b_0c_{l-1}+\sum_{i=1}^{m-1}\omega^{\beta_i}b_i,$$

where the last expression is a Cantor normal form. Comparing coefficients in the right and left ends of the series of equations, we see that for some $j$, $0 \leqq j \leqq n-1$, $\beta_0 = \alpha_j$, $b_0$ is a factor of $a_j$, and $\sum_{i=1}^{m-1}\omega^{\beta_i}b_i = \sum_{i=j+1}^{n-1}\omega^{\alpha_i}a_i$.

---

[6] See F. Hausdorff, *Mengenlehre*, 1935, p. 63.

If $\gamma_{l-1} > 0$, then

$$\sum_{i=0}^{n-1} \omega^{\alpha_i}a_i = \left(\sum_{i=0}^{m-1} \omega^{\beta_i}b_i\right)\left(\sum_{i=0}^{l-1} \omega^{\gamma_i}c_i\right) = \sum_{i=0}^{l-1} (\omega^{\beta_0+\gamma_i})c_i.$$

But since $x \geqq \omega^{\alpha_{n-1}}$, we have $\beta_0 \geqq \alpha_{n-1}$, $\beta_0 + \gamma_{l-1} > \alpha_{n-1}$, and thus the left and right ends of series of equations could not be equal. From this contradiction $\gamma_{l-1} = 0$ and by the preceding argument our condition is necessary.

Since in $\sum_{i=0}^{n-1}\omega^{\alpha_i}a_i$ there is only a finite set of coefficients and since each of these coefficients, being finite, has only a finite set of factors, there is but a finite set of left factors of the form $\omega^{\alpha_j}p_j + \sum_{i=j+1}^{n-1}\omega^{\alpha_i}a_i$, $p_j$ a factor of $a_j$; that is, there is only a finite set of left factors greater than or equal to $\omega^{\alpha_{n-1}}$. Hence we have proved:

2.2. Corollary. *The set of left factors of* $A = \sum_{i=0}^{n-1}\omega^{\alpha} a_i$ *consists of the set of all positive ordinals less than* $\omega^{\alpha_{n-1}}$ *and a finite set of ordinals not less than* $\omega^{\alpha_{n-1}}$.

2.3. Corollary. *If* $A = \sum_{i=0}^{n-1}\omega^{\alpha_i}a_i$, *the left factors of* $A$, *in order of magnitude, form a normally ordered set whose ordinal number is* $-1 + \omega^{\alpha_{n-1}} + \sum_{i=0}^{n-1}\tau(a_i)$, *where* $\tau(a_i)$ *is the number of factors of* $a_i$.

2.31. Corollary. *The ordinal number of the normally ordered set of left factors of a positive ordinal is either a finite ordinal or the sum of an additively indecomposable*[7] *ordinal and a finite ordinal.*

We can now obtain the previously stated result of Sierpiński,[8] who makes no use of the Cantor normal form:

2.4. Corollary. *The left factors of A form a closed set.*

The following corollary has been proved by Hausdorff[9] and Bennett[10] by means of the euclidean algorithm.

2.5. Corollary. *Any two ordinals greater than* 0 *have a greatest common left factor.*

We shall prove this by means of the preceding corollary. Let $A > 0$

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[7] An ordinal $\alpha > 0$ is said to be additively indecomposable when there do not exist ordinals $\beta$, $\gamma$ such that $\beta < \alpha$, $\gamma < \alpha$, and $\alpha = \beta + \gamma$. See Sierpiński (I), p. 178.

[8] Sierpiński (II).

[9] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914.

[10] A. A. Bennett, *Some arithmetic operations with transfinite ordinals*, American Mathematical Monthly, vol. 28 (1921), pp. 427–430.

have a set of left factors, $\mathfrak{A}$, and $B>0$ have a set of left factors $\mathfrak{B}$. $\mathfrak{A}\mathfrak{B}$, the intersection of $\mathfrak{A}$ and $\mathfrak{B}$, has at least one element, 1. The least upper bound, $\mathfrak{c}$, of $\mathfrak{A}\mathfrak{B}$ is a member of $\mathfrak{A}$ and a member of $\mathfrak{B}$, since both $\mathfrak{A}$ and $\mathfrak{B}$ are closed. Hence $\mathfrak{c}$ is a member of $\mathfrak{A}\mathfrak{B}$; that is, $\mathfrak{A}\mathfrak{B}$ has a maximum element. Thus $\mathfrak{c}$ is the greatest common left factor of $A$ and $B$.

The development in case of right factors is somewhat analogous. The following can easily be seen to be true.

2.6. THEOREM. *If* $A=\sum_{i=0}^{n-1}\omega^{\alpha_i}a_i$, *then a necessary and sufficient condition that* $x$ *be a right factor of* $A$ *is that either* $x=\sum_{i=0}^{n-1}(\omega^{-\beta_0+\alpha_i})a_i$, *where* $\beta_0<\alpha_{n-1}$, *or* $x=\sum_{i=0}^{j-1}(\omega^{-\alpha_j+\alpha_i})a_i+r_j$, *where* $r_j$ *is a factor of* $a_j$.

Since each exponent of the Cantor normal form has a finite set of residues and each coefficient has only a finite set of factors, we can state the following:[11]

2.7. THEOREM. *The set of all right factors of any positive ordinal is finite.*

Sierpiński has proved some of these theorems by using merely the properties of additively indecomposable ordinals. The use of the Cantor normal form not only yields more extensive results in the case of left factors, but also provides a unified approach to other problems in the theory of transfinite ordinals.

3. **The distributive laws.** Among transfinite ordinals the distributive law is always valid for one order of multiplication

$$C(A+B)=CA+CB,$$

but not always for the other order of multiplication

$$(A+B)C \qquad ? \qquad AC+BC.$$

We shall prove:

3.1. THEOREM. $(A+B)C\leqq AC+BC$.

If either $A$, $B$, or $C$ is zero, then our theorem holds. Consider $A>0$, $B>0$, and $C>0$, where

$$A=\sum_{i=0}^{n-1}\omega^{\alpha_i}a_i=\omega^{\alpha_0}a_0+A', \quad B=\sum_{i=0}^{m-1}\omega^{\beta_i}b_i=\omega^{\beta_0}b_0+B',$$

$$C=\sum_{i=0}^{l-1}\omega^{\gamma_i}c_i.$$

---

[11] Sierpiński (I).

$C$ is either finite, transfinite and limiting, or transfinite and non-limiting.

Suppose $C$ is finite and $0 < C < \omega$. If $\alpha_0 < \beta_0$, then

$$A + B = A + \omega^{\beta_0}b_0 + B' = B, \qquad (A + B)C = BC = AC + BC.$$

If $\alpha_0 = \beta_0$, then

$$A + B = \omega^{\beta_0}(a_0 + b_0) + B'$$

and

$$\begin{aligned} AC + BC &= \omega^{\alpha_0}a_0C + A' + \omega^{\beta_0}b_0C + B' \\ &= \omega^{\alpha_0}(a_0 + b_0)C + B' = (A + B)C. \end{aligned}$$

If $\alpha_0 > \beta_0$, then

$$A + B = \omega^{\alpha_0}a_0 + A' + \omega^{\beta_0}b_0 + B',$$

and

$$(A + B)C = \omega^{\alpha_0}a_0c + A' + \omega^{\beta_0}b_0 + B' = AC + B \leqq AC + BC.$$

Thus if $0 < C < \omega$, then

$$(A + B)C \leqq AC + BC.$$

Suppose $C$ is transfinite and limiting, then $\gamma_{l-1} > 0$. Let $J = \max(A, B)$; then

$$(A + B)C \leqq (J + J)C = (J2)C = J(2C) = JC \leqq AC + BC.$$

Suppose $C$ is transfinite and non-limiting, then $\gamma_{l-1} = 0$, and if we take

$$C_0 = \sum_{i=0}^{l-2} \omega^{\gamma_i}c_i, \qquad C_1 = C_{l-1},$$

then

$$C_0 + C_1 = C.$$

If $\alpha_0 \geqq \beta_0$, then

$$\begin{aligned} (A + B)C &= (A + B)(C_0 + C_1) = (A + B)C_0 + (A + B)C_1 \\ &\leqq AC_0 + AC_1 + BC_1 < AC + BC. \end{aligned}$$

If $\alpha_0 < \beta_0$, then

$$(A + B)C \leqq BC \leqq AC + BC.$$

Thus if $\gamma_{l-1} = 0$, then

$$(A + B)C \leqq AC + BC,$$

and our theorem is proved.

We can also prove the following:

3.2. COROLLARY. *A necessary and sufficient condition that*

$$(A + B)C = AC + BC$$

*for positive* $A$, $B$, *and* $C$ *is that either* $C=1$, *or* $1<C<\omega$ *and* $\alpha_0 \leqq \beta_0$, *or* $\omega \leqq C$ *and* $\alpha_0+\gamma_0<\beta_0+\gamma_0$.

This corollary follows quite easily from the reasoning found in the preceding section.

CORNELL UNIVERSITY

# THE DECOMPOSITION THEOREM FOR ABELIAN GROUPS[1]

JOEL BRENNER

Let $G$ be an abelian group such that $p^k g=0$ for all $g \in G$, $p$ prime, $k$ fixed. We prove $G$ *has a basis*, that is, a set of elements such that each $g \in G$ is uniquely expressible as a linear combination of elements of the set.[2]

THEOREM. *There exists an ascending chain of sets* $B_i$, $0 \leqq i \leqq k$, *of elements of* $G$ *with the properties*:

(i) *Every element in* $B_i$ *is of order greater than* $p^{k-i}$.

(ii) *The elements in* $B_i$ *are completely linearly independent.*

(iii) *If the order of the element* $g$ *in* $G$ *is greater than* $p^{k-i}$, *then there exists a* (*unique*) *linear combination* $z$ *of elements of* $B_i$ *such that the order of* $g-z$ *is at most* $p^{k-i}$.

Since we may choose as $B_0$ the vacuous set, we may assume that the sets $B_0, \cdots, B_s$ have already been constructed in such a way as to meet the requirements (i) to (iii). In order to construct $B_{s+1}$ we adjoin to $B_s$ any greatest subset $C$ of $G$ with the following properties.

(a) All the elements in $C$ are of order $p^{k-s}$.

(b) The join $B_{s+1}$ of the sets $B_s$ and $C$ is an independent set.

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[1] Presented to the Society, April 6, 1940.

[2] Unique in that the number of nonzero terms in an expression for $g$ is unique and only the arrangement but not the respective values of the nonzero terms may differ in two expressions for $g$.

The set $B_{s+1}$ satisfies (i) and (ii). In order to prove (iii) let $h$ be any element in $G$ whose order is at least $p^{k-s}$. If firstly the order of $h$ is exactly $p^{k-s}$, then it follows from the conditions (a) and (b) that there exists an integer $m$ and a linear combination $y=\sum a_j y_j$ of elements in $B_{s+1}$ so that $0 \neq mh = y$. Assume without loss of generality that $m$ is a power of $p < p^{k-s}$. Then $(p^{k-s}/m)mh = 0 = \sum a_j(p^{k-s}/m)y_j$. By (ii), $p^{k-s}$ divides $a_j(p^{k-s}/m)$, so that $a_j/m$ is an integer (all $j$). Therefore the order of $h - \sum(a_j/m)y_j$ is $m < p^{k-s}$. If secondly the order of $h$ is greater than $p^{k-s}$, then there exists a linear combination $z$ of elements in $B_s$ such that the order of $h-z$ is at most $p^{k-s}$, and thus (as above) $y=\sum c_j y_j$ can be found with $y_j$ in $B_{s+1}$ for which $h-z-y$ has order $< p^{k-s}$. We have shown[3] that $B_k$ is a basis of $G$.

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[3] The following contain proofs for finite groups: Andreas Speiser, *Theorie der Gruppen von endlicher Ordnung*, 3d edition, Berlin, Julius Springer, 1937, p. 46–49. R. Remak, *Über die Zerlegung der kommutativen Gruppen in zyklische teilerfremde Faktoren*, Journal für die reine und angewandte Mathematik, vol. 141. L. C. Mathewson, *A simple proof of a theorem of Kronecher*, ibid., vol. 161 (1929), p. 255. A. Korselt, ibid. N. Tschebotaroew, *Bewies der Existenz einer Basis bei Abelschen Gruppen von endlicher Ordnung*, Kasan University, Fizzico-Matematico Obchestvo Izviestiia, vol. 4 (1929–1930). Third line after $\delta$) in the proof read $\omega = \omega_s r$ for $\omega = \omega_s = r$. In the fourth paragraph following, read $\tilde{\omega}_s = q\omega_s + t$, $0 \leqq t < \omega$, $A_s \cdots t \cdots$. The following contain proofs for infinite groups: H. Pruefer, *Untersuchungen ueber die Zerlegbarkeit*, Mathematische Zeitschrift, vol. 17 (1923), p. 53; R. Baer, Compositio Mathematica, vol. 1 (1934), p. 274.

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# ON THE CONVEX SOLUTION OF A CERTAIN FUNCTIONAL EQUATION[1]

H. P. THIELMAN

The object of this paper is to give a generalization of a result obtained in a recent article by Mayer.[2] His result is a particular case (for $a=1$, $p=1$, $K=0$) of the following theorem.

THEOREM 1. *The only function which is convex for all* $x>K\geqq 0$, *and satisfies the functional equation*

$$1/f(x+a) = x^p f(x), \qquad x > 0,\ a > 0,\ p > 0, \tag{1}$$

*is*

$$F(x) = \left[\frac{\Gamma(x/2a)}{(2a)^{1/2}\Gamma((x+a)/2a)}\right]^p. \tag{2}$$

The following proof follows closely that given by Mayer in the paper to which we referred above. All real functions occurring in this paper will be assumed to be defined for all positive values of the independent variable unless otherwise specified. A function $f(x)$ is said to be convex for all $x>K$ if for every pair of values of $x$, say $x_1$ and $x_2$, for which $f(x)$ is defined, and $x_1>K$, $x_2>K$,

$$f\left(\frac{x_1+x_2}{2}\right) \leqq \tfrac{1}{2}[f(x_1)+f(x_2)]. \tag{3}$$

That $f(x)$ satisfies the functional equation (1) follows from the fundamental equation for the Gamma function, $\Gamma(x+1)=x\Gamma(x)$.

We shall now show that $F(x)$ is convex. From (2) it follows[3] that

$$\frac{F'(x)}{F(x)} = -\,ap\sum_{\nu=0}^{\infty}\frac{1}{(2a\nu+x)(2a\nu+a+x)}.$$

Since $a>0$, $p>0$, it follows that $F'(x)/F(x)$ is increasing with increasing $x$. Log $F(x)$ is therefore a convex function, and therefore $F(x)$, for the convexity of a function follows from the convexity of its logarithm as can be easily verified.

Since $F'(x)<0$, $F(x)$ is a decreasing function of $x$, and we see by (1) that

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[1] Presented to the Society, April 13, 1940.

[2] A. E. Mayer, Acta Mathematica, vol. 70 (1938), pp. 57–62.

[3] Mayer, loc. cit.

$$x^{-p} > F^2(x+a) > (x+a)^{-p}. \tag{4}$$

We can now prove the main part of our theorem: if $f(x)$ is a function convex for $x > K \geqq 0$, and if it satisfies (1) for all $x > 0$, then $f(x) \equiv F(x)$. From (1) it follows that the quotient $Q(x) = f(x)/F(x)$ satisfies the functional equations

$$Q[x + (2n-1)a] = 1/Q(x), \tag{5}$$

$$Q[x + 2na] = Q(x), \tag{6}$$

where $n$ is any positive integer. We must show that $Q(x) \equiv 1$. First we suppose that $Q(x) > 0$ for some particular value of $x$. Then for that value of $x$ by (4), (5), and (6)

$$f[x + (2k-1)a] = \frac{F[x + (2k-1)a]}{Q(x)} < \frac{1}{Q(x)[x + (2k-2)a]^{p/2}}$$

$$f[x + 2ka] = Q(x)F(x + 2ka) > \frac{Q(x)}{[x + 2ka]^{p/2}}.$$

Making use of the convexity of $f(x)$ for $x > K$, we may write for $x + (2n-1)a > K$,

$$f(x + 2na) \leqq \tfrac{1}{2}\{f[x + (2n-1)a] + f[x + (2n+1)a]\}, \tag{7}$$

and therefore

$$\frac{Q(x)}{(x + 2na)^{p/2}} < \frac{1}{2Q(x)}\left\{\frac{1}{[x + (2n-2)a]^{p/2}} + \frac{1}{(x + 2na)^{p/2}}\right\},$$

or

$$Q^2(x) < \frac{1}{2}\left\{\left[\frac{x + 2na}{x + (2n-2)a}\right]^{p/2} + 1\right\}.$$

If $Q(x) > 0$, then $Q(x+a) = 1/Q(x) > 0$, and in an exactly similar way we find that

$$Q^2(x+a) = \frac{1}{Q^2(x)} < \frac{1}{2}\left\{\left[\frac{x + (2n+1)a}{x + (2n-1)a}\right]^{p/2} + 1\right\}.$$

Since the last two displayed equations are true for all values of $n$ such that $x + (2n-1)a > K$, $Q^2(x) = 1$, and therefore $|f(x)| = F(x)$. If we assume that there exists an $x$ for which $Q(x) < 0$, an argument similar to the preceding one shows again that $|f(x)| = F(x)$. Hence we conclude that for all values of $x > 0$ $|f(x)| \equiv F(x)$. We thus see, that, for every value of $x$, $f(x)$ is equal to either plus or minus $F(x)$. If $f(x)$ were equal to minus $F(x)$ for some particular value of $x$, say $x=x_0$, then by (1) $f(x_0+na)<0$ for all $n$. Hence there would exist an $x$, say $x_1$, larger than $K$ such that $f(x_1)<0$, and $f(x_1)=-F(x_1)$. Then for every $h>0$

$$f(x_1+h) \leqq \tfrac{1}{2}[f(x_1)+f(x_1+2h)] \leqq \tfrac{1}{2}[-F(x_1)+F(x_1+2h)].$$

But $F(x)$ is decreasing, and therefore $f(x_1+h)$ would be negative for all $h>0$, and equal to $-F(x_1+h)$. This function is, however, not convex. Hence $f(x)$ can never be negative, and must be identical with $F(x)$.

A function is said to be concave for $x>K$ if it satisfies equation (3) with the inequality sign reversed. It is obvious that the only function which is concave for $x>K\geqq 0$ and satisfies the functional equation (1) is $-F(x)$.

If the condition of convexity be replaced by requirement that $f(x)$ be monotone for $x>K\geqq 0$, we can obtain the following result:

THEOREM 2. *The only function $f(x)$ which is monotone, not increasing, for all $x>K\geqq 0$, and satisfies the functional equation* (1) *is* $F(x)$.

The proof of this theorem is analogous to that of Theorem 1, with the convexity condition (7) replaced by the monotone relation

$$f(x+2na) \leqq f[x+(2n-1)a] \leqq f[x+(2n-2)a] \tag{8}$$

($n$ sufficiently large). If $x>0$ is such that $Q(x)>0$, then from (8) it follows that $Q^2(x)=1$. And if it is supposed that $Q(x)<0$ for some $x$, (8) leads to the contradictory inequalities $Q^2(x)>1$, $Q^2(x)<1$. Hence $f(x)\equiv F(x)$. The details of this proof are left to the reader.

It is also obvious from the last theorem that the only function $f(x)$ which is monotone, not decreasing, for all $x>K\geqq 0$, and satisfies (1) is $-F(x)$.

COLLEGE OF ST. THOMAS

# AN UNSYMMETRIC FUBINI THEOREM[1]

R. H. CAMERON AND W. T. MARTIN

Because of the great variety of ways in which repeated Stieltjes integrals may occur, the ordinary Fubini theorem for Lebesgue integrals has more than one analogue in Lebesgue-Stieltjes (Radon) integrals.[2] One of these (the symmetric one) is well known.[3] However, there is another form which is not symmetric, and its proof does not seem to be in the literature, although it is known to many writers, in special cases at least.[4] This form does not appear to be immediately derivable from the symmetric form, and since it is of interest in various connections, it seems worthwhile to give an explicit statement and proof. We do not follow the usual procedure of beginning with finite limits and then allowing the limits to become infinite because the passage to infinity seems to present difficulties of the same order as the direct proof of the final result itself. The immediate use of infinite limits is made possible by the fact that the symmetric Fubini theorem has already been proved with infinite limits.

THEOREM. *Let $k(x)$ be a function of bounded variation on every finite interval. Let $p(x, u)$ be Borel measurable in $(x, u)$; for almost all $x$ with respect to $k(x)$ let it be of bounded variation in $u$ over every finite $u$-interval. Denote by $V(x, u)$ the variation*[5] $V(x, u)=\int_{0+0}^{u+0}|d_v p(x, v)|$. *Assume that* $\int_{-\infty}^{\infty} V(x, u)|dk(x)|$ *exists (is finite) for all*[6] $u$. *Let $s(u)$ be Borel measurable on $(-\infty, \infty)$. Then the existence (finiteness) of either*

$$(1) \qquad \int_{-\infty}^{\infty} |s(u)|\, d_u \int_{-\infty}^{\infty} V(x, u)\, |dk(x)|$$

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[1] Presented to the Society, February 24, 1940.

[2] The integrals which occur in this paper are understood to be Lebesgue-Stieltjes (Radon) integrals. See, for example, Saks, *Theory of the Integral*, 2d revised edition, Warsaw-Lemberg, 1937, pp. 19 and 67. We note that the familiar principles of monotone and dominated convergence in the Lebesgue theory are also valid in this theory; see pages 28 and 29. In this paper we shall not admit ± ∞ as members of our number system; that is, existence implies finiteness.

[3] Saks, loc. cit., p. 81.

[4] See for instance, N. Wiener and H. R. Pitt, *On absolutely convergent Fourier-Stieltjes transforms*, Duke Mathematical Journal, vol. 4 (1938), pp. 420–436. The reader will note that a form of this theorem was used in passing from line 13 to line 14 on page 421.

[5] For negative $u$ we understand that $V(x, u)=-\int_{u+0}^{0+0}|d_v p(x, v)|$. Of course $V$ is strictly a variation only for positive $u$.

[6] Clearly it would be sufficient to require that the integral be finite for $u$ ranging over some sequence having ± ∞ as limit points.

*or*

$$\int_{-\infty}^{\infty} |dk(x)| \int_{-\infty}^{\infty} |s(u)| \cdot |d_u p(x, u)| \tag{2}$$

*implies the existence of the other, their equality, and the existence and equality of the following integrals*

$$\int_{-\infty}^{\infty} s(u) d_u \int_{-\infty}^{\infty} p(x, u) dk(x) = \int_{-\infty}^{\infty} dk(x) \int_{-\infty}^{\infty} s(u) d_u p(x, u). \tag{3}$$

We shall first prove this theorem under the following additional assumptions which we shall later successively remove:

(A) $s(u)$ is a polygonal function (that is, a continuous function whose second derivative exists and is zero except at a finite number of points),

(B) $s(u)$ is bounded and vanishes identically outside a finite interval,

(C) $s(u)$ is non-negative, $k(x)$ is monotonic increasing and $p(x, u)$ is monotonic increasing and right continuous in $u$ and $p(x, 0) = 0$.

Clearly the additional hypotheses (B) and (C) imply the existence of (1). Also they imply the existence of (2), as one sees in the following manner. Let $n$ be such that $s(u) = 0$ when $|u| > n - 1$ and such that $s(u)$ is everywhere less than or equal to $n$. Then since $V(x, u) = p(x, u)$ by (C) we see that

$$\int_{-\infty}^{\infty} s(u) d_u p(x, u) \leqq n \int_{-n}^{n} d_u p(x, u) = n[V(x, n) - V(x, -n)]. \tag{4}$$

Also

$$\int_{-\infty}^{\infty} V(x, u) dk(x) \tag{5}$$

exists by hypothesis. Hence

$$\int_{-\infty}^{\infty} dk(x) \int_{-\infty}^{\infty} s(u) d_u p(x, u) \leqq n \int_{-\infty}^{\infty} dk(x) [V(x, n) - V(x, -n)];$$

that is, (2) also exists.

Thus under the additional hypotheses (B) and (C) we need only show the equality of (1) and (2) since obviously the right and left members of (3) are equal to (1) and (2) respectively. In order to show the equality of (1) and (2) under (A), (B) and (C) we integrate by parts and use the fact that $s$ vanishes at $-n$ and at $n$. Thus

$$(6)\qquad \int_{-n}^{n} s(u)d_u \int_{-\infty}^{\infty} p(x,u)dk(x) = -\int_{-n}^{n} ds(u) \int_{-\infty}^{\infty} p(x,u)dk(x).$$

By the symmetric Fubini theorem, we obtain

$$(7)\qquad \int_{-n}^{n} ds(u) \int_{-\infty}^{\infty} p(x,u)dk(x) = \int_{-\infty}^{\infty} dk(x) \int_{-n}^{n} p(x,u)ds(u).$$

Integrating the inner integral of the right member of (7) by parts, we have

$$(8)\qquad \int_{-\infty}^{\infty} dk(x) \int_{-n}^{n} p(x,u)ds(u) = -\int_{-\infty}^{\infty} dk(x) \int_{-n}^{n} s(u)d_u p(x,u)$$

and this with (6) and (7) establishes the equality of (1) and (2) and thus in view of the preceding remarks proves the theorem under the additional assumptions (A), (B) and (C).

Now since every bounded Borel measurable function is a repeated limit of continuous (and even of polygonal) functions[7] we shall show by the repeated use of the principle of dominated convergence that the theorem still holds when we drop (A) and keep only the additional hypotheses (B) and (C). As we have noted, for this purpose it is again only necessary to prove the equality of (1) and (2). For purposes of induction let us assume that $s^*(u)=\lim_{j\to\infty} s_j(u)$ $(-\infty<u<\infty)$ where $s_1(u), s_2(u), \cdots$ is a sequence of non-negative functions each of which is Borel measurable and vanishes when $|u|>n-1$ and each of which is everywhere less than or equal to $n$. Assuming that the conclusion of the theorem holds for each $s_j(u)$ when $p$ and $k$ satisfy (C), we shall show that it also holds for the limit function $s^*(u)$. By bounded convergence

$$\lim_{j\to\infty} \int_{-n}^{n} s_j(u)d_u \int_{-\infty}^{\infty} p(x,u)dk(x) = \int_{-n}^{n} s^*(u)d_u \int_{-\infty}^{\infty} p(x,u)dk(x);$$

and since (4) holds for each $s_j(u)$ it follows by dominated convergence that

$$\lim_{j\to\infty} \int_{-\infty}^{\infty} dk(x) \int_{-n}^{n} s_j(u)d_u p(x,u) = \int_{-\infty}^{\infty} dk(x) \lim_{j\to\infty} \int_{-n}^{n} s_j(u)d_u p(x,u)$$
$$= \int_{-\infty}^{\infty} dk(x) \int_{-n}^{n} s^*(u)d_u p(x,u).$$

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[7] See, for example, de la Vallée Poussin, *Intégrales de Lebesgue* (Borel Monograph), 1916, pp. 36, 37.

Thus if the theorem holds under (B) and (C) for one Baire class it holds under (B) and (C) for the next; and hence the theorem holds under (B) and (C) for all Borel measurable functions.

In order to remove (B) (retaining only (C)) let

$$s^n(u) = \begin{cases} s(u) \text{ if } |u| \leqq n-1 \text{ and } s(u) \leqq n, \\ 0 \quad \text{otherwise.} \end{cases} \tag{9}$$

Then since $s^n(u)$ satisfies (B) and (C),

$$\int_{-\infty}^{\infty} s^n(u) d_u \int_{-\infty}^{\infty} p(x, u) dk(x) = \int_{-\infty}^{\infty} dk(x) \int_{-\infty}^{\infty} s^n(u) d_u p(x, u). \tag{10}$$

Since either (1) or (2) exists, both these expressions are bounded (in $n$). By monotone convergence

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} s^n(u) d_u \int_{-\infty}^{\infty} p(x, u) dk(x) = \int_{-\infty}^{\infty} s(u) d_u \int_{-\infty}^{\infty} p(x, u) dk(x),$$

$$\begin{aligned} \lim_{n\to\infty} \int_{-\infty}^{\infty} dk(x) \int_{-\infty}^{\infty} s^n(u) d_u p(x, u) &= \int_{-\infty}^{\infty} dk(x) \lim_{n\to\infty} \int_{-\infty}^{\infty} s^n(u) d_u p(x, u) \\ &= \int_{-\infty}^{\infty} dk(x) \int_{-\infty}^{\infty} s(u) d_u p(x, u). \end{aligned}$$

Thus the theorem holds under (C).

Finally, we can dispense with (C) and assume merely the original hypotheses of the theorem. To show this let

$$\begin{aligned} s_i(u) &= \max\{(-1)^i s(u), 0\}, & i &= 0, 1, \\ k_i(x) &= \frac{1}{2}\left\{ \int_0^{x+0} |dk(\xi)| + (-1)^i k(x+0) \right\}, & i &= 0, 1, \\ p_i(x, u) &= \tfrac{1}{2}\{V(x, u) + (-1)^i [p(x, u+0) - p(x, 0+0)]\}, & & \\ & & i &= 0, 1. \end{aligned} \tag{11}$$

Obviously all these functions are Borel measurable and satisfy (C). We first show that the existence of (1) or (2) for $s$, $p$ and $k$ implies the existence of (1) or (2) respectively for $s_i$, $p_j$, $k_l$. Since we always have

$$\int_{-\infty}^{\infty} |\phi(x)|\, df(x) \geqq \int_{-\infty}^{\infty} |\phi(x)|\, dg(x)$$

when $f(x) - g(x)$ is non-decreasing, and

$$\int_{-\infty}^{\infty}\phi(x)\,|\,df(x)\,|\geqq\int_{-\infty}^{\infty}\theta(x)\,|\,df(x)\,|$$

when $\phi(x)\geqq\theta(x)$, it follows that

$$\begin{aligned}\int_{-\infty}^{\infty}|\,s(u)\,|\,d_u\int_{-\infty}^{\infty}V(x,u)\,|\,dk(x)\,| &\geqq\int_{-\infty}^{\infty}s_i(u)d_u\int_{-\infty}^{\infty}V(x,u)\,|\,dk(x)\,|\\ &\geqq\int_{-\infty}^{\infty}s_i(u)d_u\int_{-\infty}^{\infty}p_j(x,u)\,|\,dk(x)\,|\\ &\geqq\int_{-\infty}^{\infty}s_i(u)d_u\int_{-\infty}^{\infty}p_j(x,u)dk_l(x)\end{aligned}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty}|\,dk(x)\,|\int_{-\infty}^{\infty}|\,s(u)\,|\,|\,d_up(x,u)\,| &\geqq\int_{-\infty}^{\infty}dk_l(x)\int_{-\infty}^{\infty}|\,s(u)\,|\,|\,d_up(x,u)\,|\\ &\geqq\int_{-\infty}^{\infty}dk_l(x)\int_{-\infty}^{\infty}s_i(u)\,|\,d_up(x,u)\,|\\ &\geqq\int_{-\infty}^{\infty}dk_l(x)\int_{-\infty}^{\infty}s_i(u)d_up_j(x,u).\end{aligned}$$

Thus $s_i$, $p_j$, $k_l$ satisfy the hypotheses of the theorem, as well as (C), and hence

$$\begin{aligned}(-1)^{i+j+l}&\int_{-\infty}^{\infty}s_i(u)d_u\int_{-\infty}^{\infty}p_j(x,u)dk_l(x)\\ &=(-1)^{i+j+l}\int_{-\infty}^{\infty}dk_l(x)\int_{-\infty}^{\infty}s_i(u)d_up_j(x,u).\end{aligned}$$

Adding these eight equations yields (3), and hence completes the proof of the theorem.

THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY

# LINEAR FORMS IN FUNCTION FIELDS[1]

LEONARD TORNHEIM

We shall prove algebraically an analogue for function fields[2] of a well known theorem of Minkowski on linear forms.[3]

THEOREM 1. *Let $F$ be a field and $z$ an indeterminate over $F$. Let*

$$L_i = \sum_{j=1}^{n} a_{ij}x_j, \qquad i = 1, \cdots, n, \tag{1}$$

*be $n$ linear expressions with coefficients $a_{ij}$ in $F(z)$ and with the determinant $|a_{ij}|$ of degree*[4] *$d$. Then for any set of $n$ integers $c_1, \cdots, c_n$ which satisfy the condition $\sum_{i=1}^{n} c_i > d-n$ there exists a set of values for $x_1, \cdots, x_n$ in $F[z]$ and not all zero such that each $L_i$ has degree at most $c_i$.*

First, we may assume that all of the $c_i$ are equal. For, suppose that $c$ is the maximum of the $c_i$. Write $L_i'$ for $L_i z^{c-c_i}$. The determinant of the coefficients of the $L_i'$ has degree $d' = d + \sum(c-c_i) < \sum c + n$. If there is a set of values for $x_1, \cdots, x_n$ with the property that the degree of each $L_i'$ is at most $c$, then these same values will make the degree of $L_i$ at most $c_i$.

Next, we may assume, after multiplying each $L_i$ by a suitable polynomial and by using an argument similar to that above, that all the $a_{ij}$ are in $F[z]$.

We shall now convert our system of $L_i$ by means of a transformation of determinant unity with elements in $F[z]$ into an equivalent system having $a_{ij}=0$ for $i<j$. Let $b_1$ be the g.c.d. of the $a_{1j}$; then $b_1 = \sum_{j=1}^{n} a_{1j}c_{j1}$ for appropriate $c_{j1}$ in $F[z]$. Necessarily the $c_{j1}$ are relatively prime. It is possible to find other quantities $c_{jk}$ $(k=2, \cdots, n)$ such that the determinant $|c_{jk}|$ has value unity.[5] Thus the transfor-

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[1] Presented to the Society, April 13, 1940.

[2] See M. Deuring, *Zur Theorie der Idealklassen in algebraischen Funktionenkörpern*, Mathematische Annalen, vol. 106 (1932), pp. 103–106, for a related result. I believe the results I prove are new.

[3] A bibliography of both analytic and algebraic proofs of the theorem of Minkowski on linear forms is given by E. Jacobsthal, *Der Minkowskische Linearformensatz*, Sitzungsberichte Berliner mathematischen Gesellschaft, vol. 33 (1934), pp. 62–64. See also L. J. Mordell, *Minkowski's theorem on homogeneous linear forms*, Journal of the London Mathematical Society, vol. 8 (1933), pp. 179–192.

[4] The degree of a rational function is the degree of the numerator less that of the denominator. Zero is assigned the degree minus infinity.

[5] A. A. Albert, *Normalized integral bases of algebraic number fields* I, Annals of Mathematics, (2), vol. 38 (1937), p. 926 ff. The statement is proved for rational integral $c_{jk}$ but the proof applies to any integral domain having the property that a

mation $x_j=\sum_{k=1}^n c_{jk}x'_k$ has determinant unity and hence it has a reciprocal transformation with elements in $F[z]$. The forms $L_i$ are transformed into $L'_i=\sum_{k=1}^n a'_{ik}x'_k$. Here $a'_{1k}=\sum_{j=1}^n a_{1j}c_{jk}$, and, being a linear combination of $a_{1j}$, it is divisible by their g.c.d. $b_1$; $a'_{1k}=b_1a_k$. The transformation

$$x'_1 = x'_1 - \sum_{k=2}^{n} a_k x''_k, \qquad x'_r = x''_r, \qquad\qquad r = 2, \cdots, n,$$

of determinant unity transforms the $L'_i$ into $L''_i$ with $L''_1=b_1x''_1$.

The procedure is repeated for the $n-1$ linear forms $M_i=\sum_{j=2}^n a''_{ij}x''_j$ $(i=2, \cdots, n)$. Finally, if this process is continued, the resultant transformation converts the original system (1) into one with $a_{ij}=0$ for $i<j$. As a consequence, if the degree of $a_{ii}$ is $d_i$, then $\sum d_i=d$. By using another transformation of determinant unity we may assume that the degree of each $a_{ij}$ is at most $d_i$.

Let $G_1$ be the set of all $n$-tuples $(s_1, \cdots, s_n)=s$ where the $s_i$ are in $F[z]$ and have degree not greater than $c$; hence $G_1$ is a linear set over $F$ whose order $u_1=n(c+1)$. Write $L_i(s)$ for $\sum_{j=1}^n a_{ij}s_j$. Let $G_r$ of order $u_r$ over $F$ be the linear subset of $G_1$ composed of all quantities $s$ for which $L_1, \cdots, L_{r-1}$ all take values of degree not greater than $c$. Designate by $P_r$ the set of all $L_r(s)$ with $s$ in $G_r$, and by $Q_r$ the set of all polynomials in $P_r$ of degree not exceeding $c$. Since the maximum degree possible for a polynomial in $P_r$ is $c+d_r$, the number of linearly independent polynomials of $P_r$ which are not in $Q_r$, that is, the order of $P_r/Q_r$, is less than or equal to $d_r$. Now $G_r/G_{r+1}\simeq P_r/Q_r$, a fact which follows from the mapping of $G_r$ on $P_r$ and $G_{r+1}$ on $Q_r$. Hence $[G_1:G_{n+1}]\leqq\sum_{i=1}^n d_i=d$. Therefore the order $u_{n+1}$ of $G_{n+1}$ is not less than $n(c+1)-d$. To be sure that $G_{n+1}$ has elements other than zero, we must have $u_{n+1}\geqq 1$, that is, $nc=\sum c\geqq d+1-n$.

The following theorem applies if some of the $L_i$ must be made equal to zero.

THEOREM 2. *If in Theorem* 1 *the first $m$ of the $L_i$ are to be made equal to zero and if their coefficients are in $F[z]$, then the conclusion will hold if $\sum_{i=m+1}^n c_i>d-(n-m)$.*

For, the first $m$ polynomials $s_i$ must be zero if we have the transformed system used in the proof of Theorem 1. Application of Theorem 1 for the remaining $L_i$ yields Theorem 2.

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g.c.d. of any finite number of elements exists and is linearly expressible in terms of those elements, that is, that every ideal with a finite basis is principal.

# A NOTE ON A THEOREM OF RADÓ CONCERNING THE (1, $m$) CONFORMAL MAPS OF A MULTIPLY-CONNECTED REGION INTO ITSELF[1]

MAURICE H. HEINS

Let $G_z$ denote a region in the $z$-plane and let $w=f(z)$ be a function of $z$ defined for $z\subset G_z$ which has the following properties: (1) $w=f(z)$ is analytic and single-valued for $z\subset G_z$, (2) $z\subset G_z$ implies that $f(z)\subset G_z$, (3) to each point $w_0\subset G_z$ there correspond $m$ and only $m$ points $z_0^{(k)}$ $(k=1, 2, \cdots, m)$ contained in $G_z$ such that $f(z_0^{(k)})=w_0$ $(k=1, 2, \cdots, m)$ where following the usual convention we count the $z_0^{(k)}$ according to their multiplicities. Then $w=f(z)$ is said to define a (1, $m$) *conformal map of* $G_z$ *onto itself.* Such maps have been studied by Fatou[2] and Julia[3] for the case where $G_z$ is simply-connected, and by Radó[4] who treated multiply-connected regions as well. Among the results which Radó established is the following theorem:

*Let* $G_z$ *be a region of finite connectivity* $p$ $(>1)$; *then there exists no* (1, $m$) *conformal map of* $G_z$ *onto itself for* $m>1$.

Let us remark with Radó that the theorem is no longer valid if $G_z$ is of infinite connectivity, as simple examples from the theory of the iteration of rational functions show.[5] Radó's proof of the theorem just cited is based on the possibility of mapping one-to-one and conformally a region of finite connectivity $p$, none of the components of its boundary reducing to points, onto a region of connectivity $p$, the boundary of which consists of $p$ disjoint circles. Other types of canonical regions yield the same result, notably one due to Koebe.[6] It is the object of the present note to establish Radó's theorem directly without appeal to the possibility of mapping one-to-one and conformally the region $G_z$ onto a canonical region. Our tools are the theory of iteration and a simple modification of Nevanlinna's principle of harmonic measure.[7]

Let $G_z$, the region we are going to study, have as its boundary $p$ $(>1)$ disjoint continua $\Gamma_k$ $(k=1, 2, \cdots, p)$. It is evident that we may assume without loss of generality that the $\Gamma_k$ are all closed Jordan curves. Further let $w=f(z)$ define a $(1, m)$ conformal map of $G_z$ onto itself. In Radó's paper cited above, it is shown that if $w=f(z)$ defines a $(1, m)$ conformal map of $G_z$ onto itself, then whenever $z$ tends to the boundary of $G_z$, so does $f(z)$, in such a manner that whenever $z$ tends to a given component $\Gamma_k$ of the boundary, then $f(z)$ tends to one and only one component of the boundary $\Gamma_{l_k}$ where the index $l_k$ depends on $k$. Now the relation $k \rightarrow l_k$ $(k=1, 2, \cdots, p)$ defines a permutation of the indices $1, 2, \cdots, p$.

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[1] Presented to the Society, April 27, 1940, under the title *A note on a theorem of Radó.*

[2] P. Fatou, Bulletin de la Société Mathématique de France, 1919.

[3] G. Julia, Comptes Rendus de l'Académie des Sciences, Paris, vol. 166 (1918).

[4] T. Radó, Acta Szeged, vol. 1 (1922).

[5] G. Julia, Journal de Mathématiques Pures et Appliquées, 1918.

[6] P. Koebe, Acta Mathematica, vol. 41.

[7] R. Nevanlinna, *Eindeutige analytische Funktionen*, chap. 3.

We shall understand by $f_n(z)$, *the nth iterate of* $f(z)$, that function defined by the recursive relations

$$\text{(A)} \qquad f_0(z) \equiv z, \; f_1(z) \equiv f(z), \cdots, \; f_n(z) \equiv f[f_{n-1}(z)].$$

By the definition of $w=f(z)$ it is clear that the definition given by (A) for $f_n(z)$ is meaningful. Since the relation $k \rightarrow l_k$ defines a permutation of the indices $1, 2, \cdots, p$, a suitably chosen power of this permutation is the identity. Hence for a properly chosen whole number $n_0$, $f_{n_0}(z)$ has the property that when $z$ tends to a component $\Gamma_k$ of the boundary of $G_z$, then $f_{n_0}(z)$ tends to exactly the same component $\Gamma_k$. Let us denote $f_{n_0}(z)$ by $F(z)$.

We are now in a position to demonstrate Radó's theorem directly. By the *harmonic measure of z with respect to* $\Gamma_k$, denoted by $\omega(z, \Gamma_k, G_z)$ we understand that harmonic function defined for $z \subset G_z$ which is single-valued and bounded for $z \subset G_z$ and further takes on the boundary value 1 on $\Gamma_k$ and the boundary value 0 on all the components of the boundary of $G_z$ exclusive of $\Gamma_k$.[8]

The relations

$$\text{(B)} \qquad \omega(F(z), \Gamma_k, G_z) \equiv \omega(z, \Gamma_k, G_z), \qquad k = 1, 2, \cdots, p,$$

are an immediate consequence of the principle of the maximum for harmonic functions. For, as $z$ tends to a given component of the boundary of $G_z$, $F(z)$ tends to the same component. Hence $\omega(F(z), \Gamma_k, G_z)$ has the same boundary values as $\omega(z, \Gamma_k, G_z)$, hence the identity. To establish Radó's theorem we need consider only one of these identities, say

$$\text{(C)} \qquad \omega(F(z), \Gamma_1, G_z) \equiv \omega(z, \Gamma_1, G_z).$$

Let $z_0$ be a point lying in $G_z$, and let $\omega(z_0, \Gamma_1, G_z) = \lambda_0$ $(0 < \lambda_0 < 1)$. Then our identity (C) implies

$$\omega(F_n(z_0), \Gamma_1, G_z) = \lambda_0,$$

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[8] R. Nevanlinna, ibid.

where $F_n(z)$ is the $n$th iterate of $F(z)$ for $n=1, 2, \cdots$. Hence if $z$ is on a given level curve defined by $\omega(z, \Gamma_1, G_s)=\lambda$ $(0<\lambda<1)$, $F_n(z)$ is on the same level curve for $n=1, 2, \cdots$. This permits us to conclude that no limit function of the sequence $\{F_n(z)\}$ is a constant. From this fact we infer that $w=F(z)$ defines a (1, 1) conformal map of $G_s$ onto itself, and hence so does $w=f(z)$; that is, *$m$ cannot exceed one.*

Suppose contrary to our assertion that $m>1$. Then $w=F(z)$ would define a $(1, m^{n_0})$ conformal map of $G_s$ onto itself and $w=F_n(z)$ would define a $(1, m^{nn_0})$ conformal map of $G_s$ onto itself. Let $\{F_{k_n}(z)\}$ be a convergent subsequence of $\{F_n(z)\}$ and let $F_0(z)$ denote the limit function of this subsequence. A point $w_0$ has, under the map defined by $w=F_{k_n}(z)$, $m^{k_n n_0}$ antecedents all of which lie on the level curve defined by $\omega(z, \Gamma_1, G_s)=\omega(w_0, \Gamma_1, G_s)$. But by Hurwitz's theorem, for $k_n$ sufficiently large $F_{k_n}(z)-w_0$ and $F_0(z)-w_0$ have the same number of zeros, and this is manifestly impossible. Hence $m$ cannot exceed one.

It is interesting to note that the technique employed in the present proof of Radó's theorem, that is, a combined use of the theory of iteration and of an extended form of the principle of harmonic measure (the modification consisting in the fact that we did not require the continuity of $F(z)$ on $\Gamma_1$) may be applied to other problems in the theory of the conformal mapping of multiply-connected regions. In particular, such a technique permits us to conclude that the number of (1, 1) conformal maps of a given multiply-connected region of finite connectivity $p$, where $p>2$, bounded by $p$ disjoint continua is finite. Koebe's proof of this theorem is based on the fact that the region in question can be mapped one-to-one and conformally onto a canonical region. We shall not give in the present note the details of a direct proof of this theorem. Let us remark however that it is based on the fact that for $p>2$ each of the harmonic functions $\omega(z, \Gamma_k, G_s)$ $(k=1, 2, \cdots, p)$ has precisely $p-2$ $(>0)$ critical points. A study of the level curves containing these critical points yields the desired result.

HARVARD UNIVERSITY

# NON-CYCLIC ALGEBRAS OF DEGREE FOUR AND EXPONENT TWO WITH PURE MAXIMAL SUBFIELDS[1]

ROY DUBISCH

In a recent paper[2] A. A. Albert proved the falsity of the converse of the well known proposition that a cyclic normal division algebra contains a quantity $j$ whose minimum equation is $x^n = j$ in the base field of the algebra. The proof consists in giving an example of a non-cyclic normal division algebra containing a quantity $j$ as described above. The algebra described in that example was of degree and exponent four. It is the purpose of this paper to show that the exponent does not affect the property, and this we shall do by constructing similarly an algebra of degree four whose exponent is two.

We shall actually prove the following theorem.

THEOREM. *Let $\xi$ and $\eta$ be independent indeterminants over the field $R$ of real numbers, $K = R(\xi, \eta)$. Then there exist non-cyclic normal division algebras of degree four and exponent two over $K$ such that $t^4 = \gamma$ in $K$, $t^2$ not in $K$, for some quantity $t$ in each algebra $A$.*

To make our proof we use the known[3] property stating that a normal division algebra of degree four has exponent two if and only if it is expressible as a direct product of two algebras of degree two. Therefore we may take our desired algebra $A$ to be

$$\begin{aligned} &A = B \times C = (1, i, j, ij) \times (1, x, y, xy), \\ &ji = -ij, \quad i^2 = u, \quad j^2 = a, \qquad u \neq 0,\ a \neq 0 \text{ in } K, \\ &yx = -xy, \quad x^2 = v, \quad y^2 = b, \qquad v \neq 0,\ b \neq 0 \text{ in } K. \end{aligned} \tag{1}$$

We seek first a quantity $t$ with minimum equation $t^4 = \gamma$ in $K$. Now if we take $t = a_1 i + a_2 j + a_3 ij$, where $a_1, a_2, a_3$ are in $K(x)$, and if we put

$$a_1 = c_1 + c_2 x, \quad a_2 = d_1 + d_2 x, \quad a_3 = f_1 + f_2 x, \qquad c_i, d_i, f_i \text{ in } K, \tag{2}$$

we can easily compute that if

$$a = -\frac{c_1^2 u + c_2^2 uv}{d_1^2 + d_2^2 v - f_1^2 u - f_2^2 uv}, \tag{3}$$

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[1] Presented to the Society, April 13, 1940.

[2] A. A. Albert, *Non-cyclic algebras with pure maximal subfields*, this Bulletin, vol. 44 (1938), pp. 576–579.

[3] A. A. Albert, *Normal division algebras of degree four*, Transactions of this Society, vol. 34 (1932), p. 369.

and if

$$(4) \qquad b_2 = 2c_1c_2u + 2d_1d_2a - 2f_1f_2ua \neq 0,$$

the algebra $A$ defined by (1) contains a quantity $t$ such that $t^4 = \gamma$ in $K$, $t^2$ not in $K$.

In proving $A$ a non-cyclic division algebra we use the following method.[4] Let $z^2 = \delta^2 + \epsilon^2 = \Delta$ in $K$, $\delta$ and $\epsilon$ in $K$ with $L = K(z)$ a quadratic field over $K$. The algebras $B_0 = B \times L$ and $C_0 = C \times L$ over $L$ are generalized quaternion algebras over their reference field $L$. Furthermore, $A_0 = A \times L$ is the direct product $A_0 = B_0 \times C_0$. Then it is known[5] that a necessary and sufficient condition that $A_0$ over $L$ shall be a division algebra is that the quadratic form

$$(5) \qquad Q \equiv u\lambda_1^2 + a\lambda_2^2 - ua\lambda_3^2 - (v\lambda_4^2 + b\lambda_5^2 - vb\lambda_6^2)$$

in the variables $\lambda_1, \lambda_2, \cdots, \lambda_6$ in $L$ shall not vanish for any $\lambda_1, \cdots, \lambda_6$ not all zero in $L$.

If $J = R[\xi, \eta]$ is the integral domain of all polynomials in $\xi$ and $\eta$ with real coefficients, it is obvious that we may, without loss of generality, take the $\lambda_i$ in (5) to be in $J$. Hence we may write

$$(6) \qquad \lambda_i = \alpha_i + \beta_i z, \qquad i = 1, \cdots, 6,$$

where the $\alpha_i$ and $\beta_i$ are in $J$. Then $\lambda_i^2 = (\alpha_i^2 + \beta_i^2\Delta) + 2\alpha_i\beta_i z$ so that if

$$(7) \qquad P_i = \alpha_i^2 + \beta_i^2, \qquad Q_i = 2\alpha_i\beta_i,$$

the equation $Q \equiv 0$ becomes

$$(8) \qquad \begin{aligned} &uP_1 + aP_2 - uaP_3 - vP_4 - bP_5 + vbP_6 \\ &\qquad + (uQ_1 + aQ_2 - uaQ_3 - vQ_4 - bQ_5 + vbQ_6)z \equiv 0. \end{aligned}$$

But 1 and $z$ are linearly independent[6] with respect to $K$ so that (8) implies that

$$(9) \qquad \phi(\xi, \eta) \equiv uP_1 + aP_2 - uaP_3 - vP_4 - bP_5 - vbP_6 \equiv 0.$$

Now if in (3) we let $c_1 = c_2 = d_1 = 1$, $d_2 = f_1 = f_2 = 0$, we obtain

$$(10) \qquad a = -(u + uv).$$

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[4] This is the device used by A. A. Albert in his paper *A construction of non-cyclic normal division algebras*, this Bulletin, vol. 38 (1932), pp. 452–453.

[5] A. A. Albert, *On the Wedderburn norm condition for cyclic algebras*, this Bulletin, vol. 37 (1931), p. 311, Theorem 3.

[6] It is obvious that we can have this condition hold by a suitable choice of $\delta$ and $\epsilon$.

Note that these choices also satisfy (4). Substituting this value for $a$ in (3) we have

$$(11)\quad \phi(\xi, \eta) = u(P_1 - P_2) - uvP_2 + u^2P_3 + u^2vP_3 - vP_4 - bP_5 + vbP_6.$$

With these choices for the parameters of our algebra $A$ is easy to prove by the well known method of an argument[7] on the degrees of the polynomial in (7) that $\phi(\xi, \eta)$ cannot be identically zero unless all of the coefficients are zero. From this it follows that $A_0$ over $L$ is a division algebra over $L$; so is algebra $A$ a division algebra over $K$. By its form it is a normal division algebra of degree four and exponent two over $K$.

The proof of the non-cyclic character of $A$ is exactly the same as that given by A. A. Albert in a previous paper,[8] and with this we have the desired theorem.

WILSON JUNIOR COLLEGE

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[7] A similar argument was employed by A. A. Albert in the paper last cited.

[8] This Bulletin, vol. 38 (1932), p. 454.

# NOTE ON NORMALITY IN QUASI-GROUPS

D. C. MURDOCH

An interesting problem in the theory of quasi-groups is to determine how strong an associative law must be assumed in order to obtain a theory of normal subquasi-groups similar to that of ordinary groups. The properties of normal subgroups which it is desirable to retain in the non-associative case are, first, that they form a Dedekind structure, and second, that each one gives rise to a quotient group homomorphic to the whole group. In this note we shall take the latter property as the definition of normality and show that the former follows from it under very general conditions.

We shall understand by a quasi-group $G$ a system of elements satisfying the following two postulates:

I. PRODUCT AXIOM. *Any ordered pair of elements $a$, $b$ of $G$ has a unique product $ab$ which also belongs to $G$.*

II. QUOTIENT AXIOM. *For any two elements $a$ and $b$ of $G$ there exist unique elements $x$ and $y$ of $G$ such that*

$$ax = b, \qquad ya = b.$$

We shall make one further assumption, namely that there exist left coset expansions with respect to any subquasi-group $H$ of $G$. In other words, if $H$ is any subquasi-group, the cosets $aH$ and $bH$ are either identical or have no elements in common. This assumption has been shown by Hausmann and Ore [1] to be equivalent to the following:

III. WEAK ASSOCIATIVE LAW. *If $a$ and $b$ are arbitrary elements of $G$ and $c_0$ a fixed element, let $d_0$ be determined so that $(ab)c_0 = ad_0$. Then for any $c$*

$$(ab)c = ad,$$

*where $d$ belongs to the quasi-group $\{c_0, d_0, c\}$ generated by $c_0$, $d_0$ and $c$.*

Since no confusion can arise, we shall for convenience use the term subgroup for subquasi-group and quotient group for quotient quasi-group, without implying thereby that the systems in question are associative.

It should be noted that a theory of normal subgroups has been given by Hausmann and Ore [1] based on the definition of a normal subgroup as a subgroup $H$ such that $aH = Ha$ for all $a$ in $G$. This theory requires that $G$ should satisfy associative laws considerably stronger than III in order that the normal subgroups so defined shall have the desired properties. To overcome this difficulty we shall adopt instead the following:

DEFINITION: *A subgroup $H$ of $G$ shall be called normal if*

$$(aH)(bH) = (ab)H,$$

*for all elements $a$ and $b$ of $G$.*

This is the same definition used in a previous paper by the author [2] where it was applied, however, only to quasi-groups satisfying rather special associative laws. It will be shown here that the results of that paper concerning normal subgroups can be extended to any quasi-group satisfying III.

Comparing this definition with that of Hausmann and Ore, it is seen that neither one implies the other. Under the latter all subgroups of a commutative quasi-group are normal, while under the former this property is enjoyed by the "abelian" quasi-groups [2] which satisfy the law $(ab)(cd) = (ac)(bd)$. In a Suschkewitsch quasi-group [3] (having unique right unit $e$ and satisfying $a(bc) = (ab)(ec)$, the right unit $e$ is a subgroup which is normal in the sense of this paper but not in the sense of Hausmann and Ore.

Examples of quasi-groups satisfying III but not satisfying the stronger laws previously discussed can easily be constructed by rearranging the elements in the Cayley square of a group, or known quasi-group, so as to destroy the associativity without disturbing the existence of cosets. The following, for example,

|  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 3 | 2 | 5 | 6 | 4 | 9 | 7 | 8 |
| 2 | 3 | 2 | 1 | 6 | 4 | 5 | 7 | 8 | 9 |
| 3 | 2 | 1 | 3 | 4 | 5 | 6 | 8 | 9 | 7 |
| 4 | 6 | 4 | 5 | 7 | 8 | 9 | 2 | 3 | 1 |
| 5 | 4 | 5 | 6 | 8 | 9 | 7 | 3 | 1 | 2 |
| 6 | 5 | 6 | 4 | 9 | 7 | 8 | 1 | 2 | 3 |
| 7 | 8 | 9 | 7 | 3 | 1 | 2 | 6 | 5 | 4 |
| 8 | 9 | 7 | 8 | 1 | 2 | 3 | 4 | 6 | 5 |
| 9 | 7 | 8 | 9 | 2 | 3 | 1 | 5 | 4 | 6 |

satisfies III since cosets exist with respect to all subgroups. It is easily verified that it satisfies none of the laws previously discussed by the author nor the laws $A_2$, $A_3$, $A_4$ or $A_5$ of Hausmann and Ore.

It is natural to require next that every element of a coset $aH$ shall define the same coset $aH$, or in other words that $(ah)H=aH$ for all $h$ in $H$. In this connection we have the following:

THEOREM 1. *The necessary and sufficient condition that every coset* $aH$ *may be defined by any one of its elements is that* $H$ *contain the right unit of every element of* $G$.

PROOF. If $H$ contains all right units of $G$, then $aH=(ah)H$ for all $h$ in $H$, since each coset contains the element $ah$. Conversely if $aH=(ah)H$ for all $h$ in $H$ and all $a$ in $G$, then $ah$ must belong to $(ah)H$. Hence $H$ contains the right unit of $ah$ for all $a$ in $G$ and therefore contains all right units of $G$.

The main object of this note is to prove that in any finite quasigroup the normal subgroups which contain all right units of $G$ form a Dedekind structure. With a slight strengthening of the definition of normality the result may also be proved in the infinite case. We shall assume for the present, however, that $G$ is finite.

THEOREM 2. *If* $H$ *and* $K$ *are normal subgroups of* $G$ *and contain all right units of* $G$, *their crosscut* $H\cap K=D$ *is also normal.*

PROOF. From the normality of $H$ and $K$ we have

$$(aD)(bD) \subseteq (ab)D,$$

and since $D$ is finite the equality must hold. Moreover $D$ is not void since it contains all right units of $G$.

THEOREM 3. *If* $H$ *and* $K$ *are normal and contain all right units of* $G$, *their union* $H\cup K$ *consists of all products* $hk$ *where* $h$ *belongs to* $H$ *and* $k$ *to* $K$.

The proof of this theorem is similar to that of Theorem 7 of the author's previous paper already cited. The union $H\cup K$ may therefore be written as the product $HK$.

THEOREM 4. *If* $H$ *and* $K$ *are normal, and contain all right units of* $G$, *their union is also normal.*

PROOF. Using the notation $e_a$ for the right unit of $a$, we have, from the normality of $H$ and $K$,

$$\begin{aligned}[a(HK)][b(HK)] &= [(ae_a)(HK)][(be_b)(HK)] \\ &= [(aH)K][(bH)K] = [(aH)(bH)]K = [(ab)H]K \\ &= [(ab)e_{ab}][HK] = (ab)(HK),\end{aligned}$$

which completes the proof.

THEOREM 5. *The normal subgroups of G which contain all right units of G form a Dedekind structure.*

PROOF. From Theorems 2 and 4 these subgroups form a structure. It is only necessary, therefore, to show that if $M \supset H$,

$$M \cap (H \cup K) \subset H \cup (M \cap K).$$

This follows as usual since every element of the left-hand side has the form $hk$ and belongs to $M$. Hence $k$ belongs to $M \cap K$ since $h$ belongs to $M$.

In establishing Theorem 5 we have used the finiteness of $G$ in the proofs of Theorems 2, 3 and 4. This can be avoided by strengthening the definition of normality as follows:

A subgroup $H$ shall be called normal in $G$ if for all elements $a$, $b$ of $G$ and every element $h$ of $H$ we have

$$(ah)(bH) = (ab)H, \qquad (aH)(bh) = (ab)H.$$

This of course implies the other definition and is equivalent to it in case $G$ is finite. This strengthening of the definition enables us to prove the quotient axioms for the union and crosscut in Theorems 2 and 3. For example, for the union, the equation $x(h_1k_1) = h_2k_2$ has a unique solution of the form $x = hk$. For choosing $h$ so that $hh_1 = h_2$, it follows from (2) that there exists an element $k$ of $K$ such that

$$(hk)(h_1k_1) = h_2k_2.$$

For the proof of Theorem 4 we have as in the finite case

$$a(HK) = (ae_a)(HK) \subset (aH)K.$$

But in the infinite case it is necessary to show also that $(aH)K \subset a(HK)$. This follows from III, applied in the form of Theorem 7, Chapter I, Hausmann and Ore.

From Theorem 5 the usual theorems concerning normal subgroups follow. In particular, the Jordan-Hölder theorem for principal chains, and the Schmidt-Remak decomposition theorems, as proved by Ore [4] for arbitrary Dedekind structures, will all hold. One interesting question arises which has no counterpart in ordinary groups, namely whether the subgroup $R$, generated by the right units of $G$, is necessarily normal. Although there seems to be no obvious reason to believe that this is true, I have no counterexample. If $R$ is not always normal the question arises whether infinite quasi-groups exist in which the Dedekind structure of normal subgroups has no unit element.

## BIBLIOGRAPHY

1. B. A. Hausmann and O. Ore, *Theory of Quasi-groups*, American Journal of Mathematics, vol. 59 (1937), pp. 983–1004.

2. D. C. Murdoch, *Quasi-groups which satisfy certain generalized associative laws*, American Journal of Mathematics, vol. 61 (1939), pp. 509–522.

3. A. Suschkewitsch, *On a generalization of the associative law*, Transactions of this Society, vol. 31 (1929), pp. 204–214.

4. O. Ore, *On the foundation of abstract algebra* I and II, Annals of Mathematics, vol. 36 (1935), pp. 406–437, and vol. 37 (1936), pp. 265–292.

YALE UNIVERSITY

# ON THE FIRST CASE OF FERMAT'S LAST THEOREM

D. H. AND EMMA LEHMER

In 1909 Wieferich [1] proved his celebrated criterion for the first case of Fermat's last theorem, namely:

*The equation*

$$x^p + y^p = z^p, \qquad x, y, z \textit{ prime to } p, \tag{1}$$

*has no solutions unless*

$$2^{p-1} \equiv 1 \pmod{p^2}. \tag{2}$$

Since that time numerous other criteria of the form

$$m^{p-1} \equiv 1 \pmod{p^2} \tag{3}$$

have been proved by Mirimanoff [2] (for $m=3$), Vandiver [3] (for $m=5$), Frobenius [4], Pollaczek [5], Morishima [6], and Rosser [7] for all prime values of $m \leqq 41$.

Wieferich's criterion alone has been applied by Meissner [8] and Beeger [9] for $p<16{,}000$ and was found to be satisfied only for $p=1{,}093$ and 3,511, both of which cases failed to satisfy Mirimanoff's criterion.

Until recently no effort has been made to combine these various criteria in a practical way. Mirimanoff observed, however, in 1910 that his criterion and that of Wieferich could be combined to state that equation (1) has no solutions for all primes $p$ of the form $2^\alpha 3^\beta \pm 1$ or $|2^\alpha \pm 3^\beta|$.

In the presence of more criteria this statement can be extended thus: We call a number an "$A_n$ number" (after Western) if it is divisible by no prime exceeding the $n$th prime $p_n$. If the criterion (3) has been established for all $m \leqq p_n$, then equation (1) does not hold if $p$ is the sum or difference of two $A_n$ numbers [10]. Since all the numbers less than $p_{n+1}$ are $A_n$ numbers, we may state that equation (1) has no solution for any prime in a region where the $A_n$ numbers are so dense that they do not differ by more than $2p_{n+1}-1$. This method was used in 1938 by A. E. Western [11] to show that (1) is impossible for $16{,}000<p<100{,}000$.

A more powerful method of combining the criteria was suggested recently by Rosser [12], who observes that while the congruence

$$x^{p-1} \equiv 1 \pmod{p^2} \tag{4}$$

has only $(p-1)/2$ solutions less than $p^2/2$, every $A_n$ number is a solution of (4) if (1) holds and if (3) has been established as far as $m = p_n$. Hence if $\phi_n(x)$ denotes the number of $A_n$ numbers not exceeding $x$, then

$$\phi_n(p^2/2) \leqq (p-1)/2, \tag{5}$$

a condition which cannot hold for small $p$'s if the $A_n$ numbers are sufficiently dense.

Other inequalities of this kind can be derived by separating the solutions of (4) into classes. We shall do this here only in the case in which the solutions are distinguished by their parity. For every positive odd solution $\omega < p^2/3$ of (4) there exists also a solution $(p^2-\omega)/2$ (using Wieferich's criterion) which lies between $p^2/3$ and $p^2/2$, and hence differs from $\omega$. The number $(p-1)/2$ of solutions of (4) not exceeding $p^2/2$ is then at least the number of even $A_n$ numbers less than $p^2/3$ plus twice the number of odd $A_n$ numbers less than $p^2/3$. This can be written as

$$\phi_n(p^2/3) + \phi_n^*(p^2/3) \leqq (p-1)/2 \tag{6}$$

where $\phi_n^*(x)$ denotes the number of odd $A_n$ numbers less than $x$.

If one is to apply an inequality such as (5) or (6) outside the limit of existing tables [13] of $A_n$ numbers, it is necessary to find lower bounds for $\phi_n$ and $\phi_n^*$. Rosser [12] has given a lower bound for $\phi_n(10^x)$ in the form of a polynomial $f_n(x)$ of the $n$th degree.

By an improved method [14] which makes use of Bernoulli polynomials we have constructed polynomials $P_n(x)$ and $Q_n(x)$ of degree $n$ giving lower and upper bounds for $\phi_n(10^x)$, and also a polynomial $P_{n-1}^*(x)$ of degree $n-1$ giving a lower bound for $\phi_n^*(10^x)$. As we shall actually need not $P_{n-1}^*$ but the sum $P_n + P_{n-1}^*$, we tabulate the following polynomials for $n = 13$:

| $P_{13}(x)$ | $P_{13}(x)+\overset{*}{P}_{12}(x)$ | $Q_{13}(x)$ |
|---|---|---|
| $.0^{9}1380608198\ x^{13}$ | $.0^{9}1380608198\ x^{13}$ | $.0^{9}1380608198\ x^{13}$ |
| $.0^{7}1272704088\ x^{12}$ | $.0^{7}1326732670\ x^{12}$ | $.0^{7}1326732670\ x^{12}$ |
| $.0^{6}5247670000\ x^{11}$ | $.0^{6}5691949617\ x^{11}$ | $.0^{6}5729012056\ x^{11}$ |
| $.0^{4}1277269076\ x^{10}$ | $.0^{4}1439087748\ x^{10}$ | $.0^{4}1469359894\ x^{10}$ |
| $.0^{3}2039391192\ x^{9}$ | $.0^{3}2383232393\ x^{9}$ | $.0^{3}2493297644\ x^{9}$ |
| $.0^{2}2244543403\ x^{8}$ | $.0^{2}2716892889\ x^{8}$ | $.0^{2}2952478417\ x^{8}$ |
| $.01740607057\ x^{7}$ | $.02179700460\ x^{7}$ | $.02510548659\ x^{7}$ |
| $.09545572654\ x^{6}$ | $.1235281024\ x^{6}$ | $.1557243217\ x^{6}$ |
| $.3654889719\ x^{5}$ | $.4882665632\ x^{5}$ | $.7120647453\ x^{5}$ |
| $.9452666752\ x^{4}$ | $1.302427247\ x^{4}$ | $2.432810990\ x^{4}$ |
| $1.542870198\ x^{3}$ | $2.191273599\ x^{3}$ | $6.376807029\ x^{3}$ |
| $1.361483164\ x^{2}$ | $1.996360212\ x^{2}$ | $13.24421612\ x^{2}$ |
| $.3459458265\ x$ | $.5410860433\ x$ | $21.15029718\ x$ |
| $-.2049399029$ | $-.2911608671$ | $19.69764203$ |

If we now replace (5) by

$$P_{13}(\log p^2/2) \leqq (p-1)/2,$$

we find, by actual substitution into $P_{13}$, that this inequality holds only for $p \geqq 93{,}785{,}629$, and hence (1) has no solution[1] for $p < 93{,}785{,}629$.

Since

$$Q_{13}(\log p^2/2) \leqq (p-1)/2$$

holds only for $p > 141{,}000{,}000$, it follows that the inequality (5), even if we knew the exact value of $\phi_n(x)$, could not be used beyond this limit for $n = 13$.

The inequality (6) becomes

$$P_{13}(\log p^2/3) + P_{12}^{*}(\log p^2/3) \leqq (p-1)/2.$$

This holds only for $p > 102{,}108{,}200$.

Hence the first case of Fermat's last theorem is now proved for $p < 102{,}108{,}200$.

Further criteria of type (3), when established, may be used to extend this limit by calculating approximating polynomials of higher degree from the ones given above by the method described in [14].

*Note added March 1.* Since writing the above, Dr. Rosser has kindly sent us the manuscript of his forthcoming paper [15] in which he completes Morishima's proof that the prime 43 gives also a criterion of the form (3). We have therefore calculated $P_{14}(x)$ and $P_{13}^{*}(x)$ from the $P_{13}$ and $P_{12}^{*}$ given above. These polynomials are as follows:

| $P_{14}(x)$ | $P_{13}^{*}(x)$ | $P_{14}(x)+P_{13}^{*}(x)$ |
|---|---|---|
| $.0^{11}6037145739\ x^{14}$ | | $.0^{11}6037145739\ x^{14}$ |
| $.0^{9}6683705079\ x^{13}$ | $.0^{10}2544306733\ x^{13}$ | $.0^{9}6938135752\ x^{13}$ |
| $.0^{7}3337944336\ x^{12}$ | $.0^{8}2536688143\ x^{12}$ | $.0^{7}3591613150\ x^{12}$ |
| $.0^{6}9940243443\ x^{11}$ | $.0^{6}1131551216\ x^{11}$ | $.0^{5}1107179466\ x^{11}$ |
| $.0^{4}1965570473\ x^{10}$ | $.0^{5}2980593461\ x^{10}$ | $.0^{4}2263629819\ x^{10}$ |
| $.0^{3}2719317505\ x^{9}$ | $.0^{4}5152041706\ x^{9}$ | $.0^{3}3234521676\ x^{9}$ |
| $.0^{2}2700968174\ x^{8}$ | $.0^{3}6140455952\ x^{8}$ | $.0^{2}3315013769\ x^{8}$ |
| $.01944029859\ x^{7}$ | $.0^{2}5157919367\ x^{7}$ | $.02459821796\ x^{7}$ |
| $.1009940169\ x^{6}$ | $.03064345185\ x^{6}$ | $.1316374687\ x^{6}$ |
| $.3720257684\ x^{5}$ | $.1271048263\ x^{5}$ | $.4991305947\ x^{5}$ |
| $.9365601794\ x^{4}$ | $.3559975936\ x^{4}$ | $1.292557773\ x^{4}$ |
| $1.500217863\ x^{3}$ | $.6290315435\ x^{3}$ | $2.129249406\ x^{3}$ |
| $1.299882088\ x^{2}$ | $.6014604442\ x^{2}$ | $1.901342532\ x^{2}$ |
| $.3050249350\ x$ | $.1730336327\ x$ | $.4780585677\ x$ |
| $-.2081092173$ | $-.08700841024$ | $-.2951176274$ |

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[1] The corresponding result obtained by Rosser is about 41,000,000.

We find that (5) is satisfied only if $p>230$ millions, while (6) gives the better inequality $p \geqq 253,747,889$.

The first case of Fermat's last theorem is therefore now established for all $p<253,747,889$.

## BIBLIOGRAPHY

1. A. Wieferich, Journal für die reine und angewandte Mathematik, vol. 136 (1909), pp. 293–302.
2. D. Mirimanoff, Comptes Rendus de l'Académie des Sciences, Paris, vol. 150 (1910), pp. 204–206.
3. H. S. Vandiver, Journal für die reine und angewandte Mathematik, vol. 144 (1914), pp. 314–318.
4. G. Frobenius, Sitzungsberichte der Akademie der Wissenschaften, Berlin, 1914, pp. 653–681.
5. F. Pollaczek, Sitzungsberichte der Akademie der Wissenschaften, Vienna, IIa, vol. 126 (1917), pp. 25–59.
6. T. Morishima, Japanese Journal of Mathematics, vol. 8 (1931), pp. 159–173.
7. J. B. Rosser, this Bulletin, vol. 46 (1940), pp. 299–304.
8. W. Meissner, Sitzungsberichte der Akademie der Wissenschaften, Berlin, 1913, pp. 663–667.
9. N. G. W. H. Beeger, Messenger of Mathematics, vol. 51 (1922), pp. 149–150; Nieuw Archief voor Wiskunde, (2), vol. 20 (1939), pp. 51–54.
10. This is a special case of a theorem due to Landau, l'Intermédiaire des Mathematiciens, vol. 20 (1913), p. 206. See also E. Gottschalk, Mathematische Annalen, vol. 115 (1938), pp. 157–158.
11. A. E. Western, unpublished manuscript.
12. J. B. Rosser, this Bulletin, vol. 45 (1939), pp. 636–640.
13. There is only one such table in print, namely: A. J. C. Cunningham, *Quadratic and Linear Tables*, London, 1927, pp. 162–170 (for $n=5$). Extensive manuscript tables have been prepared by Dr. A. E. Western.
14. D. H. Lehmer, *The lattice points of an n-dimensional tetrahedron*, Duke Mathematical Journal, vol. 7 (1940), pp. 341–353.
15. J. B. Rosser, *An additional criterion for the first case of Fermat's last theorem*, this Bulletin, vol. 47 (1941), pp. 109–110.

LEHIGH UNIVERSITY

# PROOF OF A THEOREM OF HALL

HARRY GOHEEN

In the Journal of the London Mathematical Society for July, 1937, Mr. Philip Hall gave a proof of the theorem, "If a group $G$ of order $g$ has a subgroup of order $m$ for every divisor $m$ of $g$ such that $(m, g/m)=1$, then $G$ is a soluble group." The proof is a very simple one in contrast to the rather difficult proof of the converse theorem which Hall had published in the same journal for April, 1927. It seems worthwhile to give a simpler proof of this converse.

THEOREM. *If $g$ is the order of a soluble group $G$ and $m$ is a divisor of $g$ such that $m$ and $g/m$ are relatively prime, then $G$ has a subgroup of order $m$ and furthermore all the subgroups of order $m$ in $G$ are conjugate under $G$.*

PROOF. Since the theorem is true by default for prime power groups, let us suppose that it is true for all soluble groups of orders less than $g$ and use the method of complete induction. Then the theorem is true for an invariant subgroup $G'$ of prime index $r$ in $G$.

If $r$ divides $g/m$, then $G'$ has a subgroup of order $m$. Since $G'$ is invariant and $(m, r)=1$, every element of order dividing $m$ in $G$ must be in $G'$. Hence all the subgroups of order $m$ in $G$ must be in $G'$ where by hypothesis they form a complete set of conjugates under $G'$ and consequently a complete set of conjugates under $G$.

If $r$ divides $m$, then $G'$ has a complete set of conjugates of order $m/r$. Since $G'$ is invariant, the subgroups of order $m/r$ in $G'$ are a complete set of conjugates under $G$. Let $M'$ be one of these subgroups of order $m/r$ in $G'$.

If $M'$ is the only subgroup of order $m/r$ in $G'$, then, it must be invariant in $G$. Then the quotient group is of order $rg/m$ and since $r$ is a Sylow divisor of the order of this quotient group, it has a subgroup of order $r$. Then $G$ has a subgroup of order $r \cdot m/r$ or $m$. On the other hand if $M'$ is not the only subgroup of order $m/r$ in $G'$, let it be one of $k$ subgroups of order $m/r$ in $G'$. Then $k$ divides the order $g/r$ of $G'$ and the normalizer of $M'$ in $G$ is of order $g/k$ divisible by $m$. Since this normalizer is a soluble group of order $g/k$ less than $g$, it has a subgroup of order $m$.

There remains only to show, for the case $r$ divides $m$, that all the subgroups of order $m$ in $G$ are conjugate under $G$. Let $M$ be a subgroup of order $m$. If there is no other subgroup of order $m$ in $G$, then the theorem is true by default. However, if $M_1$ is another subgroup of order $m$ in $G$ it will be shown that $M$ and $M_1$ are conjugate under $G$.

Let the crosscut of $M$ and $G'$ be $\Gamma$ of order $\gamma$ and the crosscut of $M_1$ and $G'$ be $\Gamma_1$ of order $\gamma_1$. Then since $G$ is generated by $M$ and $G'$ as well as by $M_1$ and $G'$ and since $G'$ is invariant under both $M$ and $M_1$ we have

$$(g/r)(m)/\gamma = g, \qquad (g/r)(m)/\gamma_1 = g,$$

whence

$$\gamma = \gamma_1 = m/r.$$

Then since $\Gamma$ and $\Gamma_1$ are of order $m/r$, they are conjugate under $G'$. If $S^{-1}\Gamma_1 S = \Gamma$, then $S^{-1}M_1S$ and $M$ have a common invariant subgroup $\Gamma$. If $\Gamma$ is invariant in $G$, the quotient group $G/\Gamma$ is of order $gr/m$. Since $r$ is a Sylow divisor, the subgroups $M/\Gamma$ and $S^{-1}M_1S/\Gamma$ of order $r$ are conjugate under $G/\Gamma$ and hence $M$ and $S^{-1}M_1S$ are conjugate under $G$ as was to be proved. If, however, $\Gamma$ is not invariant under $G$, its normalizer is a proper subgroup of $G$ containing $M$ and $S^{-1}M_1S$ which are therefore conjugate under the normalizer of $\Gamma$ as was to be proved.

Reed College

# ON REGULAR FAMILIES OF CURVES[1]

HASSLER WHITNEY

A family $F$ of non-intersecting curves filling a metric space is called *regular* if, in a neighborhood of any point $p$, it is homeomorphic with a family of straight lines. We have given in another paper[2] a necessary and sufficient condition, which we shall call (A′) (to be described below), that a family $F$ be regular. We shall prove in this note that the following condition is sufficient:

(A) Given any point $p$, and a direction on the curve through $p$, there is an arc $pq$ in this direction with the following property. For every $\epsilon > 0$ there is a $\delta > 0$ such that for any $p'$, with $\rho(p', p) < \delta$, there is an arc $p'q'$ of $C(p')$ such that

$$p'q' \subset V_\epsilon(pq), \qquad q' \subset V_\epsilon(q). \tag{1}$$

The condition (A′) is the same, except that after (1), we add:

(2) If $r'$ and $s'$ are on $p'q'$ and $\rho(r', s') < \delta$, then $\delta(r's') < \epsilon$.

From the present theorem it is clear that the families of curves recently defined by Niemitzki[3] are regular.

To prove the theorem, suppose (A) holds, but (A′) does not. Then the following is true:

(B) There is a point $p$, and a direction of the curve $C(p)$, such that for any arc $pq$ on $C(p)$ in this direction, there is an $\epsilon > 0$, such that for any $\delta > 0$, there is a point $p'$, with $\rho(p', p) < \delta$, such that for any $q'$ on $C(p')$,

(3) either $p'q' \not\subset V_\epsilon(pq)$, or $q' \not\subset V_\epsilon(q)$,

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[1] Presented to the Society, April 27, 1940.

[2] Annals of Mathematics, (2), vol. 34 (1933), pp. 244–270. We refer to this paper as RF. By RF, Theorem 7A, $F$ is regular as there defined. The converse is proved as follows. By Theorem 17A, there is a cross-section $S$ through $p$. In a neighborhood of $p$, the curves are orientable (this is easily seen, for instance, with the help of Theorem 9B). Choose an open subset $S'$ of $S$, and let $U$ be all points $q' = g'(q, \alpha)$, $q$ in $S'$, $|\alpha| < \epsilon$ (see RF, §15); $U$ is a neighborhood of $p$, expressed as the product of $S'$ and the open line segment $-\epsilon < \alpha < \epsilon$.

By a curve, we shall mean here the topological image of an open line segment or of a circle. We shall use $\rho(p, q)$ for distance, $\delta(A)$ for the diameter of the set $A$, and $V_\epsilon(A)$ for the set of all points $p$, $\rho(p, A) < \epsilon$. Let $C(p)$ mean the curve of $F$ through $p$.

[3] V. Niemytzki, Recueil Mathématique de Moscou, vol. 6 (48) (1939), pp. 283–292. We mention two further papers in the subject: H. Whitney, Duke Mathematical Journal, vol. 4 (1938), pp. 222–226, showing that if the curves fill a region in 3-space, a cross-section may be chosen so as to be a 2-cell; W. Kaplan, Duke Mathematical Journal, vol. 7 (1940), pp. 154–185, studying families filling the plane.

(4) or there are points $r'$, $s'$ on $p'q'$ such that $\rho(r', s') < \delta$, and $\delta(r's') \geqq \epsilon$.

Choose a point $p$ and a direction on $C = C(p)$, by (B). Choose $q$ on $C$ in this direction, by (A). Choose $\epsilon > 0$ by (B). For each positive integer $i$, choose $\delta_i$ by (A), with $\epsilon$ replaced by $\epsilon/i$. Choose $p_i$ by (B), with $\delta$ replaced by $\delta_i$. Choose $q_i$ by (A), with $p'$ replaced by $p_i$; then

$$(5) \qquad p_iq_i \subset V_{\epsilon/i}(pq), \qquad q_i \subset V_{\epsilon/i}(q).$$

By (B), as $\epsilon_i < \epsilon$, we may choose $p'_i$ and $q'_i$ on $p_iq_i$ so that

$$(6) \qquad \rho(p'_i, q'_i) < \delta_i, \qquad \delta(p'_iq'_i) \geqq \epsilon.$$

By (6), we may choose $r_i$ on $p'_iq'_i$ so that $\rho(p'_i, r_i) \geqq \epsilon/2$. By (5) and (6), we may choose a subsequence so that for some points $p'$ and $r$ on $pq$,

$$(7) \qquad p'_{\lambda_i} \to p', \qquad q'_{\lambda_i} \to p', \qquad r_{\lambda_i} \to r;$$

then $r \neq p'$. Say, for definiteness, that $r$ is in the direction of $q$ from $p'$. The set of such points $r$ which are limits of such sequences $\{r_{\lambda_i}\}$ forms a closed set, which, by (5), is in $p'q$; we shall let $r$ be the point furthest from $p'$. (It might be $q$.)

Assuming that (A) holds for the point $r$ and the direction away from $p'$, we shall arrive at a contradiction. Choose a point $s$ on $C$ in this direction from $r$, by (A). (If $C$ is a closed curve, it might happen that $s$ is on the arc $pr$.) Choose $r'$ and $s'$ on $C$ just behind and just in front of $r$, so that $r'$ is on neither $pp'$ nor $rs$, and $s'$ is not on $p'r$. We shall show that for any $\epsilon' > 0$ there is an integer $j$ and a point $s_j$ on $p'_jq'_j$ within $\epsilon'$ of $s'$; as $s$ is in $pq$, by (5), this will contradict the definition of $r$, and thus prove the theorem.

Set

$$(8) \qquad 4\eta = \min\,[\rho(r', rs), 2\epsilon'].$$

Choose $r_-$, $r_+$, $s_-$, $s_+$ on $C$ in the order $r_-r'r_+rs_-s's_+s$, so that $r_-$ is not in $rs$ and $s_+$ is not in $pr$ (if $C$ is closed), and so that

$$(9) \qquad r_-r'r_+ \subset V_\eta(r'), \qquad s_-s's_+ \subset V_\eta(s').$$

Set

$$(10) \qquad 2\epsilon'' = \min\,[\rho(pr_-, r_+q), \rho(rs_-, s_+s), \eta].$$

Using $r$, $s$, and $\epsilon''$, choose $\delta'' > 0$ by (A). By (7), we may choose $j$ so that

$$(11) \qquad \epsilon/j < \epsilon'', \quad \rho(p'_j, p') < \epsilon'', \quad \rho(q'_j, p') < \epsilon'', \quad \rho(r_j, r) < \delta''.$$

By the choice of $\delta''$, we may choose $s^*$ on $C(r_j)$ so that

$$(12) \qquad r_j s^* \subset V_{\epsilon''}(rs), \qquad s^* \subset V_{\epsilon''}(s).$$

As $r_j s^*$ is a connected set, (11), (12) and (10) show that there is a point $s_j$ on it such that $\rho(s_j, rs_- + s_+ s) \geqq \epsilon''$; hence, by (12), $\rho(s_j, s_- s_+) < \epsilon''$, and by (9),

$$(13) \qquad \rho(s_j, s') < \epsilon'' + \eta < 2\eta \leqq \epsilon'.$$

By (12) and (8),

$$(14) \qquad \rho(r', r_j s_j) > 2\eta.$$

By (5) and (11),

$$(15) \qquad p_j' q_j' \subset V_{\epsilon''}(pq).$$

By (11), (10), (5) and (9), there are points $p_j^*$ in $p_j' r_j$ and $q_j^*$ in $r_j q_j'$ such that

$$(16) \qquad \rho(p_j^*, r') < 2\eta, \qquad \rho(q_j^*, r') < 2\eta.$$

By this and (14), the arc $r_j s_j$ is contained in the arc $p_j^* r_j q_j^* \subset p_j' q_j'$. Hence $s_j \subset p_j' q_j'$, which, with (13), gives the contradiction.

HARVARD UNIVERSITY

# TRANSLATED FUNCTIONS AND STATISTICAL INDEPENDENCE[1]

R. P. AGNEW AND M. KAC

1. **Introduction.** Professor Steinhaus[2] proposed the question whether there exists a function $f(t)$ defined in $(-\infty, \infty)$ such that, for each sequence $\lambda_1, \lambda_2, \cdots$ of different real numbers, the "translated" functions $f(t+\lambda_1), f(t+\lambda_2), \cdots$ form a sequence of statistically independent functions.[3] We shall answer this question in the affirmative by giving concrete examples, and shall discuss some related problems.

2. **Notation and lemmas.** Let

$$\mathfrak{M}\{f(t)\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)dt \tag{1}$$

denote as usual the *mean value* of $f(t)$ in case the limit exists. If $E$ is a point set on the line $(-\infty, \infty)$ and $g(t)$ is the characteristic function of $E$, the mean value $\mathfrak{M}\{g(t)\}$ (if it exists) is called the *relative measure* of $E$ and will be denoted by $|E|$. A non-decreasing function $\sigma(\alpha)$ such that $\sigma(-\infty)=0$, $\sigma(+\infty)=1$ is called an *asymptotic distribution function* of $f(t)$ if

$$\left| \underset{t}{E}\{f(t) < \alpha\} \right| = \sigma(\alpha) \tag{2}$$

at each point of continuity of $\sigma(\alpha)$. A set $f_1(t), \cdots, f_n(t)$ of functions having asymptotic distribution functions $\sigma_1(\alpha), \cdots, \sigma_n(\alpha)$ is called *statistically independent* if

$$\left| \underset{t}{E}\{f_1(t) < \alpha_1; \cdots; f_n(t) < \alpha_n\} \right| = \sigma_1(\alpha_1) \cdots \sigma_n(\alpha_n) \tag{3}$$

for each set $\alpha_1, \cdots, \alpha_n$ of real numbers such that $\alpha_k$ is a point of continuity of $\sigma_k(\alpha)$. An infinite set of functions is called statistically independent if each finite subset is statistically independent.

Our proofs will be based on the following theorem of Kac and Steinhaus, loc. cit.

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[1] Presented to the Society, February 24, 1940.

[2] In a letter to one of the authors. The problem arose in connection with the theory of turbulence, but because of the outbreak of the war we have not been able to get details.

[3] See M. Kac and H. Steinhaus, *Sur les fonctions indépendantes* IV, Studia Mathematica, vol. 7 (1937), pp. 1–15; and P. Hartman, E. R. van Kampen, and A. Wintner, *Asymptotic distributions and statistical independence*, American Journal of Mathematics, vol. 61 (1939), pp. 477–486.

LEMMA 1. *A set* $f_1(t), \cdots, f_n(t)$ *of bounded functions having absolutely continuous distribution functions is statistically independent if and only if the equality*

$$\mathfrak{M}\left\{\prod_{j=1}^{n} f_j^{k_j}(t)\right\} = \prod_{j=1}^{n} \mathfrak{M}\{f_j^{k_j}(t)\} \tag{4}$$

*holds for each set* $k_1, k_2, \cdots, k_n$ *of positive integer exponents.*

Repeated use will be made of the following simple lemma.

LEMMA 2. *If* $\phi(t)$ *is a real differentiable function and constants* $\tau > 0$ *and* $\alpha > 0$ *exist such that* $\phi'(t)$ *is monotone and* $|\phi'(t)| > \alpha$ *over each of the intervals* $(-\infty, -\tau)$ *and* $(\tau, \infty)$, *then*

$$\mathfrak{M}\{e^{i\phi(t)}\} = 0. \tag{5}$$

Let $I_1$, $I_2$, and $I_3$ denote the integrals of $\exp i\phi(t)$ over $(-T, -\tau)$, $(-\tau, \tau)$, and $(\tau, T)$ respectively. Applying the second mean value theorem we obtain, when $T > \tau$,

$$\begin{aligned} I_3 &= \int_\tau^T \frac{\phi'(t)\cos\phi(t)}{\phi'(t)}\,dt + i\int_\tau^T \frac{\phi'(t)\sin\phi(t)}{\phi'(t)}\,dt \\ &= \frac{1}{\phi'(\tau)}\int_\tau^{\xi_1} \phi'(t)\cos\phi(t)\,dt + \frac{1}{\phi'(T)}\int_{\xi_1}^{T} \phi'(t)\cos\phi(t)\,dt \\ &\quad + \frac{i}{\phi'(\tau)}\int_\tau^{\xi_2} \phi'(t)\sin\phi(t)\,dt + \frac{i}{\phi'(T)}\int_{\xi_2}^{T} \phi'(t)\sin\phi(t)\,dt, \end{aligned}$$

and evaluating the last four integrals gives $|I_3| < 8/\alpha$. Likewise $|I_1| < 8/\alpha$, and since $|I_2| \leq 2\tau$ our lemma follows.

If $\phi(t)$ is a real polynomial in $t$ of positive degree, Lemma 2 is obviously applicable. The same is true if $\phi(t) = \sum_{j=1}^{n} a_j \exp(t+\lambda_j)^2$ where $\lambda_1, \cdots, \lambda_n$ are different real numbers and $a_1, \cdots, a_n$ are real coefficients not all 0.

3. **An example.** We can now establish our main result.

THEOREM 1. *The translations of the function*

$$f(t) = \cos e^{t^2} \tag{6}$$

*form a statistically independent set; that is, for each set* $\lambda_1, \cdots, \lambda_n$ *of different real numbers, the functions* $f(t+\lambda_1), \cdots, f(t+\lambda_n)$ *are statistically independent.*

Our first step is to verify that (4) holds. We find that

$$\begin{aligned}\prod_{j=1}^{n} \cos^{k_j} e^{(t+\lambda_j)^2} &= \prod_{j=1}^{n} 2^{-k_j}\left[\exp\left(ie^{(t+\lambda_j)^2}\right) + \exp\left(-ie^{(t+\lambda_j)^2}\right)\right]^{k_j}\\ &= \prod_{j=1}^{n} 2^{-k_j} \sum_{\alpha_j=0}^{k_j} C_{k_j,\alpha_j}\left[\exp\left(i\alpha_j e^{(t+\lambda_j)^2}\right)\right]\\ &\quad \cdot\left[\exp\left(-i(k_j-\alpha_j)e^{(t+\lambda_j)^2}\right)\right]\\ &= \prod_{j=1}^{n} 2^{-k_j} \sum_{\alpha_j=0}^{k_j} C_{k_j,\alpha_j} \exp i(2\alpha_j - k_j)e^{(t+\lambda_j)^2}\\ &= 2^{-k_1}\cdots 2^{-k_n} \sum_{\alpha_1=0}^{k_1}\cdots\sum_{\alpha_n=0}^{k_n} C_{k_1,\alpha_1}\cdots C_{k_n,\alpha_n}\\ &\quad \cdot \exp i \sum_{j=1}^{n} (2\alpha_j - k_j)e^{(t+\lambda_j)^2}.\end{aligned}$$

Since $\mathfrak{M}\{f\}$ is an additive homogeneous operation, we can apply Lemma 2 to show that

$$\mathfrak{M}\left\{\prod_{j=1}^{n} \cos^{k_j} e^{(t+\lambda_j)^2}\right\} = 0 \tag{7}$$

if at least one of $k_1, k_2, \cdots, k_n$ is odd; but

$$\mathfrak{M}\left\{\prod_{j=1}^{n} \cos^{k_j} e^{(t+\lambda_j)^2}\right\} = \prod_{j=1}^{n} 2^{-k_j}C_{k_j,k_j/2} \tag{8}$$

if $k_1, k_2, \cdots, k_n$ are all even. Noticing that the particular case $n=1$ gives

$$\begin{aligned}\mathfrak{M}\left\{\cos^{k} e^{(t+\lambda)^2}\right\} &= 0, &&k \text{ odd},\\ &= 2^{-k}C_{k,k/2}, &&k \text{ even},\end{aligned} \tag{9}$$

we see that the mean value of the product is the product of the mean values as in (4). Moreover the fact that the moments of $f(t) \equiv \cos \exp (t+\lambda)^2$ are the same as the moments of $\cos t$ implies[4] that the distribution function of $f(t)$ is the same as that of $\cos t$ and hence is absolutely continuous. Therefore we can apply Lemma 1 to obtain the conclusion of Theorem 1.

4. **Functions which oscillate less rapidly.** The function $\cos \exp t^2$ of Theorem 1 oscillates very rapidly as $|t| \to \infty$. A natural question

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[4] A. Wintner, *Über die statistische Unabhängigkeit der asymptotischen Verteilungsfunktionen inkommensurabler Partialschwingungen*, Mathematische Zeitschrift, vol. 36 (1933), pp. 618–629.

arises whether $\exp t^2$ can be replaced by a positive integer power of $t$, say $t^m$. If we put $f(t)=\cos t^m$, then the answer to the question of statistical independence of the translations $f(t+\lambda_1)$, $f(t+\lambda_2)$, $\cdots$ is only partially positive.

THEOREM 2. *If* $f(t)=\cos t^m$, *$m$ being a positive integer, then each set of $m$ different translations* $f(t+\lambda_1), \cdots, f(t+\lambda_m)$ *is a statistically independent set; but* $(m+1)$ *different numbers* $\lambda_1, \cdots, \lambda_{m+1}$ *can be chosen in such a way that the corresponding set of* $(m+1)$ *translations is not statistically independent.*

Proof of this theorem is similar to that of Theorem 1. For each $n=1, 2, \cdots$ we find on setting

$$P_n(t) = \prod_{j=1}^{n} \cos^{k_j}(t+\lambda_j)^m \tag{10}$$

that

$$\mathfrak{M}\{P_n(t)\} = 2^{-k_1}\cdots 2^{-k_n}\sum_{\alpha_1=0}^{k_1}\cdots\sum_{\alpha_n=0}^{k_n} C_{k_1,\alpha_1}\cdots C_{k_n,\alpha_1}\mathfrak{M}\{e^{i\phi(t)}\} \tag{11}$$

where $\phi(t)\equiv\phi(\lambda_1, \cdots, \lambda_n; k_1, \cdots, k_n; \alpha_1, \cdots, \alpha_n; t)$ is given by

$$\phi(t) = \sum_{j=1}^{n}(2\alpha_j - k_j)(t+\lambda_j)^m = \sum_{s=0}^{m}\left[C_{m,s}\sum_{j=1}^{n}(2\alpha_j - k_j)\lambda_j^s\right]t^{m-s}. \tag{12}$$

Applying Lemma 2, we see that $\mathfrak{M}\{\exp i\phi(t)\}\neq 0$ if and only if

$$\sum_{j=1}^{n}(2\alpha_j - k_j)\lambda_j^s = 0, \qquad s = 0, 1, \cdots, m-1. \tag{13}$$

In case $n=m$, (13) is a system of $m$ homogeneous linear equations in $m$ quantities $(2\alpha_j-k_j)$ whose determinant (Vandermonde's determinant) does not vanish; and if (13) holds, then $2\alpha_j=k_j$ when $j=1, 2, \cdots, m$. Hence, in case $n=m$, $\mathfrak{M}\{\exp i\phi(t)\}=0$ unless $k_1, \cdots, k_m$ are even and $\alpha_j=k_j/2$ when $j=1, 2, \cdots, m$. Therefore, in case $n=m$, formulas analogous to (7) and (8) hold and statistical independence of $f(t+\lambda_1), \cdots, f(t+\lambda_m)$ follows as in the proof of Theorem 1.

In case $m>1$ and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are linearly independent (in the sense that no one of $\lambda_1, \cdots, \lambda_n$ is a linear combination with rational coefficients of the remaining) then (13) again implies that $2\alpha_j=k_j$ and we conclude that $f(t+\lambda_1), \cdots, f(t+\lambda_n)$ are statistically independent. In case $m>1$, this shows existence of infinite sets of statistically independent translations of $\cos t^m$.

We complete the proof of Theorem 2 by proving the following theorem.

THEOREM 3. *If $m$ is a positive integer and $\lambda_1, \lambda_2, \cdots, \lambda_{m+1}$ are $m+1$ different rational numbers, then the set of $(m+1)$ translations*

$$\cos (t+\lambda_1)^m, \cos (t+\lambda_2)^m, \cdots, \cos (t+\lambda_{m+1})^m \tag{14}$$

*is not statistically independent.*

We use the formulas obtained by setting $n=m+1$ in (10), (11), (12), and (13). The system

$$\sum_{j=1}^{m+1} \lambda_j^s x_j = 0, \qquad s = 0, 1, \cdots, m-1, \tag{15}$$

of $m$ equations in $(m+1)$ unknowns $x_1, \cdots, x_{m+1}$ is such that if a nonzero rational value is assigned to one of the $x$'s, then the remaining $x$'s are uniquely determined and are rational. It follows that (15) has non-trivial solutions in integers. Let $X$ denote the set of points $x \equiv (x_1, \cdots, x_{m+1})$ in $(m+1)$-dimensional space for which $x_1, \cdots, x_{m+1}$ are integers, not all 0, satisfying (15). For each $x \in X$, let $\|x\|^2 = x_1^2 + \cdots + x_{m+1}^2$. Since $\|x\|^2$ is necessarily a positive integer, the minimum value of $\|x\|^2$ for $x \in X$ is an integer $R>0$. Choose $y \equiv (y_1, \cdots, y_{m+1}) \in X$ such that $\|y\|^2 = R$, and let $k_1, \cdots, k_{m+1}$ be defined by

$$k_j = |y_j|, \qquad j = 1, 2, \cdots, m+1. \tag{16}$$

If $0 \leqq \alpha_j \leqq k_j$ for each $j = 1, 2, \cdots, m+1$ while $0 < \alpha_j < k_j$ for at least one $j$, then

$$\sum_{j=1}^{m+1} (2\alpha_j - k_j)^2 < \sum_{j=1}^{m+1} k_j^2 = \sum_{j=1}^{m+1} y_j^2 = \|y\|^2 = R \tag{17}$$

and the minimal property of $R$ implies that the point

$$(2\alpha_1 - k_1, 2\alpha_2 - k_2, \cdots, 2\alpha_{n+1} - k_{n+1}) \tag{18}$$

is not in the set $X$; this means that at least one of the equations in (13) fails to hold, and accordingly $\mathfrak{M}\{\exp i\phi(t)\} = 0$. The remaining $2^{m+1}$ terms in the sum (11) are those for which each $\alpha_j$ is either 0 or $k_j$ and accordingly (18) is a point on the sphere (hypersphere) $S$ with center at $(0, \cdots, 0)$ and radius $R^{1/2}$. But the set $X$, being on each of $m$ independent planes (hyperplanes), is on their line of intersection; hence at most two of the points on $S$ can be in the set $X$. It follows that the point (18) fails to be in $X$, and hence that $\mathfrak{M}\{\exp i\phi(t)\} = 0$, unless the indices $\alpha_1, \cdots, \alpha_{n+1}$ are such that (i) $2\alpha_j - k_j = y_j$ for each $j=1, \cdots, m+1$ or (ii) $2\alpha_j - k_j = -y_j$ for each $j=1, \cdots, m+1$. In case (i), (13) holds so that $\phi(t)$ is independent of $t$, and we see that $\phi(t) = \theta$ where

$$\theta = \sum_{j=1}^{m+1} y_j \lambda_j^m; \tag{19}$$

in case (ii), $\phi(t) = -\theta$. Therefore

$$\begin{aligned} \mathfrak{M}\left\{ \prod_{j=1}^{m+1} \cos^{k_j}(t+\lambda_j)^m \right\} &= 2^{-k_1} \cdots 2^{-k_{m+1}}(\mathfrak{M}\{e^{i\theta}\} + \mathfrak{M}\{e^{-i\theta}\}) \\ &= 2^{-k_1} \cdots 2^{-k_{m+1}}(2\cos\theta) \neq 0, \end{aligned} \tag{20}$$

the last step being a consequence of the fact that the right member of (19) is rational and hence that $\theta$ cannot be an odd multiple of $\pi/2$.

At least one of $k_1, k_2, \cdots, k_{n+1}$ must be odd, for otherwise $z \equiv (y_1/2, \cdots, y_{m+1}/2)$ would be a point of $X$ with $\|z\|^2 = \|y\|^2/4 = R/4 < R$; therefore

$$\prod_{j=1}^{m+1} \mathfrak{M}\{\cos^{k_j}(t+\lambda_j)^m\} = 0. \tag{21}$$

Comparing (20) and (21), we see that (4) fails when $n=m+1$ and $f_j(t) = \cos(t+\lambda_j)^m$; hence Lemma 1 implies that the $(m+1)$ translations in (14) are not statistically independent. This completes proof of Theorems 3 and 2.

Anyone who wishes to check Theorem 3 by consideration of a simple special case will find that if $m=2$, $\lambda_1=1$, $\lambda_2=2$, $\lambda_3=3$ then $y \equiv (y_1, y_2, y_3)$ may be either $(1, -2, 1)$ or $(-1, 2, -1)$; that $k_1=1$, $k_2=2$, $k_3=1$; and that elementary formulas for products of cosines and use of Lemma 2 gives

$$\mathfrak{M}\{\cos(t+1)^2 \cos^2(t+2)^2 \cos(t+3)^2\} = 8^{-1}\cos 2, \tag{22}$$

which is a special case of (20).

5. **Other functions.** Our methods can be used to show that certain rapidly oscillating functions $f(t)$ other than the function $\cos\exp t^2$ of §3 have statistically independent translations. Our first solution of the problem of §1 was a "direct" (but too tedious to be given here) proof of the fact that the set of all translations of the function $\cos\exp\exp t^2$ is statistically independent. Our "direct" method uses the definition of statistical independence rather than the criterion of Lemma 1, and applies only to functions which oscillate very rapidly.

To handle such functions as cos $t^m$, where the question of statistical independence of translations is more delicate, we need the criterion of Lemma 1.

6. **Applications.** As soon as statistical independence of a set of functions is established, we can apply the general theory of addition of independent random variables.[5] As an illustration, we mention the following result: If $f(t) = \cos \exp t^2$, and $\lambda_1, \lambda_2, \cdots$ is a sequence of different real numbers, then

$$(23)\quad \lim_{n\to\infty} \left| \underset{t}{E}\{f(t+\lambda_1)+\cdots+f(t+\lambda_n) < \alpha n^{1/2}\} \right| = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\alpha} e^{-u^2} du.$$

Cornell University

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[5] See Kac and Steinhaus, loc. cit., and further references given there. Also P. Lévy, *Théorie de l'Addition des Variables Aléatoires*, Paris, Gauthier-Villars, 1937.

# NOTE ON A CERTAIN TYPE OF DIOPHANTINE SYSTEM[1]

E. T. BELL

**1. The quadratic case.** Several writers[2] have considered the problem of making $a^2x^2+dx$, $b^2x^2+ex$, $\cdots$ simultaneously squares, $a$, $d$, $b$, $e$, $\cdots$ being constants, without discussing the conditions under which an integer solution exists. The fact that the coefficients of $x^2$ are squares is of no consequence. It is shown in §2 how necessary and sufficient conditions for the existence of integer solutions in much more general problems (§2, (9), (13)) can be determined and how, when these conditions are satisfied, the solutions may be found. The details are given first for three very special cases, (1), (5), (6) below.

If all letters denote integers, and $a$, $b$, $c$, $d$ are constant, we seek necessary and sufficient conditions that the system

$$(1) \qquad ax^2 + bx = y^2, \qquad cx^2 + dx = z^2$$

shall have a solution $x$, $y$, $z$. We shall assume that $abcd \neq 0$, as the excluded possibilities require only slight modifications, all of which are included in the general method.

The required conditions are that $b$, $d$ be simultaneously representable in two forms of degree seven. Precisely, (1) has a solution in integers $x$, $y$, $z$ if and only if integers $f$, $g$, $h$, $k$, $m$, $n$ exist such that

$$(2) \qquad b = hf^2(m^2 - ag^2k^2), \qquad d = hg^2(n^2 - cf^2k^2).$$

Provided such $f$, $g$, $\cdots$ exist, the complete solution of (1) is

$$x = hf^2g^2k^2, \qquad y = hf^2gkm, \qquad z = hfg^2kn,$$

where $f$, $g$, $h$, $k$, $m$, $n$ run through all solutions of (2).

To prove this, we rewrite (1) as

$$x(ax + b) = y^2, \qquad x(cx + d) = z^2,$$

which is of the form

$$(3) \qquad xu = y^2, \qquad xv = z^2,$$

with

$$u \equiv ax + b, \qquad v \equiv cx + d.$$

The complete integer solutions of the respective equations in (3), with $x$, $y$, $z$, $u$, $v$ unrestricted variables, are

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[1] Presented to the Society, April 6, 1940.

[2] Summary of results to 1920 in L. E. Dickson, *History of the Theory of Numbers*, vol. 2, 1920, chap. 18.

$$x = r_1 s_1^2, \qquad u = r_1 t_1^2, \qquad y = r_1 s_1 t_1;$$
$$x = r_2 s_2^2, \qquad v = r_2 t_2^2, \qquad z = r_2 s_2 t_2,$$

in which all the letters with suffixes denote arbitrary integers. Since in the system (3) the two values of $x$ must be equal, we have

$$r_1 s_1^2 = r_2 s_2^2, \tag{4}$$

of which the complete integer solution is

$$r_1 = hf^2, \qquad s_1 = kg, \qquad r_2 = hg^2, \qquad s_2 = kf,$$

where $f$, $g$, $h$, $k$ are arbitrary integers. From the (restricted) values $u \equiv ax+b$, $v \equiv cx+d$ of $u$, $v$, we find the stated conditions on $b$, $d$, after the change of notation $(t_1, t_2) \to (m, n)$. The completeness of the solutions of (3), (4) implies the completeness of the solution of (1) when $f, g, \cdots$ run through all solutions of (2). This treatment is possible only because (3), equivalent to (1), is a pure multiplicative system. A full discussion of such systems, with a non-tentative method for finding their complete solutions, was given in a former paper.[3]

The above solutions of (3), (4) were obtained by this method.

Proceeding similarly with

$$ax^2 + bx = y^2, \qquad cx^2 + dx = z^2, \qquad ex^2 + jx = w^2, \tag{5}$$

in which $a$, $b$, $c$, $d$, $e$, $j$ are constant integers, different from zero (not an essential restriction), we find that all solutions $x$, $y$, $z$, $w$ are given by

$$x = hf^2g^2k^2p^2q^2r^2s^2, \qquad y = hf^2gkpq^2rs^2l,$$
$$z = hf^2g^2k^2pqrsm, \qquad w = hfg^2kpq^2r^2sn,$$

where $h, \cdots, m$ run through all integer solutions of

$$b = hf^2q^2s^2(l^2 - ak^2g^2p^2r^2), \qquad d = hf^2g^2k^2(m^2 - cp^2q^2r^2s^2),$$
$$j = hg^2q^2r^2(n^2 - ek^2f^2p^2s^2).$$

Here the simultaneous representation of $b$, $d$, $j$ which is necessary and sufficient for the existence of a solution of (5) is by forms of degree fifteen. This may suggest that anything approaching a complete solution of the systems quoted[2] is not to be expected in the near future. However, any number of solvable systems can be constructed immediately from the conditions, and likewise for such systems having infinities of solutions.

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[3] American Journal of Mathematics, vol. 55 (1933), pp. 50–66.

The pure multiplicative system corresponding to (3) for (5) is

$$r_1 s_1^2 = r_2 s_2^2 = r_3 s_3^2,$$

of which the complete solution is

$$r_1 = hf^2q^2s^2, \qquad r_2 = hf^2g^2k^2, \qquad r_3 = hg^2q^2r^2,$$
$$s_1 = kgpr, \qquad s_2 = pqrs, \qquad s_3 = kfps.$$

The special features of (1), (5) and the systems cited[2] which make it possible to treat the equations as a pure multiplicative system are that each equation lacks a constant term in one of its members and that its other member consists of a single term. The second of these restrictions is removed in §2. It is immaterial that the single term has the coefficient unity; for multiplicative systems with arbitrary constant integer coefficients can also be solved completely by a straightforward, non-tentative process.[3]

For example, the pure multiplicative system corresponding to the system

$$ax^2 + bx = ry^2, \qquad cx^2 + dx = sz^2, \tag{6}$$

in which the constant integers $a$, $b$, $c$, $d$, $r$, $s$ are all different from zero, is

$$xu = ry^2, \qquad xv = sz^2. \tag{7}$$

We first solve (7) as if all the letters were variable integers, and find the complete solution

$$\begin{aligned} x &= tr_1r_2^2r_4^2s_1s_2^2s_4^2x_1x_2^2y_1^2y_2^2, \quad y = r_2s_1s_2s_4x_1x_2y_1^2y_2z_1, \\ z &= r_1r_2r_4s_2x_1x_2y_1^2y_2z_2, \\ u &= r_2r_3s_1x_1y_1^2z_1^2, \qquad v = r_1s_2s_3x_1y_2^2z_2^2, \\ r &= tr_1r_2r_3r_4^2, \qquad s = ts_1s_2s_3s_4^2, \end{aligned} \tag{8}$$

in which $t$ and all the letters with suffixes denote arbitrary integers. If now $r$, $s$ are constant, we resolve them in all possible ways into products of five factors as indicated above, and it suffices (as appears in the method cited[3]) to take $t$ equal to the G.C.D. of $r$, $s$. With each such pair of resolutions of $r$, $s$ there is associated a solution $x$, $y$, $z$, $u$, $v$ as given in (8); all the solutions of (7), with $r$, $s$ constant, fall into sets determined by these pairs. Proceeding as in (1)–(4), we find that (6) has an integer solution $x$, $y$, $z$ if and only if integers $x_1$, $x_2$, $y_1$, $y_2$, $z_1$, $z_2$ exist such that, for some resolution of $r$, $s$ as in (8),

$$b = r_2 s_1 x_1 y_1^2 (r_3 z_1^2 - a t r_1 r_2 r_4^2 s_2^2 s_4^2 x_2^2 y_2^2),$$

$$d = r_1 s_2 x_1 y_2^2 (s_3 z_2^2 - c t r_2^2 r_4^2 s_1 s_2 s_4^2 x_2^2 y_1^2);$$

and if these conditions are satisfied, the complete solution $x$, $y$, $z$ of (7) is as given in (8), in which $t$ and the $r_i$, $s_i$ refer to all resolutions of the constants $r$, $s$ of the type indicated. The extension to a system of $n$ $(>2)$ quadratics is immediate.

2. **The general case.** The notation is as follows.

$P_i(x)$ is a polynomial in $x$ of degree $m_i + r_i$, $m_i > 0$, $r_i > 0$, with constant integer coefficients.

The term of lowest degree in $P_i(x)$ is $a_i x^{m_i}$, $a_i \neq 0$.

$$P_i(x) \equiv x^{m_i} Q_i(x), \qquad Q_i(x) \equiv R_i(x) + a_i,$$

so that $R_i(x)$ is a polynomial in $x$ of degree $r_i > 0$ with constant integer coefficients and no constant term.

$b_i$, $n_i$, $s$ denote constant integers, $b_i \neq 0$, $n_i > 0$, $s > 1$.

By the method exemplified in §1, we may obtain necessary and sufficient conditions that the system

$$(9) \qquad P_i(x) = b_i y_i^{n_i}, \qquad i = 1, \cdots, s,$$

shall have a solution in integers $x$, $y_1, \cdots, y_s$. The conditions concern the representability of the constant integers $a_1, \cdots, a_s$ in certain forms; the complete solution of (9) is known when all these representations of $a_1, \cdots, a_s$ are known.

Rewriting (9) as

$$(10) \qquad x^{m_i} [R_i(x) + a_i] = b_i y_i^{n_i}, \qquad i = 1, \cdots, s,$$

we associate with (10) the pure multiplicative system

$$(11) \qquad x^{m_i} z_i = b_i y^{n_i}, \qquad i = 1, \cdots, s.$$

The complete integer solution $x$, $z_i$, $y_i$, $i = 1, \cdots, s$, of (11) is found as in §1 in terms of $t$, $\equiv t(m_1, \cdots, m_s, n_1, \cdots, n_s)$, parameters; say the solution is

$$(12) \qquad x = f(u_1, \cdots, u_t), \quad z_i = g_i(u_1, \cdots, u_t), \quad y_i = h_i(u_1, \cdots, u_t),$$
$$i = 1, \cdots, s.$$

The $f$, $g_i$, $h_i$ are power products in $u_1, \cdots, u_t$ with integer coefficients which are power products formed from the factors in certain resolutions of the constants $b_1, \cdots, b_s$ into products; the forms of $f$, $g_i$, $h_i$ and the coefficients described depend only on the given constant integers $m_i$, $n_i$. Necessary and sufficient conditions that the system (9) have a solution in integers $x$, $y_i$, $i=1, \cdots, s$, are found as in §1, and refer to the representation of the constants $a_i$ in certain forms. Comparing (10), (11), we find

$$a_i = g_i(u_1, \cdots, u_t) - R_i(f(u_1, \cdots, u_t)), \qquad u = 1, \cdots, s,$$

as the required conditions. When these are satisfied, the solutions are obtained from (12).

An obvious modification gives necessary and sufficient conditions for the existence of an integer solution, and so on, of the more general system

$$c_iP_i(x) = b_iT_i(y_i), \qquad i = 1, \cdots, s, \tag{13}$$

in which $c_i$, $b_i$ are constants, $P_i(x)$ is as before, and $T_i(y_i)$ is a polynomial in $y_i$ of degree $n_i+t_i$, $n_i>0$, $t_i>0$, in which the term of lowest degree is $d_iy_i^{t_i}$, $d_i\neq 0$. The associated pure multiplicative system is

$$c_ix^{m_i}z_i = b_iy_i^{n_i}w_i, \qquad i = 1, \cdots, s;$$

the conditions concern the representability of the $a_i$, $d_i$ in certain forms. As in all cases the conditions are both necessary and sufficient, the complete solution of the given system is equivalent to finding the total representation of the coefficients $a_i$, $d_i$ in a determinable system of forms.

California Institute of Technology

# POWER SERIES THE ROOTS OF WHOSE PARTIAL SUMS LIE IN A SECTOR[1]

LOUIS WEISNER

If the roots of the partial sums of a power series $f(z)=\sum a_n z^n$ lie in a sector with vertex at the origin and aperture $\alpha<2\pi$, the power series cannot have a positive finite radius of convergence.[2] But if $f(z)$ is an entire function, the roots of its partial sums may lie in such a sector. The question arises: what restrictions are imposed on $f(z)$ by the requirement that $\alpha$ be sufficiently small, say $\alpha<\pi$? According to a theorem of Pólya the order of $f(z)$ must be not greater than 1 if the radius of convergence of the power series is positive.[3] Without this assumption the investigation which follows shows that *if* $\alpha<\pi$, $f(z)$ *is an entire function of order* 0. This result was obtained by Pólya for the case in which $\alpha=0$.[4]

LEMMA. *If the complex numbers* $z_1, \cdots, z_n$ $(z_1\cdots z_n\neq 0)$ *lie in a sector with vertex at the origin and aperture* $\alpha<\pi$, *then*

$$\frac{n\cos\alpha/2}{\left|\sum_{k=1}^{n} z_k^{-1}\right|} \leqq \left|z_1\cdots z_n\right|^{1/n} \leqq \frac{1}{n}\sec\alpha/2\left|\sum_{k=1}^{n} z_k\right|. \tag{1}$$

*When* $\alpha=0$ *equality occurs if and only if* $z_1=\cdots=z_n$. *When* $\alpha>0$ *equality occurs if and only if* $n$ *is even and* $n/2$ *of the numbers are equal to* $re^{i\phi}$ $(r>0; 0\leqq\phi<2\pi)$ *and the other* $n/2$ *numbers are equal to* $re^{i(\phi+\alpha)}$.

Suppose first that the sector is $-\alpha/2\leqq \operatorname{am} z\leqq\alpha/2$. Let the $n$ numbers be

$$z_k=\left|z_k\right|e^{i\theta_k}, \qquad -\alpha/2\leqq\theta_k\leqq\alpha/2;\ k=1,\cdots,n.$$

Since

$$\sum_{k=1}^{n} z_k=\sum_{k=1}^{n}\left|z_k\right|\cos\theta_k+i\sum_{k=1}^{n}\left|z_k\right|\sin\theta_k, \tag{2}$$

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[1] Presented to the Society, April 27, 1940.

[2] This follows from Jentzsch's theorem: every point on the circle of convergence of a power series is a limit point of roots of its partial sums. See R. Jentzsch, *Untersuchungen zur Theorie der Folgen analytischer Funktionen*, Acta Mathematica, vol. 41 (1917), p. 219; E. C. Titchmarsh, *Theory of Functions*, 1932, p. 238.

[3] G. Pólya, *Ueber Annäherung durch Polynome deren sämtliche Wurzeln in einen Winkelraum fallen*, Nachrichten der Gesellschaft der Wissenschaften zu Göttingen, 1913, pp. 325–330.

[4] G. Pólya, *Ueber Annäherung durch Polynome mit lauter reellen Wurzeln*, Rendiconti del Circolo Matematico di Palermo, vol. 36 (1913), pp. 279–295.

$$(3) \qquad \left|\sum_{k=1}^{n} z_k\right| \geqq \sum_{k=1}^{n} |z_k| \cos\theta_k \geqq \cos\alpha/2 \sum_{k=1}^{n} |z_k|.$$

Now

$$(4) \qquad \frac{1}{n}\sum_{k=1}^{n} |z_k| \geqq |z_1 \cdots z_n|^{1/n}.$$

Consequently

$$\frac{1}{n}\left|\sum_{k=1}^{n} z_k\right| \geqq \cos\alpha/2\, |z_1 \cdots z_n|^{1/n}.$$

Since the numbers $z_1^{-1}, \cdots, z_n^{-1}$ also lie in the sector $-\alpha/2 \leqq \operatorname{am} z \leqq \alpha/2$, we have

$$\frac{1}{n}\left|\sum_{k=1}^{n} z_k^{-1}\right| \geqq \cos\alpha/2\, |z_1^{-1} \cdots z_n^{-1}|^{1/n}.$$

Combining the last two inequalities, (1) results.

When $\alpha=0$, (1) reduces to the well known relation among the harmonic, geometric and arithmetic means of $n$ positive numbers. Here equality occurs if and only if $z_1 = \cdots = z_n$.

If equality occurs in (1) it also occurs in (3) and (4). By (4), $|z_1| = \cdots = |z_n|$. By (3), $\cos\theta_k = \cos\alpha/2$; hence $\theta_k = \pm\alpha/2$ $(k=1, \cdots, n)$. By (2), $\sum_{k=1}^{n} \sin\theta_k = 0$. Therefore if $\alpha > 0$, $n$ must be even, and $n/2$ of the numbers equal $re^{-i\alpha/2}$, while the other $n/2$ numbers equal $re^{i\alpha/2}$. Conversely, when these conditions are satisfied, equality is attained in (1).

If the numbers are in the sector $\phi \leqq \operatorname{am} z \leqq \operatorname{am}(\phi+\alpha)$, we apply the transformation

$$z' = e^{-i(\alpha/2+\phi)}z,$$

which rotates this sector into the sector $-\alpha/2 \leqq \operatorname{am} z \leqq \alpha/2$ without affecting the value of any member of (1).

THEOREM. *If, for each $n \geqq n_0$, the roots of the partial sum of degree $n$ of the formal power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ lie in some sector with vertex at the origin and aperture $\alpha < \pi$,[5] then $f(z)$ is an entire function of order* 0.

The case in which $f(z)$ is a polynomial is trivial and is excluded from consideration. We shall suppose $a_0 \neq 0$; otherwise a power of $z$ could be removed from $f(z)$ without affecting the theorem. Let

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[5] While $\alpha$ is independent of $n$, we do not require that there shall be one sector which contains the roots of all the partial sums of degree $n \geqq n_0$; the lines bounding the sector may be different for different values of $n$.

$$f_n(z) = \sum_{k=0}^{n} a_k z^k, \qquad n \geq n_0.$$

By the Gauss-Lucas theorem the roots of $f_n'(z)$ are also in the sector which contains the roots of $f_n(z)$, and the only roots of $f_n'(z)$ that lie on the boundary of the sector are multiple roots of $f_n(z)$; hence $f_n'(0) \neq 0$. Repeated applications of this argument yield the result that $a_k \neq 0$ $(k=0, 1, \cdots)$.

According to the lemma, if $z_1, \cdots, z_n$ denote the zeros of $f_n(z)$,

$$nc\left|\frac{a_0}{a_1}\right| \leq \left|\frac{a_0}{a_n}\right|^{1/n} \leq \frac{1}{nc}\left|\frac{a_{n-1}}{a_n}\right|, \qquad c = \cos \alpha/2;\ n \geq n_0. \tag{5}$$

From the first two members of this inequality it follows that $|a_n|^{-1/n} \to \infty$ with $n$. Therefore $f(z)$ is an entire function. If $\rho$ is its order,

$$\frac{1}{\rho} = \liminf_{n\to\infty} \frac{\log |a_n|^{-1}}{n \log n}. \tag{6}$$

From the last two members of (5) we have

$$\frac{1}{n}\log\frac{1}{|a_n|} - \frac{1}{n-1}\log\frac{1}{|a_{n-1}|} \geq \frac{1}{n(n-1)}\log |a_0| + \frac{1}{n-1}\log nc. \tag{7}$$

Let $m = \max(n_0, 4)$, $n > m$. Substituting $n = m, m+1, \cdots, n$ in (7), and adding, we obtain

$$\frac{1}{n}\log\frac{1}{|a_n|} \geq \frac{1}{m-1}\log\frac{1}{|a_{m-1}|} + \log |a_0| \sum_{s=m}^{n}\frac{1}{s(s-1)} + \sum_{s=m}^{n}\frac{\log s}{s-1} + \log c \sum_{s=m}^{n}\frac{1}{s-1}.$$

Now

$$\sum_{s=m}^{n}\frac{\log s}{s-1} > \sum_{s=m}^{n}\frac{\log(s-1)}{s-1} > \frac{1}{2}\log^2 n - \frac{1}{2}\log^2(m-1),$$

$$\sum_{s=m}^{n}\frac{1}{s-1} > \log n - \log(m-1).$$

Consequently

$$\frac{1}{n}\log\frac{1}{|a_n|} > A + \frac{1}{2}\log^2 n + \log c \log n, \tag{8}$$

where $A$ is bounded as $n \to \infty$. Comparing (6) and (8), we conclude that $\rho = 0$.

Hunter College of the City of New York

# ON A RESULT OF HUA FOR CUBIC POLYNOMIALS[1]

ALVIN SUGAR

In a paper by L. K. Hua,[2] we find the following principal result.

THEOREM. *For any positive integer* $\epsilon$, *every integer can be expressed in infinitely many ways as a sum of seven values of the cubic function*

$$f(x) = \epsilon(x^3 - x)/6 + x$$

*for integral values of* $x$; *also, every integer can be expressed in infinitely many ways as a sum of seven values of*

$$F(x) = (x^3 - x)/6$$

*for integral values of* $x$.[3]

In this paper we get a better result by applying a known identity for cubes to generalizations of the above polynomials. We state our results in the following two theorems. Note that, in Theorem 1, $\epsilon$ may be positive or negative, or, for that matter, zero.

Unless otherwise stated all letters in this paper stand for integers, positive, negative, or zero.

THEOREM 1. *For any* $\epsilon$, $c$, *and any* $k$ *prime to* $\epsilon$, *every integer can be expressed in infinitely many ways as a sum of five values of the function*

$$p(x) = \epsilon(x^3 - x)/6 + kx + c$$

*for integral values of* $x$.

THEOREM 2. *For any* $k$ *and* $c$, *every integer can be expressed as a sum of four values of the function*

$$P(x) = (x^3 - x)/6 + kx + c$$

*for integral values of* $x$.

Theorem 1 is trivially true when $\epsilon = 0$. For, in this case, $(\epsilon, k) = 1$ implies $k = 1$. Henceforth we take $\epsilon \neq 0$.

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[1] Presented to the Society, April 27, 1940.

[2] *On the representation of integers by the sums of seven cubic functions*, Tôhoku Mathematical Journal, vol. 41 (1935–1936), pp. 361–366.

[3] Hua fails to mention in his formulation of this theorem whether his $\epsilon$ may take negative values. It seems that Hua implicitly assumed $\epsilon$ positive, as was noted by Pall in his review of the Hua paper in the Zentralblatt für Mathematik, vol. 14, p. 10. (This assumption was probably unnecessary, however.)

Let us first prove Theorem 1 for Hua's polynomial $f(x)$. Since $x^3-x$ is divisible by 6, it is evident that

$$\frac{\epsilon}{6}(x^3-x)+x\equiv x \pmod{|\epsilon|}, \tag{1}$$

for any integer $x$. Hence it is evident that there exist $|\epsilon|$ incongruent values modulo $|\epsilon|$ of $f$. Consequently for any integer $n$ it is true that there exist integers $t$ and $m$ such that

$$n=f(t)+m\epsilon. \tag{2}$$

We can show that $m\epsilon$ is a sum of four values of $f$ by employing the following identity:

$$f(m+1)+f(m-1)+2f(-m)=m\epsilon. \tag{3}$$

From (2) and (3), we get

$$n=f(t)+f(m+1)+f(m-1)+2f(-m), \tag{4}$$

which tells us that every integer is a sum of five values of $f(x)$, $x$ integral. Since we can replace $t$ in (2) by any number congruent to it modulo $|\epsilon|$, it follows that the representation (4) can be made in infinitely many ways.

With but slight modification of this proof, we can obtain the more general result stated in Theorem 1. In this case (1) becomes $p(x)\equiv kx+c \pmod{|\epsilon|}$. And, since $k$ is prime to $\epsilon$, $kx+c$ ranges over a complete set of residues modulo $|\epsilon|$ when $x$ does. Now if we replace $f$ by $p$ in (3), the right member of this identity becomes $m\epsilon+4c$. From these facts we can deduce Theorem 1.

If we replace $f$ by $P$ in (3), the right member of our identity becomes $m+4c$, which ranges over all of the integers with $m$; hence we have proved Theorem 2. It is evident from Theorem 1, with $\epsilon=1$, that we could duplicate Hua's "infinitely many ways" in our Theorem 2 if we increased the number of summands to five.

COOPER UNION INSTITUTE OF TECHNOLOGY

# DIVISORS OF ZERO IN MATRIC RINGS

NEAL H. MCCOY

1. **Introduction.** An element $a$ of a ring $S$ is a *divisor of zero* in $S$ if there exists a nonzero element $x$ of $S$ such that $ax=0$, or a nonzero element $y$ of $S$ such that $ya=0$. The purpose of the present note is to obtain several theorems about divisors of zero in matric rings which, although quite elementary in character, have apparently not been previously noted. Throughout, unless otherwise stated, $R$ will be used to denote an arbitrary commutative ring with unit element. Let $R_n$ denote the ring of all matrices of order $n$ with elements in $R$, and $R[\lambda]$ the ring of polynomials in the indeterminate $\lambda$ with coefficients in $R$. If $A$ is an element of $R_n$, we shall denote by $f(\lambda)=|\lambda-A|$ the *characteristic polynomial* of $A$, and thus $f(\lambda)$ is an element of $R[\lambda]$, with leading coefficient 1. The ideal $\mathfrak{m}$ of all elements $g(\lambda)$ such that $g(A)=0$ is the *minimum ideal* of $A$. If the minors of $\lambda-A$ of order $n-1$ are denoted by $h_{ij}(\lambda)$ $(i, j=1, 2, \cdots, n)$, it has been shown in a previous paper[1] that $g(\lambda)\equiv 0$ $(\mathfrak{m})$, if and only if

$$g(\lambda)h_{ij}(\lambda) \equiv 0 \quad (f(\lambda)), \qquad i, j = 1, 2, \cdots, n.$$

If $R[A]$ denotes the subring of $R_n$ generated by $A$ together with the unit element of $R_n$, which we identify with the unit element of $R$, then the elements of $R[A]$ are the polynomials in $A$ with coefficients in $R$. It is quite easy to show that $A$ is a divisor of zero in $R_n$ if and only if $|A|$ is a divisor of zero in $R$. But we shall show, in §2, that $A$ is actually a divisor of zero in $R[A]$ if it is a divisor of zero in $R_n$—a fact which is almost trivial if $R$ is a field. This theorem is used in the following section in which we define, by means of the Sylvester determinant, the *resultant* $\mathcal{R}(f, g)$ of two elements $f(\lambda)$ and $g(\lambda)$ of $R[\lambda]$ and show, following Frobenius, that if $f(\lambda)$ has leading coefficient unity,

$$\mathcal{R}(f, g) = | g(A) |,$$

where $A$ is *any* matrix having $f(\lambda)$ as characteristic polynomial. It then follows readily that an element $g(\lambda)$ of $R[\lambda]$ is *prime to*[2] $\mathfrak{m}$ if and only if it is prime to $f(\lambda)$, which in turn is true if and only if $\mathfrak{R}(f,g)$ is not a divisor of zero in $R$.

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[1] Neal H. McCoy, *Concerning matrices with elements in a commutative ring*, this Bulletin, vol. 45 (1939), pp. 280–284. Hereafter, this paper will be referred to as M.

[2] For definitions of this term, see W. Krull, *Idealtheorie in Ringen ohne Endlichkeitsbedingung*, Mathematische Annalen, vol. 101 (1929), pp. 729–744. This will be referred to later as K. A definition will also be found in §3 of the present paper.

If $A$ is such that $\mathfrak{m}=(f(\lambda))$, we may say, following Sylvester, that $A$ is *not derogatory*. Let us, as above, denote the minors of $\lambda-A$ of order $n-1$ by $h_{ij}(\lambda)$, and set

$$\mathfrak{a} = (h_{11}(\lambda), h_{12}(\lambda), \cdots, h_{nn}(\lambda)).$$

We conclude our remarks by showing that, if each ideal in $R$ has a finite basis, $A$ is not derogatory if and only if $\mathfrak{a}$ contains an element of $R$ which is not a divisor of zero in $R$.

## 2. Divisors of zero in $R[A]$. We shall now prove

THEOREM 1. *An element $A$ of $R_n$ is a divisor of zero in $R[A]$ if and only if $|A|$ is a divisor of zero in $R$.*

The necessity follows easily by a familiar argument. Suppose that $A$ is a divisor of zero in $R[A]$ and hence that there exists a nonzero element $X$ of $R[A]$ such that $AX=0$. In other words, if $A=(a_{ij})$, the following system of equations,

$$\sum_{j=1}^{n} a_{ij}x_j = 0, \qquad i = 1, 2, \cdots, n,$$

has a solution $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ in $R$ with some $\bar{x}_\alpha \neq 0$. Multiply these equations in order by the respective cofactors of $a_{1\alpha}, a_{2\alpha}, \cdots, a_{n\alpha}$ in $|A|$, and add. There results $\bar{x}_\alpha |A| = 0$, and hence $|A|$ is a divisor of zero in $R$.

We now pass to the second part of the proof. Accordingly, we assume that there is an element $k\neq 0$ of $R$ such that $k|A|=0$, and shall show that $A$ is a divisor of zero in $R[A]$. Our method is to assume that $A$ is not a divisor of zero in $R[A]$ and obtain a contradiction. First, we prove the following

LEMMA. *If $B$ and $C$ are elements of $R_n$ and $k$ is an element of $R$ such that $kB=kC$, then $k|B|=k|C|$.*

This follows from the fact that $kb_{ij}=kc_{ij}$, where $B=(b_{ij})$ and $C=(c_{ij})$, together with the following calculation. If $b=b_{i_1 1}b_{i_2 2}\cdots b_{i_n n}$ is any term, except possibly for sign, in the expansion of $|B|$, and $c=c_{i_1 1}c_{i_2 2}\cdots c_{i_n n}$ the corresponding term in the expansion of $|C|$, then

$$\begin{aligned} kb &= (kb_{i_1 1})b_{i_2 2}\cdots b_{i_n n} = (kc_{i_1 1})b_{i_2 2}\cdots b_{i_n n} \\ &= c_{i_1 1}(kb_{i_2 2})b_{i_3 3}\cdots b_{i_n n} = c_{i_1 1}c_{i_2 2}(kb_{i_3 3})\cdots b_{i_n n}, \end{aligned}$$

and a continuation of this process shows finally that this is equal to $kc$.

We now turn to the proof of Theorem 1. The matrix $A$ satisfies its characteristic equation,[3] and hence there is a relation of the form

$$A^n + a_1A^{n-1} + \cdots + a_n = 0, \tag{1}$$

where $a_n = \pm|A|$, and thus $ka_n = 0$. Hence, from (1), we get an equation of the form

$$kA(A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}) = 0.$$

But, by our assumption that $A$ is not a divisor of zero in $R[A]$, it follows that

$$k(A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}) = 0, \tag{2}$$

or, as we may write it,

$$kA(A^{n-2} + a_1A^{n-3} + \cdots + a_{n-2}) = -ka_{n-1}.$$

Now, applying the lemma, we see that

$$k|A| \cdot |A^{n-2} + a_1A^{n-3} + \cdots + a_{n-2})| = k(-1)^n a_{n-1}^n.$$

But $k|A| = 0$, and therefore $ka_{n-1}^n = 0$. Suppose $ka_{n-1}^j \neq 0$, but $ka_{n-1}^{j+1} = 0$ $(j \geqq 0)$. Then $k' = ka_{n-1}^j \neq 0$ has the property that $k'a_n = k'a_{n-1} = 0$, and hence from (2),

$$k'A(A^{n-2} + \cdots + a_{n-2}) = 0.$$

A repetition of this argument shows that there is an element $k'' \neq 0$ of $R$, which is a multiple of $k'$, such that $k''a_n = k''a_{n-1} = k''a_{n-2} = 0$. Continuing this process we finally get an element $l \neq 0$ of $R$ such that $lA = 0$, which violates our assumption that $A$ is not a divisor of zero in $R[A]$. The theorem is therefore established.

Our method of proof also establishes the following

COROLLARY. *If $A$ is a divisor of zero in $R_n$, there exists an element $g(A)$ of $R[A]$ such that*

$$Ag(A) = 0, \qquad g(A) \neq 0,$$

*where $g(\lambda)$ is an element of $R[\lambda]$ with the property that, for some positive integer $i$,*

$$\lambda^i g(\lambda) = kf(\lambda),$$

*$f(\lambda)$ being the characteristic polynomial of $A$ and $k$ an element of $R$.*

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[3] See M, p. 282.

3. **Resultants and their properties.** We first make some preliminary remarks.[4] Let $\mathfrak{a}$ be an ideal in the commutative ring $R$, with unit element. An element $b$ of $R$ is said to be *prime to* $\mathfrak{a}$ if $bx \equiv 0$ $(\mathfrak{a})$ implies that $x \equiv 0$ $(\mathfrak{a})$, otherwise $b$ is *not prime to* $\mathfrak{a}$. An element $b$ is prime to the element $a$ if $b$ is prime to the principal ideal $(a)$. If every element of an ideal $\mathfrak{b}$ is not prime to $\mathfrak{a}$, but every proper divisor of $\mathfrak{b}$ contains an element prime to $\mathfrak{a}$, then $\mathfrak{b}$ is necessarily a prime ideal which we shall call a *maximal prime ideal belonging to* $\mathfrak{a}$. Krull has shown, by methods which are necessarily transfinite if $R$ is unrestricted, that every ideal has maximal prime ideal divisors and that *the elements of R which are not prime to* $\mathfrak{a}$ *are precisely the elements which are contained in some maximal prime ideal belonging to*[5] $\mathfrak{a}$.

We now consider the ring $R[\lambda]$, where $\lambda$ is an indeterminate, and let

$$f(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n, \qquad g(\lambda) = b_0\lambda^m + b_1\lambda^{m-1} + \cdots + b_m$$

be two elements of $R[\lambda]$, the leading coefficient of $f(\lambda)$ being 1. In case $R$ is a field, the vanishing of the resultant of $f(\lambda)$ and $g(\lambda)$ furnishes a necessary and sufficient condition that $f(\lambda)$ and $g(\lambda)$ be not prime to each other.[6] We shall now obtain an analogous result in case $R$ is unrestricted.

Let us denote the resultant of $f(\lambda)$ and $g(\lambda)$ by $\mathfrak{R}(f, g)$ and define it in a purely formal way by the Sylvester determinant

$$\mathfrak{R}(f, g) = \left| \begin{array}{cccccccc} 1 & a_1 & \cdot & \cdot & \cdot & a_n & & \\ & 1 & a_1 & \cdot & \cdot & \cdot & a_n & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & 1 & a_1 & \cdot & \cdot & \cdot & a_n \\ b_0 & b_1 & \cdot & \cdot & \cdot & b_m & & \\ & b_0 & b_1 & \cdot & \cdot & \cdot & b_m & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & b_0 & b_1 & \cdot & \cdot & \cdot & b_m \end{array} \right|,$$

the blank spaces consisting of zeros.

The connection with the preceding part of the paper is indicated by

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[4] Cf. K, p. 732.

[5] In fact, it can be shown that if in $R$ every ideal has a finite basis, the maximal prime ideals belonging to $\mathfrak{a}$ are precisely the maximal among the prime ideals associated with the primary ideals in a normalized decomposition of $\mathfrak{a}$ into the intersection of primary ideals. All of these prime ideals are the *prime ideals belonging to* $\mathfrak{a}$.

[6] In case R is a field, this implies that $f(\lambda)$ and $g(\lambda)$ have a common factor of degree at least one.

THEOREM 2. *If $A$ is any element of $R_n$ which has characteristic polynomial $f(\lambda)$, then*

$$\mathfrak{R}(f, g) = |g(A)|. \tag{3}$$

This is an almost immediate consequence of Frobenius' theorem for the case in which $R$ is the field of complex numbers.[7] For if we consider the elements of a matrix $A$, as well as the coefficients of $g(\lambda)$ to be independent complex variables, relation (3) is seen to be an identity in these variables and thus remains true if these variables are replaced by elements of[8] $R$.

Now let $A$ be any fixed element of $R_n$ with characteristic polynomial $f(\lambda)$. The ideal $\mathfrak{m}$ of all elements $h(\lambda)$ of $R[\lambda]$ such that $h(A)=0$ is the minimum ideal of $A$. It is now clear that an element $g(\lambda)$ of $R[\lambda]$ is prime to $\mathfrak{m}$ if and only if $g(A)$ is not a divisor of zero in $R[A]$, and Theorem 1 states that this is the case if and only if $|g(A)|$ is not a divisor of zero in $R$. We shall now prove

THEOREM 3. *An element $g(\lambda)$ of $R[\lambda]$ is prime to $\mathfrak{m}$ if and only if it is prime to $f(\lambda)$.*

By the preceding remarks, and Theorem 2, we see that $g(\lambda)$ is prime to $\mathfrak{m}$ if and only if $\mathfrak{R}(f, g)$ is not a divisor of zero in $R$. But clearly $\mathfrak{R}(f, g)$ depends only on $f(\lambda)$ and $g(\lambda)$ and not on the particular choice of matrix $A$ with characteristic polynomial $f(\lambda)$. Let $A'$ be a matrix with characteristic polynomial $f(\lambda)$, and such that its minimum ideal is[9] $\mathfrak{m}'=(f(\lambda))$. Then $g(\lambda)$ is prime to $\mathfrak{m}'$ if and only if $\mathfrak{R}(f, g)$ is not a divisor of zero in $R$. But it was found above that this is precisely the condition that $g(\lambda)$ be prime to $\mathfrak{m}$, and the theorem is established.

Since $g(\lambda)$ is prime to $\mathfrak{m}$ if and only if $\mathfrak{R}(f, g)$ is not a divisor of zero in $R$, we have incidentally proved the following result which is independent of the theory of matrices:

THEOREM 4. *If $R$ is a commutative ring with unit element and $\lambda$ is an indeterminate, an element $g(\lambda)$ of $R[\lambda]$ is prime to the element $f(\lambda)$, with leading coefficient unity, if and only if $\mathfrak{R}(f, g)$ is not a divisor of zero in $R$.*

**4. A characterization of matrices which are not derogatory.** Let

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[7] G. Frobenius, *Ueber lineare Substitutionen und bilineare Formen*, Journal für die reine und angewandte Mathematik, vol. 84 (1878), p. 11.

[8] Cf. M, p. 281.

[9] This will certainly be the case if $A'$ is the companion matrix of $f(\lambda)$. See relation (4) of §4.

$A$ be an element of $R_n$, with characteristic polynomial $f(\lambda)$ and minimum ideal $\mathfrak{m}$. Following Sylvester, we may say that $A$ is *not derogatory* if $\mathfrak{m} = (f(\lambda))$. Let $h_{ij}(\lambda)$ denote the minors of $\lambda - A$ of order $n-1$. It was remarked above that an element $g(\lambda)$ of $R[\lambda]$ is an element of $\mathfrak{m}$ if and only if

$$g(\lambda)h_{ij}(\lambda) \equiv 0 \quad (f(\lambda)), \qquad i, j = 1, 2, \cdots, n. \tag{4}$$

We shall now make use of this fact in a proof of the following theorem:

THEOREM 5. *If in $R$ each ideal has a finite basis, the matrix $A$ is not derogatory if and only if the ideal*

$$\mathfrak{a} = (h_{11}(\lambda), h_{12}(\lambda), \cdots, h_{nn}(\lambda))$$

*contains an element of $R$ which is not a divisor of zero.*

Suppose $b$ is an element of $R$ which is not a divisor of zero and that $b \equiv 0$ $(\mathfrak{a})$. Then, from (4), it follows that if $g(\lambda) \equiv 0$ $(\mathfrak{m})$, then $bg(\lambda) \equiv 0$ $(f(\lambda))$. But $f(\lambda)$ has leading coefficient 1, and from this fact, and the fact that $b$ is not a divisor of zero, an easy calculation shows that $g(\lambda) \equiv 0$ $(f(\lambda))$. Since always $f(\lambda) \equiv 0$ $(\mathfrak{m})$, this shows that $A$ is nonderogatory.

Now let us assume that $A$ is not derogatory which means that $g(\lambda)\mathfrak{a} \equiv 0$ $(f(\lambda))$ implies that $g(\lambda) \equiv 0$ $(f(\lambda))$. Under hypothesis of a finite basis for each ideal in $R$, this means that $\mathfrak{a}$ is not divisible by any prime ideal belonging to[10] $(f(\lambda))$. This implies, in turn, that there is an element $q(\lambda)$ of $\mathfrak{a}$ which is not in any prime ideal belonging to $(f(\lambda))$ and is thus prime to[11] $f(\lambda)$. But since $q(\lambda)$ is prime to $f(\lambda)$, it follows by Theorem 4 that $\mathcal{R}(f, q) = |q(A)|$ is not a divisor of zero in $R$. Let $|q(A)| = a$. Extend $R$ to a ring $R'$ consisting of elements of the form $r/a^i$ $(i = 0, 1, \cdots; r \text{ in } R)$.[12] In $R'$, $a$ has an inverse and therefore $q(A)$ has an inverse[13] which is a polynomial in $A$ with coefficients in $R'$, say

$$p(A) = (b_0/a^{i_0})A^k + (b_1/a^{i_1})A^{k-1} + \cdots + (b_k/a^{i_k}).$$

Since $q(A)p(A) = 1$, in $R$ we have a relation of the form

$$q(A)(c_0A^k + c_1A^{k-1} + \cdots + c_k) = a^l = b,$$

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[10] Cf. van der Waerden, *Moderne Algebra*, vol. 2, p. 41.

[11] In fact, it may be shown by induction on $k$ that if every element of an ideal $\mathfrak{a}$ is contained in some one of the prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_k$, then $\mathfrak{a}$ is divisible by some one of the $\mathfrak{p}_i$.

[12] See R. Holzer, *Zur Theorie der primären Ringe*, Mathematische Annalen, vol. 96 (1927), p. 722.

[13] Cf. M, p. 282.

where $b$ is not a divisor of zero. If we set

$$t(\lambda) = c_0\lambda^k + c_1\lambda^{k-1} + \cdots + c_k,$$

we have in $R[\lambda]$

$$q(\lambda)t(\lambda) \equiv b \quad (\mathfrak{m}),$$

which, by hypothesis, implies that

$$q(\lambda)t(\lambda) \equiv b \quad (f(\lambda)).$$

Now $q(\lambda) \equiv 0$ $(\mathfrak{a})$ and clearly also $f(\lambda) \equiv 0$ $(\mathfrak{a})$, from which it follows that $b \equiv 0$ $(\mathfrak{a})$. The theorem is therefore established.

SMITH COLLEGE

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## THE NOVEMBER MEETING IN LOS ANGELES

The three hundred seventy-fourth meeting of the American Mathematical Society was held at the University of California, Los Angeles, California, on Saturday, November 23, 1940. The attendance was about one hundred including the following fifty members of the Society:

O. W. Albert, H. M. Bacon, Harry Bateman, Clifford Bell, H. F. Blichfeldt, J. L. Brenner, J. G. Charney, Myrtie Collier, P. H. Daus, T. C. Doyle, G. C. Evans, A. L. Foster, H. E. Glazier, W. H. Glenn, E. C. Goldsworthy, H. J. Hamilton, E. R. Hedrick, P. G. Hoel, G. H. Hunt, Glenn James, J. P. LaSalle, D. H. Lehmer, S. H. Levy, Hans Lewy, G. F. McEwen, Rhoda Manning, W. A. Manning, A. D. Michal, Jerzy Neyman, W. G. Palm, W. T. Puckett, V. V. Quilliam, P. V. Reichelderfer, A. C. Schaeffer, G. E. F. Sherwood, D. V. Steed, P. M. Swingle, Gabor Szegö, A. E. Taylor, B. P. Taylor, T. Y. Thomas, S. E. Urner, F. A. Valentine, Morgan Ward, D. E. Whelan, W. F. Whitmore, W. M. Whyburn, J. R. Woolson, E. R. Worthington, Max Zorn.

A general session was held in the morning. The meeting was called to order at 9:30 A.M., and was presided over at different times by Professor G. C. Evans and Provost E. R. Hedrick. By invitation of the Program Committee, Dr. A. C. Schaeffer of Stanford University delivered an hour address on *Inequalities of A. Markoff and S. Bernstein for polynomials and related functions*.

Professor Gabor Szegö announced that Stanford University would celebrate its fiftieth anniversary next year and proposed to hold symposia in mathematics (topology, probability, and analysis) at that time.

In the afternoon a symposium on aeronautics was held under the direction of Professor Harry Bateman of the California Institute of Technology, who also presided. As Dr. Stewart was unable to be present, the chairman read the abstract of his paper and Professor von Kármán discussed it.

Following the meeting Provost and Mrs. Hedrick served tea at their home for members of the Society and their friends.

The titles and cross references to abstracts of papers read at the meeting are given below. Papers 11–17 were read in the symposium, and the symposium speakers were introduced by Professor Bateman. Abstracts whose numbers are followed by *t* were read by title.

1. H. J. Hamilton: *On monotonic and convex solutions of certain difference equations*. (Abstract 47-1-27.)

2. G. F. McEwen: *Statistical problems of the range divided by the mean in samples of size n*. (Abstract 47-1-77.)

3. J. P. LaSalle: *A note on pseudo-normed linear spaces.* (Abstract 47-1-33.)

4. D. H. Lehmer: *A general summation formula for numerical functions.* (Abstract 47-1-82.)

5. P. M. Swingle: *Indecomposable connexes.* (Abstract 47-1-109.)

6. A. E. Taylor: *Conjugations of complex Banach spaces.* (Abstract 47-1-48.)

7. W. F. Whitmore: *Convergence theorems for functions of two complex variables.* II. (Abstract 47-1-53.)

8. J. R. Woolson: *A theory of projections in complex Banach spaces.* Preliminary report. (Abstract 47-1-55.)

9. Max Zorn: *Idempotency of infinite cardinals.* (Abstract 47-1-88.)

10. Jerzy Neyman: *A statistical problem in mass production and routine analyses.* (Abstract 47-1-78.)

11. Harry Bateman: *Aerodynamical effects of changes in the fundamental equations.* (Abstract 47-1-56.)

12. J. Bjerknes: *Some uses of mathematics in meteorology.* (Abstract 47-1-58.)

13. H. J. Stewart: *Steady state oscillations in an atmosphere on a rotating sphere.* Preliminary report. (Abstract 47-1-64.)

14. Theodore von Kármán and W. R. Sears: *Solution of the problem of the airplane wing of finite span by expression in eigen-functions.* Preliminary report. (Abstract 47-1-61.)

15. M. A. Biot: *Finite difference equations applied to aircraft engine vibrations.* (Abstract 47-1-57.)

16. W. D. Rannie: *Tensor methods in the theory of turbulence.* Preliminary report. (Abstract 47-1-62.)

17. F. H. Clauser: *Exact solutions of the equations for the flow of a compressible fluid.* Preliminary report. (Abstract 47-3-149.)

18. R. M. Robinson: *On the simultaneous approximation of two real numbers.* (Abstract 47-1-84-*t*.)

19. G. A. Baker: *Maximum likelihood estimation of the ratio of the components of nonhomogeneous populations.* (Abstract 47-1-76-*t*.)

20. G. A. Baker: *Fundamental distributions of errors for agricultural field trials.* (Abstract 47-1-75-*t*.)

21. J. P. LaSalle: *Pseudo-normed linear sets over valued rings.* I. (Abstract 47-1-34-*t*.)

22. E. T. Bell: *Selective equations.* (Abstract 47-1-79-*t*.)

23. Morgan Ward: *The fundamental theorem of arithmetic.* (Abstract 47-1-12-*t*.)

T. M. Putnam,
*Associate Secretary*

# THE ANNUAL MEETING OF THE SOCIETY

The forty-seventh Annual Meeting of the American Mathematical Society was held at Louisiana State University, Baton Rouge, Louisiana, from Monday, December 30, 1940 to Wednesday, January 1, 1941, in conjunction with meetings of the Mathematical Association of America and the National Council of Teachers of Mathematics.

There were four general and six sectional sessions of the Society at which three addresses and ninety research papers (forty in person) were given. All sessions were held in Nicholson Hall of Louisiana State University.

Arrangements for the meetings were made by a committee of which Professor S. T. Sanders was chairman and Professors W. V. Parker and N. E. Rutt were the local members. Through the generosity of Louisiana State University, rooms were available in three of the University dormitories.

The attendance was about 500, including the following one hundred ninety-three members of the Society:

Leonidas Alaoglu, A. A. Aucoin, W. L. Ayres, R. W. Babcock, I. A. Barnett, R. E. Basye, P. O. Bell, E. E. Betz, S. F. Bibb, D. H. Blackwell, H. E. Bray, R. W. Brink, J. C. Brixey, F. L. Brooks, R. H. Bruck, H. E. Buchanan, R. S. Burington, J. E. Burnam, L. E. Bush, W. E. Byrne, W. D. Cairns, C. C. Camp, M. E. Carlen, W. B. Carver, J. E. Case, C. R. Cassity, J. H. Chanler, E. W. Chittenden, M. D. Clement, L. M. Coffin, L. W. Cohen, Nancy Cole, J. B. Coleman, E. G. H. Comfort, H. H. Conwell, L. P. Copeland, W. A. Cordrey, N. A. Court, W. H. H. Cowles, M. J. Cox, G. F. Cramer, D. R. Curtiss, J. H. Curtiss, R. D. Daugherty, W. M. Davis, C. C. Dearman, L. J. Deck, Douglas Derry, L. L. Dines, J. L. Dorroh, W. L. Duren, John Dyer-Bennet, E. D. Eaves, Samuel Eilenberg, E. R. Elliott, H. J. Ettlinger, G. C. Evans, W. K. Feller, B. F. Finkel, H. T. Fleddermann, T. C. Fry, H. K. Fulmer, R. E. Gaskell, B. E. Gatewood, F. C. Gentry, J. S. Gold, Cornelius Gouwens, C. H. Graves, J. W. Green, R. E. Greenwood, D. W. Hall, O. H. Hamilton, E. H. Hanson, O. G. Harrold, W. L. Hart, W. W. Hart, M. C. Hartley, J. O. Hassler, K. E. Hazard, L. B. Hedge, G. A. Hedlund, R. A. Hefner, M. L. Hill, I. M. Hostetter, M. G. Humphreys, Witold Hurewicz, W. A. Hurwitz, Dunham Jackson, R. L. Jackson, Nathan Jacobson, Fritz John, F. B. Jones, M. L. Kales, H. S. Kaltenborn, D. E. Kearney, J. L. Kelley, A. J. Kempner, E. C. Kennedy, E. S. Kennedy, G. S. Ketchum, P. W. Ketchum, J. R. Kline, E. C. Klipple, J. C. Knipp, C. C. Krieger, M. E. Ladue, J. P. LaSalle, C. G. Latimer, Walter Leighton, R. G. Lubben, Dorothy McCoy, Dora McFarland, Roy MacKay, Saunders MacLane, A. E. Many, William Marshall, L. E. Mehlenbacher, A. N. Milgram, D. D. Miller, W. L. Miser, Virginia Modesitt, Deane Montgomery, Richard Morris, T. A. Mossman, E. J. Moulton, S. B. Myers, J. P. Nash, Albert Neuhaus, C. V. Newsom, E. N. Nilson, N. M. Oboukhoff, Rufus Oldenburger, E. G. Olds, E. R. Ott, F. W. Owens, H. B. Owens, W. V. Parker, E. W. Pehrson, I. E. Perlin, O. J. Peterson, B. J. Pettis, G. B. Price, A. S. Rayl, Maxwell Reade, D. P. Richardson, R. G. D. Richardson, F. D. Rigby, R. F. Rinehart, C. V. Robinson, J. B. Rosser, N. E. Rutt, S. T. Sanders, Samuel Saslaw, G. E.

Schweigert, J. A. G. Shirk, M. M. Slotnick, C. D. Smith, C. E. Smith, E. R. Smith, H. L. Smith, H. W. Smith, W. F. Smith, W. S. Snyder, R. H. Sorgenfrey, C. E. Springer, D. W. Starr, N. E. Steenrod, R. C. Stephens, R. W. Stokes, E. B. Stouffer, W. T. Stratton, Otto Szász, J. D. Tamarkin, J. M. Thomas, A. W. Tucker, L. I. Wade, T. L. Wade, C. C. Wagner, R. W. Wagner, R. K. Wakerling, R. J. Walker, J. A. Ward, L. E. Ward, W. G. Warnock, Marjorie Watson, M. J. Weiss, F. P. Welch, M. E. Wescott, R. L. Wilder, F. B. Wiley, R. M. Winger, F. L. Wren, Leo Zippin.

The meetings of the Society opened Monday afternoon with two sections, Analysis, and Algebra and Topology, with Professors G. B. Price and L. W. Cohen presiding. This was followed by a general session at which Dr. Leo Zippin gave an address entitled *Topology of rigid motions.* Professor J. R. Kline presided.

On Monday evening there were two sections, Algebra with Professor Nathan Jacobson presiding, and Geometry with Professor S. B. Myers presiding.

On Tuesday morning there was a general session for short papers at which President G. C. Evans presided. Following this Professor Saunders MacLane gave an address entitled *Extensions of groups* at a general session at which Professor J. D. Tamarkin presided.

The annual business meeting and election of officers was held on Wednesday morning. President G. C. Evans presided. This was followed by a general session at which Professor G. C. Evans gave his retiring address as President of the Society. The title of this address was *Surfaces of minimum capacity.* Vice President T. C. Fry presided at this address.

The sessions of the Society closed on Wednesday afternoon with two sections, Analysis and Topology, with Professors H. J. Ettlinger and N. E. Steenrod presiding.

Sessions of the Mathematical Association of America were held on Wednesday morning, jointly with the National Council of Teachers of Mathematics, and on Thursday morning. The National Council of Teachers of Mathematics held sessions on Monday morning, afternoon and evening, Tuesday morning, and Wednesday morning.

On Monday afternoon a tea was given in the Faculty Club by President and Mrs. P. M. Hebert of Louisiana State University.

On Tuesday afternoon there was an excursion by automobile to visit some fine antebellum homes and gardens near St. Francisville.

A joint dinner for members of all three mathematical organizations and their guests was held on Wednesday evening in Foster Hall. The attendance was two hundred ninety-one. Professor F. A. Rickey was toastmaster. He introduced Dr. P. M. Hebert, President of Louisiana State University, who welcomed the mathematicians on behalf of the

University. Other speakers, representing the three organizations, were Professors H. E. Buchanan and C. V. Newsom, and Miss Mary A. Potter. Between the speeches Miss Marguerite Muncelle entertained with music on the harp and Miss Weslyn Stephens with vocal solos. President G. C. Evans presented to Dean R. G. D. Richardson, on behalf of the Council of the Society, a beautiful hand-illuminated copy of the resolution of appreciation adopted by the Council on the occasion of his retirement as Secretary of the Society, together with a silver coffee set and for Mrs. Richardson a hand bag. President Evans also read telegrams of congratulations to Dean Richardson from the Department of Mathematics of the University of Wisconsin and from Professor Marston Morse, the newly elected President of the Society. At the close of the dinner Professor R. M. Winger presented resolutions, which were adopted unanimously, thanking Louisiana State University and the members of its Department of Mathematics for their excellent arrangements and gracious hospitality.

At the meeting of the Board of Trustees at 6:00 P.M., December 30, 1940, in the library of the Faculty Club of Louisiana State University, there was no quorum present. An adjourned meeting, therefore, was held on January 6, 1941, at 5:00 P.M. in the Men's Faculty Club of Columbia University, New York City.

The Council met on Tuesday, December 31, at 7:30 P.M. in the Faculty Club at Louisiana State University; an adjourned meeting was held at 4:15 P.M. on Wednesday at the same place.

The Secretary announced the election of the following eighteen persons to membership in the Society:

Dr. Alfredo Baños, Jr., Institute of Physics, National University of Mexico;
Mr. Virgil P. Barta, Portland, Ore.;
Professor Clair J. Blackall, College of St. Thomas, St. Paul, Minn.;
Mr. Aaron Herschfeld, U. S. Social Security Board, Washington, D. C.;
Mr. Gerhard Paul Hochschild, Princeton University;
Dr. Houston T. Karnes, Louisiana State University;
Mr. Jacob Louis Miller, Brooklyn, N. Y.;
Mr. Vernon V. Quilliam, Adohr Milk Farms, Los Angeles, Calif.;
Mr. John S. Ripandelli, Forest Hills, Long Island, N. Y.;
Dr. Raphaël Salem, Montreal, Quebec, Canada;
Professor Conway C. Sams, Mars Hill College, Mars Hill, N. C.;
Dr. Kenneth Charles Schraut, University of Dayton;
Professor Sire, University of Lyon, Lyon, France;
Professor Frederick Franklin Stephan, Cornell University;
Dr. Alfred Tarski, Harvard University;
Sister Mary Felice Vaudreuil, Mount Mary College, Milwaukee, Wis.;
Brother Bernard Alfred Welch, Manhattan College, New York, N. Y.;
Professor Antoni Zygmund, Mount Holyoke College.

It was reported that the following had been elected as nominees on the Institutional Memberships of the institutions indicated:

Brooklyn College: Mr. Bernard Greenspan.
Brown University: Dr. George Polya.
University of California: Miss Julia Hall Bowman, Messrs. George Bernard Dantzig, Stanley Phillips Frankel, Robert J. Levit, James Vernon Lewis, Martin Lewis Nelson, and Samuel A. Schaaf.
University of California at Los Angeles: Messrs. Clarence Maurice Ablow and Robert Simple Arthur.
University of Cincinnati: Mr. Harry Sears Kieval.
College of the City of New York: Mr. Joshua Barlaz.
Columbia University: Messrs. Lawrence Annenberg and Richard M. Cohen, Miss Martha Isabel Fields, Messrs. Abraham Hillman and Chung-Tsi Hsu.
Cornell University: Messrs. Norman Gustav Gunderson and Theodore Hailperin.
Harvard University: Professor Percy Williams Bridgman, Messrs. Paul Ernest Guenther and Irving Kaplansky, Professor Edwin Crawford Kemble, Messrs. George Whitelaw Mackey and Frank Moore Stewart.
University of Illinois: Messrs. Thad Paul Alton, William Franklin Atchison, Ernest Evred Blanche, Dickran Hagop Erkiletian, Arthur Edward Hallerberg, and Richard W. Hamming, Miss Mary Jeannette Hurst, Mr. Paul Ben Johnson, Miss Lois Kiefer, Messrs. Ernest Reuben Kolsrud, Brooks Javins Lockhart, Charles Willard Mathews, Gottfried Emanuel Noether, and James Ferris Price.
Institute for Advanced Study: Professor Shizuo Kakutani.
Iowa State College: Mr. Charles J. Thorne.
State University of Iowa: Mr. Franklin E. Satterthwaite.
The Johns Hopkins University: Messrs. Clagett Bowie, Carl Henry Bradshaw, and Earl Alexander Coddington, Miss Mary Theresa Curran, Dr. Frederick Weiss Light, Jr., Mr. Bernard Sherman.
University of Kentucky: Mrs. Aughtum S. Howard.
Massachusetts Institute of Technology: Messrs. Kenneth James Arnold, Elliott Lyndon Buell, and Henry Earl Singleton.
University of Michigan: Messrs. Ivor Schilansky and Bernard Vinograde.
Michigan State College of Agriculture and Applied Science: Mr. Joseph Willard Sheedy.
Northwestern University: Messrs. Arthur Saastad and Zenon Szatrowski.
University of Pennsylvania: Messrs. Nathan Jacob Fine and Joseph Lehner.
Pennsylvania State College: Dr. Allen Byron Cunningham.
The College of St. Thomas: Professor Rolf Landshoff.
Smith College: Miss Alexandra Illmer.
University of Toronto: Messrs. Nathan Saul Mendelsohn and Albert Wallace Walker.
University of Virginia: Messrs. Truman Arthur Botts and Walter Helbig Gottschalk.
University of Washington: Mr. Fumio Yagi.
University of Wisconsin: Messrs. John Oswald Danielson, Robert Stewart Gardner, Allison Potter Loomer, Nels David Nelson, Thomas Edmond Oberbeck, Lowell J. Paige, and Robert L. Wilson.
Yale University: Messrs. Andrew Mattei Gleason, Gerald Robinson MacLane, and Sidney Trachtenberg.

The following appointments by President G. C. Evans or the Council were reported: as Committee on Arrangements for the Spring Meeting in Washington, D. C., May 2–3, 1941, Professors F. E. Johnston (chairman), E. J. Finan, T. R. Hollcroft, Florence M. Mears, J. H. Taylor, E. W. Titt, and F. M. Weida; as Committee on Arrangements for the Annual Meeting of 1941 at Lehigh University, Dean Tomlinson Fort (chairman), Professors T. R. Hollcroft, Everett Pitcher, G. E. Raynor, and W. M. Smith; as representative of the Society at the inauguration of Samuel Nowell Stevens as sixth president of Grinnell College (Grinnell, Iowa) on October 25, Professor W. J. Rusk; as representative of the Society at the inauguration of Raymon M. Kistler as president of Beaver College (Glenside, Pa.) on November 8, Professor Anna Pell Wheeler; as representative of the Society at the inauguration of Carl Stephens Ell as president of Northeastern University (Boston, Mass.) on November 19, Professor E. V. Huntington; as representative of the Society at the inauguration of James Franklin Findlay as president of Drury College (Springfield, Mo.) on November 29, Dr. F. T. H'Doubler; as an additional member of the Committee on Publicity, Professor R. J. Walker; as auditors of the Society's books for 1940, Professors R. G. Archibald and A. E. Meder, Jr.; as tellers for the election at the 1940 Annual Meeting, Dr. E. R. Lorch and Professor W. V. Parker; as new members of the Committees to Select Hour Speakers: for Annual and Summer Meetings, Professor M. H. Ingraham (committee now consists of Professors J. R. Kline, chairman, Einar Hille, C. R. Adams, and M. H. Ingraham); for Western Sectional Meetings, Professor W. J. Trjitzinsky (committee now consists of Professors W. L. Ayres, chairman, E. W. Chittenden, and W. J. Trjitzinsky); for Far Western Sectional Meetings, Professor T. Y. Thomas (committee now consists of Professors T. M. Putnam, chairman, Gabor Szegö, and T. Y. Thomas); as Colloquium speaker for 1943, Professor E. J. McShane; as representatives on the Editorial Board of the Duke Mathematical Journal for 1941–1943, Professors F. J. Murray and Morgan Ward.

The Secretary reported that the ordinary membership in the Society is now 2,336, including 213 nominees of institutional members and 78 life members. There are also 87 institutional members. The total attendance of members at all meetings in 1940 was 1,441; the number of papers read was 498; the number of invited addresses was 13; the number of members attending at least one meeting was 921.

At the annual election which closed on January 1, and at which 384 votes were cast, the following officers were elected:

*President*, Professor Marston Morse.

*Vice President*, Professor T. Y. Thomas.

*Secretary*, Professor J. R. Kline.

*Associate Secretary*, Professor T. R. Hollcroft.

*Treasurer*, Professor B. P. Gill.

*Member of the Editorial Committee of the Bulletin*, Professor L. M. Graves.

*Member of the Editorial Committee of the Transactions*, Professor Einar Hille.

*Member of the Editorial Committee of the American Journal of Mathematics*, Professor Hermann Weyl.

*Member of the Editorial Committee of the Colloquium Publications*, Professor G. T. Whyburn.

*Members of the Council at Large*, Professors L. R. Ford, G. B. Price, Tibor Radó, A. W. Tucker, and Morgan Ward.

*Members of the Board of Trustees*, Professors W. R. Longley, Marston Morse, and G. W. Mullins, Dean R. G. D. Richardson, and Dr. Warren Weaver.

At the business meeting following the annual election of officers, the following resolution, adopted by the Council at the Summer Meeting at Hanover, was read, and by a unanimous rising vote, the Society added its approval of this Council action:

### TO ROLAND GEORGE DWIGHT RICHARDSON

*The American Mathematical Society has been exceedingly fortunate in having had three devoted and able secretaries. Thomas Scott Fiske (1888–1895), Frank Nelson Cole (1895–1920), and Roland George Dwight Richardson (1921–1940). The achievements of the first two have been noted elsewhere in the archives of the Society. It is desire of the Council of the Society at its present meeting to record its grateful appreciation of the services of Richardson who will retire from office in December 1940. He was elected to the secretaryship in 1921, in the period of danger and uncertainty following the world war of 1914–1918. It was a time in the history of the Society when the never-ending tension between too slowly increasing financial resources and the costs of continuously increasing facilities for encouragement of mathematical research was most acute. By his persistent continuation of the campaigns for membership, his watchful interest in the resources of the Society, his sympathetic understanding of our research needs, and his constant and unifyingly friendly influence in the management of our affairs, Richardson has taken a most prominent part in insuring the financial safety and developing the scientific influence of the Society. He has devoted himself,* *often to the limit of his strength, to our interests. In contrast to these self-sacrificing services our comments here must inevitably seem inadequate and incomplete. Richardson's real memorial lies, however, in the affection and admiration for him which are widespread among the present members of the Society, and universal among those who have known him personally and worked with him. For future generations of members the Society itself and its scientific resources will constitute an ineffaceable reminder of his effective interest.*

*In view of these invaluable services the Council hereby directs that a copy of these statements shall be spread upon the records of the Society, and that a copy shall also be sent, as an expression of our high regard, to our retiring Secretary.*

The reports of the Treasurer and Auditors showed a balance of $6,966.39 in the general account and exclusive of the balances in the Colloquium, Sinking Fund, and other special funds. The Society's investments including Endowment Fund, Mathematical Reviews Fund, Colloquium Fund, and other special funds had a market value on November 30, 1940 of approximately $170,000. The interest income for the year was $6,909.41; dues from Institutional Memberships amounted to $6,705.00; dues from Contributing Memberships $1,114.92; dues from Ordinary Memberships $14,380.96. The Colloquium Fund showed a balance on hand of $9,718.95. The total received from the sale of the Society's publications was $18,148.02. The regular account of Mathematical Reviews showed income from subscriptions of $7,849.48, from investments of $2,469.99, and disbursements of approximately $11,800.00. The Board of Trustees adopted a budget for 1941 showing authorized expenditures and estimated receipts of the Society as $50,700 and $41,900 respectively. A sum of $1,000 from the income of the Marion Reilly Fund was appropriated for Mathematical Reviews and a similar sum for a volume of the Colloquium Publications. A copy of the Treasurer's report will, on request, be sent to any member of the Society.

On recommendation of the Committee on Places of Meetings (Professor E. W. Chittenden, chairman), the Annual Meeting of 1941 was set at Lehigh University, Bethlehem, Pa., December 30–January 1, and the Annual Meeting of 1942 for New York City in connection with the sessions of the American Association for the Advancement of Science. Times and places of meetings during 1941 were also set as follows: fall midwestern meeting at Kansas State College, Manhattan, on November 21–22, and fall far western meeting at Pasadena on November 22.

The Librarian reported that the Library of the Society now contains about 9721 volumes.

The American Journal of Mathematics, which is published jointly by the Johns Hopkins University and the Society, and to which the Society contributes an annual subvention of $2,500, printed 912 pages during 1940.

The Committee on War Preparedness presented a report. A copy may be obtained on request to Secretary Kline. The present membership of the Committee is as follows:

*Chairman*, Professor Marston Morse.

*Subcommittee on Research*, Professors Dunham Jackson (chairman), E. J. McShane, M. H. Stone, J. H. Van Vleck, Norbert Wiener, S. S. Wilks.

*Subcommittee on Preparation for Research*, Professors M. H. Stone (chairman), B. O. Koopman, R. E. Langer, Hans Lewy, F. D. Murnaghan, H. P. Robertson.

*Subcommitee on Education for Service*, Professors W. L. Hart (chairman), R. S. Burington, J. L. Coolidge, H. B. Curry, E. C. Goldsworthy, F. L. Griffin, M. H. Ingraham, E. J. Moulton.

*Consultants*, Aeronautics—Professor Harry Bateman, Chief; Ballistics—Professor John von Neumann, Chief, Professor W. T. Reid; Computation—Professor Norbert Wiener, Chief; Cryptanalysis—Professor H. T. Engstrom, Chief, A. A. Albert, W. A. Hurwitz, Solomon Kullback, Oystein Ore; Industry—Dr. T. C. Fry, Chief; Probability and Statistics—Professor S. S. Wilks, Chief.

The Secretary reported that during his term of office the membership of the Society had more than tripled (733–2,336).

President Marston Morse appointed as the Committee on Nomination of Officers and Members of the Council for 1942, Professors G. A. Bliss (chairman), G. C. Evans, Philip Franklin, T. H. Hildebrandt, and P. A. Smith.

The Committee on Scientific Aids to Learning with a view to aiding the use of microfilm as well as to supporting Mathematical Reviews has appropriated $10,000 for purchase of Student Microfilm Readers (see this Bulletin, p. 1). It was reported that over 6,000 notices regarding the Student Microfilm Readers had been sent to individuals and libraries and that applications for more than 350 of these machines had been received by Mathematical Reviews during the last ten days of 1940.

Revision of the By-Laws to take care of the Society's arrangements for Mathematical Reviews, to reduce the number of ex-officio voting members of the Council, and to make other minor changes, was recommended for favorable action; the revised By-Laws are to be presented for adoption at the February meeting of the Society.

A committee consisting of Professors E. W. Chittenden (chairman), T. H. Hildebrandt, H. P. Robertson, J. D. Tamarkin, and J. L. Walsh was appointed to consider and to make recommendations to the Council concerning the possible introduction of a new series of mathematical books.

The Committee on Addresses in Applied Mathematics reported through its chairman, Professor Richard Courant, recommendations for a vigorous program in connection with various meetings of mathematicians.

Titles and cross references to the abstracts of papers read at the regular sessions follow below. The papers were read as follows: Papers numbered 1–6 in the section for Analysis on Monday afternoon; papers 7–11 in the section for Algebra and Topology on Monday afternoon; papers 12–17 in the section for Algebra on Monday evening; papers 18–23 in the section for Geometry on Monday evening; papers 24–29 in the general session on Tuesday morning; papers 30–34 in the section for Analysis on Wednesday afternoon; papers 35–40 in the section for Topology on Wednesday afternoon; and papers 41–90, whose abstract numbers are followed by the letter *t*, were read by title. Mr. R. L. Swain and Miss Harlan C. Miller were introduced by Professor R. L. Moore, Mr. Nelson Robinson by Professor W. V. Parker, Professor Szolem Mandelbrojt by Professor H. E. Bray, and Dr. Russell Cowan by Professor W. G. Warnock. Paper 2 was presented by Mr. E. N. Nelson, and paper 18 by Dr. C. V. Robinson.

1. J. W. Green: *A special type of conformal map.* (Abstract 47-1-25.)

2. J. L. Walsh and E. N. Nilson: *Approximation to an analytic function by functions analytic and bounded in a region.* (Abstract 47-1-52.)

3. W. S. Snyder: *On independence of the path for line integrals of continuous functions.* (Abstract 47-1-45.)

4. J. P. LaSalle: *Pseudo-normed linear sets over valued rings.* II. (Abstract 47-1-95.)

5. I. E. Perlin: *Indefinitely differentiable functions of several real variables.* (Abstract 47-1-40.)

6. L. B. Hedge: *Transformations of multiple Fourier series.* Preliminary report. (Abstract 47-1-28.)

7. N. E. Rutt: *Rectangular arrays of combinations.* (Abstract 47-1-8.)

8. R. W. Wagner: *The differentials of analytic matrix functions.* (Abstract 47-1-11.)

9. R. H. Sorgenfrey: *Concerning triodic continua.* (Abstract 47-1-105.)

10. R. L. Swain: *Distance axioms in Moore spaces.* (Abstract 47-1-106.)

11. J. L. Kelley: *A theorem on transformations.* (Abstract 47-3-178.)

12. T. L. Wade: *Tensor algebra and Young's symmetry operators.* (Abstract 47-1-10.)

13. C. C. Camp: *A root cubing method of solving equations.* (Abstract 47-1-2.)

14. L. I. Wade: *Certain quantities transcendental over the field* $GF(p^n, x)$. (Abstract 47-1-87.)

15. Douglas Derry: *On the boundary case of Minkowski's linear form theorem.* Preliminary report. (Abstract 47-1-81.)

16. Rufus Oldenburger: *The minimal number problem.* (Abstract 47-1-6.)

17. R. H. Bruck: *The structure of the rational representations of a wide class of linear groups.* (Abstract 47-1-1.)

18. L. M. Blumenthal and C. V. Robinson: *Helly theorems on the sphere.* (Abstract 47-1-67.)

19. N. A. Court: *On the skew cubic.* (Abstract 47-1-73.)

20. E. R. Ott: *A locus determined by an algebraic correspondence.* (Abstract 46-11-497.)

21. P. O. Bell: *On differential geometry intrinsically connected with a surface element of projective arc length.* (Abstract 47-1-65.)

22. C. R. Cassity: *The double points of a pencil of cubics invariant under the quadratic transformation.* (Abstract 47-1-68.)

23. Nelson Robinson: *On the contact of a quartic surface with a general analytic surface.* (Abstract 47-3-164.)

24. L. L. Dines: *On the mapping of quadratic forms.* (Abstract 47-1-19.)

25. J. B. Rosser: *Explicit bounds for some functions of prime numbers.* (Abstract 47-1-85.)

26. S. B. Myers: *Complete Riemannian manifolds of positive mean curvature.* (Abstract 47-1-74.)

27. J. M. Thomas: *Orderly differential systems.* (Abstract 47-1-49.)

28. A. N. Milgram: *Extensions of decompositions.* (Abstract 47-1-97.)

29. Szolem Mandelbrojt: *On a Dirichlet series.* (Abstract 47-3-137.)

30. B. E. Gatewood: *Thermal stresses in regions bounded by epitrochords.* (Abstract 47-1-59.)

31. Russell Cowan: *A method of solving a linear difference equation with polynomial coefficients of degree m.* (Abstract 47-1-16.)

32. J. H. Curtiss: *Trigonometric interpolation by means of the complex Lagrange polynomial.* (Abstract 47-1-18.)

33. Fritz John: *On the character of solutions of hyperbolic equations.* Preliminary report. (Abstract 47-1-32.)

34. Otto Szász: *On the partial sums of harmonic developments and related power series.* (Abstract 47-1-47.)

35. Harlan C. Miller: *Concerning certain types of end points of compact continua.* (Abstract 47-1-98.)

36. G. E. Schweigert: *Border transformations.* (Abstract 47-1-104.)

37. O. G. Harrold: *On a class of continuous maps.* (Abstract 47-1-92.)

38. R. L. Wilder: *Characterizations of those euclidean domains whose boundaries are* $lc^k$. (Abstract 47-1-112.)

39. Samuel Eilenberg: *Imbedding of spaces into euclidean spaces.* Preliminary report. (Abstract 47-1-90.)

40. F. B. Jones: *Topologically flat spaces.* (Abstract 47-1-93.)

41. S. E. Warschawski: *On conformal mapping of infinite strips.* (Abstract 47-3-146-*t*.)

42. R. P. Boas and D. V. Widder: *Completely convex functions.* Preliminary report. (Abstract 47-1-15-*t*.)

43. W. D. Duthie: *Segments in ordered sets.* (Abstract 47-1-3-*t*.)

44. P. M. Swingle: *An abstraction of types of connected sets.* (Abstract 47-1-108-*t*.)

45. L. M. Blumenthal: *Betweenness in metric ptolemaic spaces.* (Abstract 47-1-66-*t*.)

46. J. J. Eachus: *Classification of solutions and of pairs of solutions of* $y'''+2p(x)y'+p'(x)y=0$ *by means of boundary conditions.* (Abstract 47-1-21-*t*.)

47. C. N. Moore: *On the Cesàro and Abel-Poisson summability of the differentiated double Fourier series.* (Abstract 47-1-38-*t*.)

48. H. L. Garabedian: *Relations between hypergeometric methods of summation.* (Abstract 47-1-24-*t*.)

49. C. W. Vickery: *A new proof of a theorem of Chittenden.* (Abstract 47-1-111-*t*.)

50. J. Marcinkiewicz and Antoni Zygmund: *On the behavior of trigonometric and power series.* (Abstract 47-1-36-*t*.)

51. Hyman Serbin: *Upper bounds for the remainder of certain power series.* (Abstract 47-1-44-*t*.)

52. H. S. Wall: *A theorem on real functions bounded in the unit circle.* (Abstract 47-1-51-*t*.)

53. R. E. Greenwood: *Hankel and other extensions of Dirichlet series.* (Abstract 47-1-26-*t*.)

54. Walter Leighton: *On the convergence of continued fractions.* (Abstract 47-1-35-*t*.)

55. Warren Ambrose: *Representation of ergodic flows.* Preliminary report. (Abstract 47-1-14-*t*.)

56. A. E. Heins: *On the solution of partial difference equations. Two interval "boundary conditions."* Preliminary report. (Abstract 47-1-29-*t*.)

57. N. E. Rutt: *Sets with a certain sort of derived set.* (Abstract 47-1-103-*t*.)

58. Nathaniel Coburn: *A note on conformal geometry.* (Abstract 47-1-69-*t*.)

59. Nathaniel Coburn: *Unitary spaces with corresponding geodesics.* (Abstract 47-1-70-*t*.)

60. H. M. Schwartz: *Convex functions and the law of the mean.* (Abstract 47-1-42-*t*.)

61. H. M. Schwartz: *Sequences of Stieltjes integrals.* (Abstract 47-1-43-*t*.)

62. Arthur Sard: *The measure of the critical values of differentiable maps of euclidean spaces.* (Abstract 47-1-41-*t*.)

63. M. F. Smiley: *Measurability and distributivity in the theory of lattices.* (Abstract 47-1-9-*t*.)

64. Jesse Douglas: *Solution of the inverse problem of the calculus of variations.* (Abstract 47-1-20-*t*.)

65. J. H. Curtiss: *Degree of polynomial approximation on a lemniscate.* (Abstract 47-1-17-*t*.)

66. N. A. Court: *On the harmonic pole.* (Abstract 47-1-72-*t*.)

67. G. D. Nichols: *A sufficient condition for Cesàro summability.* (Abstract 47-1-39-*t*.)

68. R. L. Wilder: *On the domains complementary to continua having certain avoidability properties.* (Abstract 47-1-113-*t*.)

69. R. L. Wilder: *The duality between the S-properties of closed sets and their complements in the euclidean n-sphere.* (Abstract 47-1-114-*t*.)

70. Hermann Weyl: *Theory of reduction for arithmetical equivalence.* II. (Abstract 47-1-13-*t*.)

71. C. J. Everett: *An extension theory for rings.* (Abstract 47-1-4-*t*.)

72. D. V. Widder: *Necessary and sufficient conditions for the representation of a function in Lidstone series.* (Abstract 47-1-54-*t*.)

73. Eric Reissner: *A new derivation of the equations for the deformation of elastic shells.* (Abstract 47-1-63-*t*.)

74. Harlan C. Miller: *On the existence of a certain type of irreducible continuum.* (Abstract 47-1-99-*t*.)

75. Harlan C. Miller and R. L. Swain: *Properties of two intersecting arcs.* (Abstract 47-1-100-*t*.)

76. R. L. Swain: *Linear metric space.* (Abstract 47-1-107-*t*.)

77. Abraham Spitzbart: *Approximation in the sense of least pth powers by polynomials with a single auxiliary condition of interpolation.* (Abstract 47-1-46-*t*.)

78. F. B. Hildebrand: *The approximate solution of singular integral equations arising in engineering practice.* (Abstract 47-1-60-*t*.)

79. Leonard Carlitz: *Finite differences and polynomials in a Galois field.* (Abstract 47-1-80-*t*.)

80. Samuel Eilenberg: *Monotone families of manifolds.* (Abstract 47-1-91-*t*.)

81. S. B. Myers: *Compact groups whose elements are of finite order.* (Abstract 47-1-101-*t*.)

82. M. R. Hestenes: *Extension of the range of a differentiable function.* (Abstract 47-1-30-*t*.)

83. Richard Courant: *Critical points and unstable minimal surfaces.* (Abstract 47-1-71-*t*.)

84. W. J. Trjitzinsky: *Analytic theory of singular elliptic partial differential equations.* (Abstract 47-1-50-*t*.)

85. Gordon Pall: *Simultaneous representation in a quadratic and linear form.* (Abstract 47-1-83-*t*.)

86. N. M. Oboukhoff: *On the validity of total differential as the principal part of the increment of a function of two or more variables.* (Abstract 47-3-138-*t*.)

87. Oscar Zariski: *Pencils on an algebraic variety and a new proof of a theorem of Bertini.* (Abstract 47-3-167-*t*.)

88. Edward Kasner and J. J. DeCicco: *The classification of analytic arcs or elements under the group of arbitrary point transformations.* (Abstract 47-3-162-*t*.)

89. R. P. Agnew: *On methods of summability and mass functions determined by hypergeometric coefficients.* (Abstract 47-3-125-*t*.)

90. E. S. Allen and Harvey Diehl: *The enumeration of glycols.* (Abstract 47-3-147-*t*.)

W. L. AYRES,
*Associate Secretary*

# ROBERT EDOUARD MORITZ—IN MEMORIAM

The sudden death of Professor Moritz in Seattle on December 28, 1940, not only marked the close of a life that had measured the allotted span, but terminated an honorable career of nearly half a century in the service of mathematics and higher education. Retired to half time two years before, he had taught through the fall quarter and had made plans to spend the winter in California. He occupied the position of head of the department of mathematics at the University of Washington for over thirty years, at various periods of which astronomy was included. He witnessed the growth of the university population from a total of some eight hundred for the academic year to approximately eleven thousand, including one thousand graduate students, in a single quarter. It was largely due to his high standards, energy and vision that the department of mathematics was competently manned and adequately equipped with library and other facilities to keep pace with this development.

Robert Edouard Moritz was born at Christianstahl in Schleswig-Holstein, the son of Karl R. and Maria Stahlhut Moritz on June 2, 1868. His early education was received in the public schools of Germany. At the age of twelve he came to the United States with his parents who settled on a farm in Nebraska. On the casual advice of a friend, young Moritz entered Hastings Academy in 1885, expecting to spend only a single term. But one term sealed his fate. By dint of the utmost frugality, of doing his own cooking, of sharply curtailing the normal hours of sleep, he was able to eke out his meagre earnings to cover the bare necessities and to remain in school—remain for seven years, at the end of which he was graduated from Hastings College in 1892. "He worked his way through college" has become a commonplace in American biography. It sounds like a simple formula, and there are of course many versions. But anyone lacking the actual experience can have only a faint conception of what those few words might connote in the short-grass region of Nebraska in the late eighties and early nineties—those years of acute and prolonged distress in the prairie states, under the scourge of hot winds, crop failures and mortgage foreclosures.

After a year spent in graduate study at the University of Chicago he returned to his Alma Mater where he held the chair of mathematics and physics from 1893 to 1898. In the meantime he attended the University of Chicago during two summer quarters, completing the requirements for the master's degree in 1896. That same summer he married Miss Cassia L. Kennedy of Aurora, Nebraska, who with two sons survive him—one, Harold K. Moritz is associate professor of civil engineering at the University of Washington, the other, Donald B. Moritz lives in Minneapolis. An appointment as instructor in mathematics enabled him to continue his graduate work at the University of Nebraska, which awarded him the Ph.D. in 1901. The next year he studied abroad, chiefly under H. Weber and Th. Reye at the University of Strassburg, earning a second Ph.D. in 1902. Returning to the University of Nebraska, he served as adjunct professor for one year and as assistant professor for another.

In 1904 he was appointed professor and head of the Department of Mathematics and Astronomy at the University of Washington, succeeding the late Arthur Ranum. He threw himself at once into the scientific and educational work of the community and state. One of his first duties was to repair the university telescope, the lenses having been stolen the preceding summer. He prepared for a Seattle Sunday paper an article with diagrams on the forthcoming solar eclipse. And he opened the observatory to the public on certain nights, a custom that has prevailed to the present time. Holding a profound conviction of the value of mathematics in the curriculum, he wrote numerous articles for educational journals, both state and national. He was a principal speaker at the three-day sessions of one of the county institutes, discussing various aspects of the teaching of elementary mathematics on several of the programs. His final address seems to have made a deep impression for it was printed in full by popular demand in the local weekly paper.

The bibliography of his published books and papers contains about 75 titles, including his educational papers which comprise about a third of the total. In addition, he was a clever problem solver and for a dozen years beginning in 1920 he was a regular contributor to the problem department of the American Mathematical Monthly. His mathematical articles range over the fields of analysis, number theory, geometry and economics. Perhaps the most noteworthy are: *Generalization of the differentiation process* [Quotientiation], American Journal, 1902; *Ueber gewisse Infinitesimal-Operationen der hoeheren Operations-Stufen* (three parts totaling 60 pages). Tôhoku Mathematical Journal, 1921, 1922, 1923; and the monograph on *Cyclic-Harmonic Curves*, University of Washington Publications in Mathematics, 1923, pp. 1–58 plus 21 plates. The last is a discussion of the curves represented by the polar equation $\rho = a \cos (p/q)\theta + k$, where $p$ and $q$ are integers. A machine for drawing these curves, when $p, q = 1, 2, \cdots, 10$, built according to his design in the University shops, is in the possession of the Department of Mathematics.

His book on *Plane and Spherical Trigonometry* (John Wiley and Sons, 1911, 1913) is one of the most comprehensive treatises on trigonometry ever published in America. His *Short Course in Mathematics* (Macmillan, 1919) was an outgrowth of the course prescribed for the Students' Army Training Corps in the fall of 1918 when the University was turned into a training camp for soldiers, sailors and marines. Perhaps the best known is his *Memorabilia Mathematica* (Macmillan, 1914) the results of the painstaking labor of ten years, which is read and valued by every lover of mathematics.

He was an expert teacher, especially for the stronger students, whom he stimulated to their best efforts. Many factors unite to make the great teacher but a major secret of his success was his manifest sincerity and perennial enthusiasm.

He, usually accompanied by Mrs. Moritz, traveled widely—went several times to Europe, twice to the Orient, once around the world. They visited Alaska and Mexico and explored every section of the United States in their Model T.

He was a member of the American Mathematical Society since 1904 and a charter member of the Mathematical Association of America, serving on the first board of trustees, 1916–1918. He maintained his mathematical interest and activity to the last, reading a paper at the recent international congress at Oslo. His latest paper was presented before the departmental Research and Journal Club last spring on the centroid of a simplex in $n$-dimensional space.

R. M. WINGER

# BOOK REVIEWS

*Stencils for Solving* $x^2 \equiv a$ (*mod* $m$). By Raphael M. Robinson. Berkeley and Los Angeles, University of California Press, 1940. 14 pp., 272 cards.

These stencils enable one to find directly all the solutions of the given congruence for $m \leqq 3000$. The range of $m$ may be extended by additional computation.

In brief, the theory back of the construction is as follows: to solve the congruence we must find a $y$ such that $a+my$ is a quadratic residue of all numbers $E$; that is, if $m$ is prime to $E$, $a/m+y$ and $m$ must have the same quadratic character (mod $E$) when the quadratic character for $E=16$ is the value of $m$ (mod 8). It is not hard to see that we need consider only $y \leqq m/4$. Each card contains the numbers from 1 to 750, it is headed with a value of $E$ (9, 16 or one of the seven primes from 5 to 23 inclusive), a value of $a/m$ (mod $E$) and a quadratic character of $m$ (mod $E$). Holes are punched in the values of $y$ for which $a/m+y$ has the same quadratic character as $m$ (mod $E$). Then to solve $x^2 \equiv a$ (mod $m$) one selects the cards appropriate to $a$ and $m$ for the nine values of $E$, stacks them and looks through the holes.

The stencils are very simple in principle and convert drudgery into fun.

BURTON W. JONES

*The Consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis with the Axioms of Set Theory.* By Kurt Gödel. (Annals of Mathematics Studies, no. 3.) Princeton, University Press; London, Humphrey Milford and Oxford University Press, 1940. 66 pp. $1.25.

*Abstract.* In this study it is proved that the axiom of choice and Cantor's generalized continuum-hypothesis are consistent with the other axioms of set theory if these latter axioms are themselves consistent. The system $\Sigma$ of axioms here adopted for set theory is essentially that of P. Bernays (Journal of Symbolic Logic, vol. 2, p. 65).

In $\Sigma$ the primitive notions are: *class*, *set*, and the relation $\epsilon$ between class and class, class and set, set and class, or set and set. The first two axioms of $\Sigma$ specify the relation between classes and sets: Axiom A1. Every set is a class. Axiom A2. Every class, which is a member of some class, is a set. As a consequence of this distinction between class and set, there exists a *universal class* $V$ of all sets in the system $\Sigma$. The remaining axioms of $\Sigma$ are of a form conventional in ordinary

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set theory, and deal with intersection, complement, power set, etc.

Chapter I of this study includes a statement of the axioms of $\Sigma$ together with a few preliminary notions, such as *ordered pair* and *single-valued class* of ordered pairs. Chapter II is concerned with the following topics: extension of propositional functions, definition and properties of relations and functions, and properties of operations with classes and sets. Chapter III is devoted to an exposition of ordinal numbers (based on the work of J. von Neumann) in which an ordinal $\alpha$ is the class of all ordinals less than $\alpha$; for example, $0=$the null set, $1=\{0\}$, $2=\{0, 1\}$, etc. Chapter IV develops the theory of cardinal numbers. In Chapters V and VI a model $\Delta$ is constructed for the system $\Sigma$ on the assumption that $\Sigma$ is consistent, for the classes and sets of $\Delta$ are merely those classes and sets of $\Sigma$ characterized as *constructible* (the definition of constructibility being based on certain properties of ordinal numbers). Let $L$ denote the class of all constructible sets in the universal class $V$. In Chapter VII it is proved that, when the axioms of $\Sigma$ are applied just to the model $\Delta$, $V$ being the class of all sets in $\Delta$ and $L$ being the class of sets constructible within $\Delta$, then $V=L$, i.e., every set in $\Delta$ is constructible. Thus, if there exists a model for the system $\Sigma$, then there also exists a model for the system $\Sigma$ augmented by the axiom $V=L$, this latter model consisting of the classes and sets constructible within the given model for $\Sigma$. The existence of this latter model shows that, if the system $\Sigma$ is consistent, then the augmented system $\Sigma'$ is consistent, where $\Sigma'$ comprises the axioms of $\Sigma$ together with the axiom $V=L$. In the last chapter it is proved that the axiom of choice and Cantor's generalized continuum hypothesis are deducible as theorems in the system $\Sigma'$.

While a number of subjects are discussed in this study, the material actually developed is largely confined to just those lemmas needed to attain the final result.

C. C. TORRANCE

*Metric Differential Geometry of Curves and Surfaces*. By Ernest Preston Lane. Chicago, University Press, 1940. 216+8 pp. $3.00.

This book is designed as a text for beginning graduate students; it is not intended to be an exhaustive treatise on the subject. The order and content of the material has been used by the author in his classes for several years. It has therefore been thoroughly subjected to the acid tests of the classroom for teachability and for readability. It should on that account be of much service to the classroom teacher and to the student in his independent reading.

Although the material selected is classic, the point of view in presentation is definitely modern. Definitions are so given as to be valid in projective space or in hyperspace if such definitions are possible. Scattered throughout the text are some 330 exercises. Many of these exercises are mere applications of the material in the text proper, but many require developments found in the textual material in the classic treatises. Several of these exercises show the influence of projective methods and material on the author.

The subject matter is well indicated by the chapter headings: Chapter I, Curves; II, The Moving Trihedron; III, Surfaces; IV, Curves on Surfaces; V, Curvature; VI, Transformations of Surfaces. A short bibliography of the classic treatises on Metric Differential Geometry is appended.

Of especial interest is the treatment of the moving trihedron and its application to curves in one-to-one correspondence. Let a curve $C'$ be in one-to-one correspondence with a given curve $C$ so that $P'$ on $C'$ corresponds to $P$ on $C$. The direction cosines of the tangent line, principal normal, and binormal of $C'$ at $P'$ referred to the local trihedron of $C$ at $P$, are derived as functions of the local coordinates of $P'$, the derivatives of the same, and the curvature and torsion of $C$ at $P$. Applications are made to involutes, evolutes, parallel curves, transformation of Combescure, Bertrand curves. Other applications are left to the student as exercises.

In the chapter on surfaces, a lucid exposition is given of envelopes of one parameter families of surfaces. As often occurs, loci associated with space curves are simple when written in local coordinates. Methods of finding envelopes applicable in this case are given. A fresh treatment is also made of ruled surfaces other than developables.

In the chapter on curves on surfaces, the relation between orthogonality, conjugacy, and harmonic separation is stressed. In this chapter too, the Gauss and Weingarten differential equations and the Gauss-Codazzi integrability conditions are derived.

The discussion of the Gaussian and mean curvatures is well presented in Chapter V, the problems of the chapter excellently supplementing the textual material.

Conformal mapping, applicability, equiareal maps, and parallel surfaces are briefly discussed in the concluding chapter. If time permitted for such a discussion, material in this Chapter VI could no doubt have been unified in a manner similar to that used for curves in one-to-one correspondence by using the methods developed by the author in his paper *The moving trihedron*, Transactions of this Society, vol. 36 (1934), pp. 696–710.

Although some teachers may fail to find some of the familiar material, the text is ample for a first course. Moreover the exercises give much of the factual material of the missing topics.

The result (7.12) in exercise 2, p. 32 seems to be incorrect. Problem 17 on page 121 would have been more accurately expressed by speaking of an *analytic* function of a complex variable. The exercises 4 and 5 on page 140 are incorrect. The University of Chicago Press may well feel proud of this book in every respect. It is my hope that the time taken in its preparation will be repaid by its wide use. Written by a master of oral and written exposition it should receive wide acclaim.

VERNON G. GROVE

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*Lattice Theory.* By Garrett Birkhoff. (American Mathematical Society Colloquium Publications, vol. 25.) New York, American Mathematical Society, 1940. 6+155 pp. $2.50.

This is the first book on the far-reaching subject of lattices. The author has succeeded in giving a comprehensive, yet not too terse, account of the theory of lattices and its relation to other branches of mathematics. The general plan is to devote six chapters to the abstract theory, and three to applications. Chapter I deals with partially ordered systems, Chapter II with lattices and their general properties; the important modular axiom is assumed in Chapter III, and the axiom of complementation is added in Chapter IV. Distributive lattices and Boolean algebras end the abstract theory in Chapters V and VI. Thus the author takes the reader through the most important general classes of lattices, by imposing successively restrictive conditions. The last three chapters apply lattice theory to function theory, logic and probability theory.

A partially ordered system is introduced in Chapter I as a set $L$ with a binary relation $\geqq$ on $L$ which is reflexive and transitive and has the property that $x \geqq y$, $y \geqq x$ implies $x = y$. Most important in connection with general partially ordered systems are the principle of duality, viz., that $L$ together with the converse $\leqq$ of the relation $\geqq$ forms a partially ordered system, and a study of the so-called chain conditions, one or both of which are satisfied in many examples. The ascending chain condition asserts that no infinite sequence $(a_i)$ with $a_i < a_{i+1}$ exists; the descending chain condition is dual. Many examples of a partially ordered system are cited, a few of which are the set of all subsets of a set, classes of distinguished subsets of a set, real numbers, integers (relative to the relation of divisibility), partitions, and topologies on a space.

Most of the familiar examples of a partially ordered system are lattices, i.e., there exist for any two elements $x$, $y \in L$ a gr.l.b. $x \cup y$ and l.u.b. $x \cap y$. Indeed, many examples form *complete* lattices, i.e., lattices in which every subset has a gr.l.b. and l.u.b. The prototype of complete lattices is the lattice of all subsets of a set having a given extensionally attainable property (E. H. Moore), i.e., all *closed* sets relative to an abstract closure operation $\overline{A}$ having the properties $\overline{A} \geqq A$, $\overline{\overline{A}} = \overline{A}$, and $\overline{A} \geqq \overline{B}$ when $A \geqq B$. The MacNeille solution to the problem of embedding a partially ordered system in a complete lattice is given, and a number of other topics are discussed, including interesting questions regarding free lattices. Finally, it is shown that there exists in a lattice an intrinsic topology defined wholly in terms of the order relation.

The modular axiom states that $a \leqq c$ implies $(a \cup b) \cap c = a \cup (b \cap c)$. Among the examples cited which satisfy the modular axiom are the set of all invariant subgroups of a group (with operators), the set of all subspaces of a linear space, the set of right (left) or two-sided ideals in a ring, and the linear subspaces of an abstract projective geometry. The main result of Chapter III is that modularity implies the Jordan chain condition, which states that two principal chains connecting the same elements of the lattice are equal in length.

Complemented modular lattices are those lattices with zero $O$ and unit $I$ in which each element $a$ has at least one "complement" $x$ such that $a \cup x = I$, $a \cap x = O$. The fundamental result in Chapter IV characterizes abstract projective geometries (in the sense of Veblen and Young) as complemented modular lattices which are *simple* (in the sense of abstract algebra). This result is due to the author. The generalization to "continuous geometries" due to von Neumann is briefly described.

Chapters V and VI contain a study of distributive lattices, i.e., those in which $(a \cup b) \cap c = (a \cap c) \cup (b \cap c)$, and boolean algebras, i.e., complemented distributive lattices. The representation theory, which connects abstract boolean algebras with algebras of sets, and the applications of boolean algebras to topology are due chiefly to Stone.

The application in Chapter VII of lattice theory to function theory is more in the nature of an abstract theory of linear spaces which are partially ordered. Applications to logic in Chapter VIII are quite natural; logics considered are Brouwerian logic, classical logic and quantum logic. The final chapter continues Chapter VIII: A probability functional on a boolean algebra is a positive functional $p(x)$ such that $p(x \cup y) + p(x \cap y) = p(x) + p(y)$, and $p(O) = 0$, $p(I) = 1$. Probability functionals are found to lie in a vector lattice; the properties of this

# NOTES

On Friday, December 27, there was held at the University of Pennsylvania the meeting of Section A of the Association for the Advancement of Science. At that time Professor Marston Morse of the Institute for Advanced Study gave his address as retiring Chairman of the Section. He spoke on *A mathematical theory of equilibrium with applications to minimal surfaces*. By special invitation Dr. C. G. Rossby of the United States Weather Bureau addressed the Section on *The mathematical problems in meteorology*. Papers read before Section A at these meetings were, by special arrangement, regarded as having been read before the Society by title. In this manner papers 61, 62, 84 and 90 of the report of the annual meeting were presented. About one hundred persons were present at the meetings.

Stanford University is celebrating its fiftieth anniversary during the year 1941. In this connection a larger than usual summer quarter program is planned. Professor G. T. Whyburn from the University of Virginia will give a course on Combinatorial Topology, Professor Einar Hille from Yale University a course on Selected Topics in Theory of Functions. During the summer, presumably in the first part of August, a two-day symposium is planned with conferences on Topology, Calculus of Probability and Statistics, Theory of Functions, and general subjects. The following speakers have been secured so far: E. T. Bell, H. F. Blichfeldt, Einar Hille, A. D. Michal, Jerzy Neyman, George Polya, G. T. Whyburn, Antoni Zygmund. The program will be announced later.

The National Council of Teachers of Mathematics held its twenty-first annual meeting in Atlantic City on February 21–22, 1941. Details of the program and places of meeting are given in the February issue of The Mathematics Teacher.

*A Survey of Methods of Apportionment in Congress* by Professor E. V. Huntington has recently been published. Copies can be obtained from the Superintendent of Documents, Washington, D. C. for ten cents each.

A plaque has recently been presented to Washington University by the United States Naval Academy in honor of William Chauvenet who was appointed professor of mathematics and astronomy at this University in 1852 and chancellor in 1862. The plaque is a replica of one at the Academy and it bears the inscription: "William Chauvenet, Professor of Mathematics, United States Navy and President of the Academic Board from 1847 to 1850, largely through whose efforts and plan the Naval Academy was established and organized at Annapolis."

The Sylvester Medal of the Royal Society has been awarded to Professor G. H. Hardy of the University of Cambridge for his important contributions to pure mathematics.

Professor Abraham Cohen of Johns Hopkins University has been given the title of professor emeritus.

Professor R. A. Fisher of University College, London, will be visiting professor of experimental statistics at North Carolina State College during the summer session June 16–July 25, 1941.

Dr. A. D. Bradley of Hunter College has been promoted to an assistant professorship.

Dr. A. A. F. Brown has accepted a position in the statistical department of L. Bamberger and Company in Newark, New Jersey.

Dr. G. W. Brown has taken a position in the research department of R. H. Macy and Company of New York City.

Assistant Professor Gertrude M. Cox of Iowa State College has been appointed professor of experimental statistics and director of the statistical laboratory at North Carolina State College.

Mr. R. D. Dorsett of the University of Oklahoma has been promoted to an assistant professorship.

Mr. L. T. Dunlap of Pennsylvania State College has been promoted to an assistant professorship.

Assistant Professor E. J. Finan of the Catholic University of America has been promoted to an associate professorship.

Dr. H. C. Fryer of Kansas State College has been promoted to an assistant professorship.

Associate Professor Gordon Fuller of Alabama Polytechnic Institute has been promoted to a professorship.

Dr. Beatrice L. Hagen of Pennsylvania State College has been promoted to an assistant professorship.

Assistant Professor J. J. L. Hinrichsen of Iowa State College has been promoted to an associate professorship.

Assistant Professor F. H. Hodge of Purdue University has retired.

Dr. L. P. Hutchison of Indiana Technical College has been appointed to an assistant professorship at The Citadel, Charleston, South Carolina.

Dr. Fulton Koehler of the University of Minnesota has accepted a position in the actuarial department of the Northwestern National Life Insurance Company.

Dr. Szolem Mandelbrojt, formerly of the College of France, has been appointed visiting lecturer at the Rice Institute.

Dr. G. E. Moore of the University of Illinois has been promoted to an assistant professorship.

Dr. H. R. Phalen has been appointed to an associate professorship at the College of William and Mary.

Professor H. R. Pyle of Earlham College has been appointed to a professorship at Whittier College.

Associate Professor C. B. Read of the University of Wichita has been promoted to a professorship.

Dr. S. T. Sanders, Jr., of Delta State Teachers College has been appointed to a professorship at Southwestern Louisiana Institute.

Associate Professor I. M. Sheffer of Pennsylvania State College has been promoted to a professorship.

Assistant Professor D. T. Sigley of Kansas State College has been promoted to an associate professorship.

Assistant Professor S. B. Townes of the University of Oklahoma has been promoted to an associate professorship.

Assistant Professor G. C. Vedova of St. John's College has been appointed to an assistant professorship at the University of Maryland.

Dr. Henry Wallman of the Institute for Advanced Study has been appointed to an assistant professorship at the University of North Carolina.

Associate Professor Morgan Ward of the California Institute of Technology has been promoted to a professorship.

Dr. G. C. Webber of the University of Delaware has been promoted to an assistant professorship.

Professor Oscar Zariski of Johns Hopkins University is on leave of absence for 1940–1941 and is at Harvard as a visiting lecturer.

Dr. M. M. Day of the University of Illinois has been made an associate in mathematics.

The following appointments to instructorships are announced: University of California at Los Angeles: Dr. Ralph Byrne, Jr.; Columbia University: Dr. Walter Strodt; Cornell University: Mr. N. G. Gunderson, Mr. R. R. R. Luckey; University of Dayton: Dr. K. C. Schraut; Georgia School of Technology: Mr. L. B. Williams; University of Illinois: Dr. R. S. Pate; Iowa State Teachers College: Dr. H. C. Trimble; Itasca Junior College: Dr. H. L. Olson; University of Kansas: Mr. L. R. Shobé; Northwestern University: Mr. H. E. Burns; Ohio State University: Dr. W. J. Schart; Oregon State College: Dr. P. C. Hammer; Rutgers University: Dr. J. H. Giese; University of Texas: Mr. J. R. Foote, Mr. H. J. Jones, Dr. A. M. Mood, Mr. I. C. Roberts; Trinity College, Washington, D. C.: Dr. Mary Varnhorn; Vermont Junior College: Mr. D. F. Johnson; University of Wichita: Mr. Edison Greer; University of Wisconsin: Dr. C. B. Smith; College of Wooster: Mr. M. P. Fobes.

The death on December 4, 1940 of Mr. E. H. Hayes, formerly mathematical tutor of New College, Oxford, at the age of eighty-six years has been reported.

The death of Dr. Myron Mathisson, formerly of the University of Warsaw, is reported in Nature.

Professor I. Schur of the University of Berlin died on January 10, 1941, in Tel Aviv.

Rev. Canon Alfred Young, formerly lecturer in mathematics in Selwyn College, Cambridge, died on December 19, 1940, at the age of sixty-seven years.

Dean W. C. Graustein of Harvard University died January 22, 1941. He had been a member of the Society since 1911. A more adequate notice will appear in a later issue.

Professor G. A. Knapp of Maryville College, Maryville, Tennessee, died in November, 1940.

Assistant Professor C. R. Wilson of Rutgers University died in July, 1940. He had been a member of the Society since 1929.

# ABSTRACTS OF PAPERS

## SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

## ALGEBRA

### 115. J. H. Bell: *Topics related to the factorization of matrices.*

This paper deals with two problems that are of particular interest in the solution of the unilateral matrix equation. Let $\mathfrak{D}$ be a quasi-field and $\mathfrak{D}[\lambda]$ the polynomial ring of $\mathfrak{D}$, where $\lambda$ is a commutative indeterminate. The first problem is: Under what conditions is the matrix $A$ with elements in $\mathfrak{D}[\lambda]$ the left associate of a matrix which is proper of degree $k$ in $\lambda$? A necessary and sufficient condition is obtained. The proof may be shortened in the case that $\mathfrak{D}$ is a field $\mathfrak{F}$. The second problem, which is considered only in the case of a field $\mathfrak{F}$, is: Given two nonsingular matrices $A$ and $B$ with elements in $\mathfrak{F}[\lambda]$; if $A$ and $B$ are not left associates, what other proper right divisors of the least common left multiple of $A$ and $B$ exist which are not left associates of $A$ or $B$? Various results are obtained for the second problem. These results may be used, with the result obtained in solving the first problem, to show the existence of families of solutions of the unilateral matrix equation, if two solutions satisfying certain conditions are known to exist. (Received January 23, 1941.)

### 116. A. H. Clifford: *Semigroups admitting relative inverses.*

A semigroup is a system $S$ closed under an associative binary operation: $(ab)c = a(bc)$. $S$ is said to admit relative inverses if for each $a$ in $S$ there exists an element $e$ of $S$ such that (1) $ea = ae = a$, and (2) $aa' = a'a = e$ for some $a'$ in $S$. The "gross structure" of such an $S$ is determined by showing that $S$ is the class sum of mutually disjoint semigroups $S_\alpha$ of known structure (A. Suschkewitsch, D. Rees) which can be arranged in a semi-lattice $P$ such that the set-product $S_\alpha S_\beta$ of any two of them is wholly contained in the "cross-cut" $S_\gamma$ of $S_\alpha$ and $S_\beta$ in $P$. If the idempotent elements of $S$ commute with each other, then the $S_\alpha$ are groups, and the structure of $S$ can be given completely. (Received January 13, 1941.)

### 117. A. H. Clifford and Saunders MacLane: *Factor sets of a group in its abstract unit group.*

This paper is an investigation of the structure of the set of group extensions of a certain abstract unit group. This group is one which arises in class field theory and has as group of operators a certain finite group $\Gamma$. If $\Gamma$ is solvable, it is shown by an explicit reduction that the group of group extensions is isomorphic to Schur's multiplicator for the group $\Gamma$. (Received January 28, 1941.)

### 118. Marshall Hall: *A problem in partitions.*

Non-void subsets $a_1, \cdots, a_n$ of a set $S(x_1, \cdots, x_n)$ determine a partition matrix

$(a_{ij})$ in which $a_{ij}=0$ or 1 according as $a_i$ and $a_j$ overlap or not. The question has arisen as to whether every $n^2$ matrix $(a_{ij})$ in which $a_{ii}=1$, $a_{ij}=a_{ji}=0$ or 1 is the partition matrix of $n$ subsets of $n$ objects. This is shown to be true for $n=1, 2, 3, 4$, but not for $n\geqq 5$. It is shown that for $n\geqq 5$ every such matrix is a partition matrix but may require as many as $n^2/4$ ($n$ even) or $(n^2-1)/4$ ($n$ odd) objects in $S$. These values are "best possible" and it is shown that a matrix requiring the maximum number of objects is unique to within an equivalence amounting to renumbering $a_1, \cdots, a_n$. (Received January 24, 1941.)

### 119. E. R. Kolchin: *On the basis theorem for differential systems.* Preliminary report.

The basis theorem of J. F. Ritt and H. W. Raudenbush for systems of differential polynomials is proved by a new method making no use of ascending sets, but using the axiom of choice. The new proof justifies its existence by the greater generality of its results. For example, in addition to the systems for which the theorem has already been proved, systems of forms with coefficients in a differential field of characteristic zero, the theorem is proved for systems of forms with coefficients in the ring of integers, coefficients in the ring of integral polynomials in $x$, and coefficients in the field of integers modulo a prime. (Received January 25, 1941.)

### 120. Howard Levi: *On the ideal theory and structure of differential polynomials.*

In the first part of this paper the differential ideal $\Sigma$ generated by the form $y^p$ is investigated. It is understood that $p$ is a positive integer greater than unity, and that the underlying ring is that of all differential polynomials in the unknown $y$ whose coefficients are in some differential domain of integrity which contains the rational numbers. Necessary and sufficient conditions for the membership of a form in $\Sigma$ are obtained. These results are used to show that the criteria (sufficient) of J. F. Ritt for the essentiality of manifolds (Annals of Mathematics, (2), vol. 37 (1936), pp. 556–560, and American Journal of Mathematics, vol. 60 (1938), pp. 5–14) are valid in the abstract case. Extensions are made to cases not discussed by him. Finally the differential ideal generated by the form $uv$ in the ring of forms in the unknowns $u$ and $v$ is examined. Necessary and sufficient conditions for membership in this ideal are obtained. In particular it is shown that for all positive integers $r$ and $s$ the form $u^r v_r^s$ is not in the ideal ($v_r$ being the $r$th derivative of $v$). This implies that the ideal in question has no representation as the intersection of differential ideals. (Received January 24, 1941.)

### 121. Oystein Ore: *Theory of monomial groups.*

If $\{x_i\}$ denotes a set of variables and $H$ some arbitrary group, then the set of all linear transformations $x_i \rightarrow h_i x_j$ where the factors $h_i$ belong to $H$, form a group called the complete monomial group. The present paper contains a study of these groups and gives among other results the determination of all normal subgroups and the group of automorphisms. (Received January 23, 1941.)

### 122. Oystein Ore: *Transformation of sets.*

The set of all one-to-one correspondences of a set $S$ to itself form a group $\Sigma_s$. It is shown that for an arbitrary infinite set this group is complete, that is, all automorphisms are inner automorphisms. (Received January 23, 1941.)

### 123. P. C. Rosenbloom: *Post algebras: I. Postulates and general properties.*

Post algebras are the algebras which bear the same relation to the $n$-valued logics defined by Post (American Journal of Mathematics, vol. 43 (1921), pp. 180–185) as Boolean algebras bear to the usual 2-valued logic. They are investigated here purely from the algebraic standpoint with no consideration of their interpretation as logic. The first postulate sets for these systems are introduced and the most important general properties are deduced. A fundamental theorem analogous to the Boolean expansion of functions in normal form is proved. The definition of "prime elements" is analogous to Huntington's definition for the Boolean algebras. "Powers of primes" are also defined. A theorem analogous to the fundamental theorem of arithmetic is proved to the effect that "in a Post algebra with a finite number of elements, every element is uniquely factorable into a product of powers of primes, disregarding order and repetition of factors." An arithmetic interpretation generalizing Sheffer's "Boolean numbers" is given. Several unsolved problems are discussed. (Received January 17, 1941.)

### 124. Ernst Snapper: *Structure of linear sets.*

It is shown that the linear sets of a vector space of arbitrary dimension over an integral domain in which every ideal has a finite basis admit a Noether decomposition into "primary" linear sets. The "associated prime ideals" of the largest primary components are uniquely determined invariants of the linear set. The proofs are based on definitions of "quotient ideal" of a linear set by a linear set, of "quotient linear set" of a linear set by an ideal, and of "product linear set" of an ideal and a linear set. The quotient ideal of a linear set by the whole space is called its "essential scalar ideal" and is fundamental in the definition of primary linear set. The radical of the essential scalar ideal of a primary linear set is prime and is called its associated prime ideal. The "isolated component linear sets" are uniquely determined by their corresponding prime ideals and the theory becomes the ordinary ideal theory in the case of dimension one. Also, this investigation gives rise to the notions of scalar ideal, almost-primarity, radical and essential radical, closed set, and dense set. (Received January 14, 1941.)

## ANALYSIS

### 125. R. P. Agnew: *On methods of summability and mass functions determined by hypergeometric coefficients.*

Let $\alpha$, $\beta$, $\gamma$ be complex constants and $\gamma \neq 0, -1, -2, \cdots$. Let $\lambda_n(\alpha, \beta, \gamma)$, $n = 0, 1, 2, \cdots$, be the coefficients in the power series expansion $\sum \lambda_n z^n$ of the hypergeometric function $F(\alpha, \beta, \gamma; z)$. Let $(H, \alpha, \beta, \gamma)$ be the Hurwitz-Silverman-Hausdorff method of summability generated by the sequence $\lambda_n(\alpha, \beta\ \gamma)$. (See Garabedian and Wall, Transactions of this Society, vol. 48 (1940), pp. 195–201.) Let $C_r$ denote the Cesàro method of order $r$. For certain ranges of the parameters it is shown that $(H, \alpha, \beta, \gamma) = C_{\alpha-1}^{-1} C_{\beta-1}^{-1} C_{\gamma-1}$ and that $(H, \alpha, \beta, \gamma)$ is equivalent to $C_{\gamma-\alpha-\beta+1}$. These results determine conditions under which $(H, \alpha, \beta, \gamma)$ is regular and $\lambda_n(\alpha, \beta, \gamma)$ is the moment sequence of a regular mass function. (Received December 31, 1940.)

### 126. E. F. Beckenbach: *On almost subharmonic functions.*

It is shown that certain integral inequalities which, in the case of continuous functions, are known to characterize subharmonic functions and functions whose logarithms are subharmonic, no longer characterize these functions when mere summability is assumed. Instead, the inequalities characterize functions which, in the terminology of Szpilrajn, are almost subharmonic or have almost subharmonic logarithms. (Received January 27, 1941.)

### 127. Stefan Bergman: *On a class of linear operators applicable to functions of a complex variable.*

The totality of functions obtained from the analytic functions of one complex variable regular at the origin by applying the operator $u=\mathrm{P}(f)=\int_{-1}^{+1}\mathrm{E}(z, \bar{z}, t)f[z(1-t^2)/2](1-t^2)^{-1/2}dt$ is called the "class of functions" $\mathcal{C}(\mathrm{E})$. The author studies the properties of functions $u(z, \bar{z})\in\mathcal{C}(\mathrm{E})$, where $\mathrm{E}=1+z\bar{z}t^2\mathrm{E}_1(z, \bar{z}, t)$. Functions $u$ possess the following property: to every point $Z$ there exists a function $H(z, \bar{z}; Z)\in\mathcal{C}(\mathrm{E})$ such that every $u$ satisfies the functional equation $u(z, \bar{z})=(2\pi i)^{-1}\int_{\mathfrak{a}^1}R(Z; u)H(z, \bar{z}; Z)dZ$; $R(Z, u)$ being a certain operator, and $\mathfrak{a}^1$ an arbitrary closed differentiable curve lying in the regularity domain of $u(z, \bar{z})$. The theorems of the theory of analytic functions of one complex variable are classified from a certain point of view, and the duality between theorems for functions of one complex variable and those for the class $\mathcal{C}(\mathrm{E})$ is studied. In this way the author obtains results on the development of $u$ in series $\sum_{\nu=1}^{\infty}a_\nu u_\nu$ in certain domains, $\{u_\nu\}$ being a special set of functions depending on the domain, and further results on the connection between the coefficients $A_{mn}$ of the development $u=\sum A_{mn}z^m\bar{z}^n$ and properties of $u$, and so on. For example, functions satisfying a linear partial differential equation of elliptic type form a class $\mathcal{C}(\mathrm{E})$ corresponding to a particular $\mathrm{E}$. (Received January 22, 1941.)

### 128. Lipman Bers: *Analytic functions of two complex variables in domains bounded by two analytic hypersurfaces.*

Let $\mathfrak{M}^4$ be a domain of the $z_1$, $z_2$-space $(z_i=x_i+iy_i)$ bounded by two analytic hypersurfaces whose intersection (the *distinguished boundary surface* of $\mathfrak{M}^4$) forms a closed surface $\mathfrak{F}^2$: $z_i=Z_i(\lambda_1, \lambda_2)$, $0\leqq\lambda_i\leqq 2\pi$. If certain hypotheses (chiefly about the existence of a suitable set of subdomains of the same type) are made, one can generalize theorems proved for simple special cases of such domains (S. Bergman, Compositio Mathematica, vol. 6 (1939), p. 305; S. Bergman and J. Marcinkiewicz, Fundamenta Mathematicae, vol. 33 (1939), p. 75; L. Bers, Comptes Rendus de l'Académie des Sciences, Paris, vol. 208 (1939), p. 1273 and p. 1475). If $U$ is a biharmonic function in $\mathfrak{M}^4$ which possesses continuous boundary values $u$ on $\mathfrak{F}^2$, then $U(z_1, z_2)=\int_0^{2\pi}\int_0^{2\pi}\Omega(z_1, z_2; \lambda_1, \lambda_2\ u(\lambda_1, \lambda_2)d\lambda_1d\lambda_2$ where $\Omega$ depends only upon the domain. Biharmonic functions which satisfy certain conditions (for example all non-negative functions) may be represented in the form: $U(z_1, z_2)=\int_0^{2\pi}\int_0^{2\pi}\Omega(z_1, z_2; \lambda_1, \lambda_2)d\omega_{\lambda_1,\lambda_2}$; $U$ possesses a finite non-tangential boundary value in every point of $\mathfrak{F}^2$ where the absolute additive set function $\omega$ possesses a finite strong derivative. It follows from these theorems that a function $f(z_1, z_2)$ which may be represented in $\mathfrak{M}^4$ as a quotient of two bounded analytic functions possesses almost everywhere on $\mathfrak{F}^2$ finite non-tangential boundary values. If these boundary values are constant on a set of positive two-dimensional measure, $f$ is constant. (Received January 22, 1941.)

### 129. R. P. Boas: *A note on functions of exponential type.*

It has been shown by D. V. Widder (Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 657–659) that a function $f(x)$ such that $(-1)^nf^{(2n)}(x)\geqq 0$ for $n=0, 1, 2, \cdots$ and $0\leqq x\leqq 1$ must coincide over $(0, 1)$ with an entire function which is at most of the first order (and of type at most $\pi$ if it is of order 1). In this note a simple proof of Widder's result is obtained by integrating $\int f(x) \sin \pi x \, dx$ repeatedly by parts; the integrated terms are all non-negative, and consequently an upper bound for $\int |f^{(2n)}(x)| \sin \pi x \, dx$ follows; from this an adequate estimate of $|f^{(2n)}(x)|$ itself is easily obtained. The same method leads to simple proofs of theorems of I. J. Schoenberg (this Bulletin, vol. 42 (1936), pp. 284–288) and J. M. Whittaker and H. Poritsky (Whittaker, Proceedings of the London Mathematical Society, (2), vol. 36 (1933), p. 455; Poritsky, Transactions of this Society, vol. 34 (1932), p. 287). (Received January 21, 1941.)

### 130. H. L. Garabedian: *Hausdorff methods of summation which include all of the Cesàro methods.*

In this paper are exhibited Hausdorff methods of summation which have the rare property of being more effective than all of the Cesàro methods of real orders. The regular mass functions associated with these Hausdorff methods are notably $\phi_\alpha(u) = 1-e^{\alpha u/(u-1)}$, $\alpha>0$, and $\phi_\beta(u)=A\int_0^u(1-t)^{\beta-2}e^{t/(t-1)}dt$, $\beta\geqq 0$, where $A$ is a normalizing factor such that $\phi_\beta(1)=1$. The conjecture is made that summability $[H, \phi_\beta(u)]$ increases in efficiency with increasing $\beta$. (Received December 26, 1940.)

### 131. Abe Gelbart: *On the behavior of a function of two complex variables in the neighborhood of an isolated essential singularity.*

In this paper the author obtains certain upper and lower bounds for classes of functions of two complex variables in the neighborhood of an isolated essential singularity. Given a sequence of functions $\{h_\nu(z_1, z_2)\}$ regular in a finite four-dimensional domain $\overline{\mathfrak{M}}^4$ such that the analytic surfaces $\{h_\nu(z_1, z_2)=0\}$ converge to the surface $z_1=g(z_2)$ which passes through the domain $\mathfrak{M}^4$. It is assumed that the analytic surfaces $h_\nu(z_1, z_2)=0$ converge to $z_1=g(z_2)$ with sufficient rapidity. With the aid of theorems of the Runge type for functions of two complex variables for multiply-connected four-dimensional domains (see abstract 46-9-418), the author obtains an upper bound for the moduli of functions of two complex variables belonging to a class of functions analytic in $\overline{\mathfrak{A}}^4=\overline{\mathfrak{M}}^4-G^4$, $G^4=E\,[z_1=g(z_2)+\rho e^{i\lambda}, 0\leqq\lambda\leqq 2\pi, 0<\rho<\epsilon]$, and having the same zeros as $\prod_{\nu=1}^{\infty} h_\nu(z_1, z_2)$. Using a technique similar to that of Bergman (Mathematische Annalen, vol. 109 (1934), pp. 324–348), upper and lower bounds are obtained for an arbitrary function meromorphic in the neighborhood of the analytic surface $z_1=g(z_2)$ and having the same zeros and poles as $\prod_{\nu=1}^{\infty} h_\nu(z_1, z_2)/p_\nu(z_1, z_2)$ wherein $p_\nu(z_1, z_2)$ are functions regular in the domain $\overline{\mathfrak{M}}^4$. (Received January 23, 1941.)

### 132. H. H. Goldstine: *The prolongment and representation of multilinear functionals.*

It is first shown that every Banach space is linearly equivalent to a linear subset of the space of all bounded and real-valued functions defined on a suitably chosen range. Every linear continuous functional on this space of functions is representable by a Lebesgue-Stieltjes integral. Making use of this theorem, the Hahn-Banach theorem can be proved for multilinear, continuous functionals; that is, the range of definition of every such functional can be extended to the entire space without changing the norm of the functional. At the same time it is shown that these functionals are representable as generalized Fréchet integrals. (Cf. Goldstine, *Bilinear functionals on the space of bounded, measurable functions*, this Bulletin, vol. 43 (1937), pp. 528–531.) (Received December 16, 1940.)

## 133. P. R. Halmos: *On the decomposition of measures.*

If $S$ is a strictly separable measure space and $A$ is a strictly separable Borel field of measurable sets then, except possibly for a set (in $A$) of measure zero, $S$ is a direct sum of measure spaces $Y_x$ formed with respect to a measure space $X$ in such a way that the Borel field of all measurable sets depending on $x$ alone coincides with the given Borel field $A$. This general decomposition theorem yields an easy proof of a result of von Neumann on the decomposition of an arbitrary measure preserving transformation into ergodic parts. Another application of the theorem is to the spectral theory of measure preserving transformations: it is possible by means of it to construct examples of transformations with mixed spectrum—that is, transformations having neither pure point spectrum nor pure continuous spectrum—and to decompose, at least partially, an arbitrary transformation into transformations with pure spectrum. (Received January 22, 1941.)

## 134. D. H. Hyers: *On the stability of the linear functional equation.*

Let $E$ and $E'$ be linear normed spaces and let $\delta$ be a positive number. A transformation $f(x)$ of $E$ into $E'$ will be called $\delta$-linear if $\|f(x+y)-f(x)-f(y)\|<\delta$ for all $x$ and $y$ in $E$, and linear if $f(x+y)=f(x)+f(y)$. S. Ulam has proposed the following problem of the "stability" of the last equation: Does there exist for each $\epsilon>0$ a $\delta>0$ such that to each $\delta$-linear transformation $f(x)$ there corresponds a linear transformation $g(x)$ satisfying the inequality $\|f(x)-g(x)\|\leqq\epsilon$ for all $x$ in $E$? In this paper this question is answered in the affirmative for the cases where $E$ and $E'$ are finite dimensional spaces or real Hilbert space, and where $f(x)$ is continuous. It is shown that $\delta$ may be taken equal to $\epsilon$ in these cases. As Ulam has pointed out, the problem has many generalizations to be investigated. (Received January 6, 1941.)

## 135. H. N. Laden: *On Kryloff-Stayermann interpolation.*

This paper deals with an interpolation polynomial $F_n(x)$ of degree less than or equal to $4n-1$ which, at the preassigned abscissas $x_1, x_2, \cdots, x_n$ takes preassigned values $y_1, y_2, \cdots, y_n$ respectively and is such that $F_n'(x_i)=F_n''(x_i)=F_n'''(x_i)=0$, $(i=1, 2, \cdots, n)$. If the abscissas are the zeros of the $n$th Jacobi polynomial and $y_i=f(x_i)$, $(i=1, 2, \cdots, n)$, where $f(x)$ is an arbitrary function continuous on $[-1, +1]$, it is shown that $F_n(x)$ converges to $f(x)$ as $n\rightarrow\infty$ uniformly on any closed interval wholly inside $[-1, +1]$. The convergence is uniform on $[-1, +1]$ if proper restrictions are placed on the parameters $\alpha$, $\beta$ of the general Jacobi polynomial. Similar results are obtained for interpolation on an infinite interval by means of Laguerre and Hermite polynomials. The first part of the results provides a correction and extension of a theorem of Kryloff and Stayermann. The methods employed are due to Fejér, Shohat, and Szegö. (Received January 11, 1941.)

## 136. Dorothy Maharam: *On measure in abstract sets.*

Let $m$ be a Boolean $\sigma$-ring of sets; $n$, a $\sigma$-ideal in $m$; $\Phi$, a class of $\sigma$-isomorphisms between principal ideals in $m/n$. It is assumed that $\Phi$ contains all products and inverses of its elements. Define the elements of $m$ ($n$) to be measurable (null) sets, the isomorphisms of $\Phi$ to be measure-preserving transformations, and the measure of an element $a$ of $m/n$ to be the totality of division images of $a$ under $\Phi$. The element $a$ is bounded if $a$ is not division-equivalent to any proper sub-element of itself. The set $C$ of bounded measure values is then described; in particular, conditions are given under which $C$ is isomorphic to a set of positive numbers. (Received January 22, 1941.)

### 137. Szolem Mandelbrojt: *On a Dirichlet series.*

This paper gives necessary and sufficient conditions on $M(\sigma)$ such that there exist a function $f(s)=\sum a_n n^{-s}$, $f(s)\neq 0$, $f(-2q)=0$ $(q\geqq 1)$, $\max |f(\sigma+it)|<M(\sigma)$. (Received December 16, 1940.)

### 138. N. M. Oboukhoff: *On the validity of total differential as the principal part of the increment of a function of two or more variables.*

K. Weierstrass defined total differential as principal part of the increment of a function. Let $du=Adx+Bdy$ where $A$ and $B$ are finite functions of $x$ and $y$ only, $|dx|$ and $|dy|$ being infinitely small or just small enough relative to $|A|$ and $|B|$. If the terms are all positive or negative, $du$ retains its rank of principal part. However there are cases in which this principal part degenerates. Suppose $du=Adx-Bdy$ and $Adx>Bdy>0$ or $Bdy>Adx>0$. Let $|Adx-Bdy|=K(Adx+Bdy)$, where $0\leqq K\leqq 1$. If $K$ is finite or great enough relative to $|dx|$ and $|dy|$ then $du$ does not degenerate. However it does if $K$ vanishes or is of the same order of magnitude as $|dx|$ and $|dy|$. Thus $K=\big|\,|dx/dy|\,|A|-|B|\,\big|:(|dx/dy|\,|A|+|B|)$; $du$ degenerates if $|dx/dy|=|B/A|\pm\eta$, where positive $\eta$ either vanishes or is of the same order of magnitude as $|dx|$ and $|dy|$. For all other values of $|dx/dy|$ from zero to infinity (where $dy$ vanishes) principal part holds. The favorable cases are greatly predominant. Closely similar results obtain for functions of several variables. (Received December 5, 1940.)

### 139. George Polya: *On converse gap theorems.*

*Let* $\lambda_1, \lambda_2, \lambda_3, \cdots$ *be integers,* $0<\lambda_1<\lambda_2<\cdots$. *The* necessary *and sufficient condition that* each *analytic function defined by a series of the form* $a_1z^{\lambda_1}+a_2z^{\lambda_2}+\cdots+a_nz^{\lambda_n}+\cdots$ *having a finite radius of convergence be uniform and its domain of definition be simply connected is that* (1) $\liminf_{n\to\infty} n\lambda_n^{-1}=0$. The much deeper half of this theorem concerning the sufficiency of the condition (1) has been proved previously (Annals of Mathematics, (2), vol. 34 (1933), pp. 731–777). The necessity of the condition (1) may be shown by the series (2) $\sum_{m=1}^{\infty}\mathcal{J}(m)z^m/m=\sum_{n=1}^{\infty}(-1)^{\lambda_n}z^{\lambda_n}/\lambda_n^2\mathcal{G}'(\lambda_n)$ where $\mathcal{J}(z)$ and $\mathcal{G}(z)$ are defined by $\pi z\mathcal{J}(z)\mathcal{G}(z)=\sin \pi z$, $\mathcal{G}(z)=\prod_{n=1}^{\infty}(1-z^2/\lambda_n^2)$. If (1) is not fulfilled, the analytic continuation of the series (2) is infinitely many valued (for proving this use Mathematische Zeitschrift, vol. 29 (1929), pp. 604–608). An analogous theorem has been proved by the author in the Comptes Rendus de l'Académie des Sciences, Paris, vol. 208 (1939), pp. 709–711. (Received January 20, 1941.)

### 140. Tibor Radó: *On the semi-continuity of double integrals in the calculus of variations.*

The purpose of this paper is to extend the results of McShane (*On integrals over surfaces in parametric form*, Annals of Mathematics, (2), vol. 34 (1933), pp. 815–838) to a wide class $K$ of surfaces which may be characterized as follows: A surface $S$, of the type considered by McShane, belongs to the class $K$ if it admits of a representation where its Lebesgue area is given by the usual integral formula. All the results of McShane are shown to remain valid for this class $K$. (Received December 16, 1940.)

### 141. Raphaël Salem: *On absolute convergence of trigonometrical series.* I.

The following theorems are proved: If the series (S) $\sum \rho_n \cos(nx-\alpha_n)$, $(\rho_n\geqq 0)$ converges absolutely at two points $x_0$, $x_1$, the series $\sum\rho_n|\sin n(x_1-x_0)|$ converges. If

$\sum \rho_n = \infty$ the series (S) cannot converge absolutely at more than one point if $\rho_{n+1} \leqq \rho_n$ or even if $\rho_{n+p}/\rho_n$ is bounded $(p>0)$. If $\sum \rho_n = \infty$, the series $\sum \rho_n \cos (k_n x - \alpha_n)$ cannot converge absolutely at two points whose abscissae differ from $\delta$ if the numbers $k_n\delta$ are uniformly distributed (mod $2\pi$). If $\sum \rho_n = \infty$ and the series (S) converges absolutely in a perfect set $N$, then for every bounded function $F$ non-decreasing, constant in every interval contiguous to $N$ but increasing from one interval to another, the cosine Fourier-Stieltjes coefficients of even rank of $dF$ have their upper limit equal to $F(2\pi) - F(0)$. This condition is sufficient to ensure the absolute convergence of (S) "almost everywhere" in $N$, with $\sum \rho_n = \infty$. Further properties of sets of the type $N$ are discussed. (Received January 30, 1941.)

### 142. Raphaël Salem: *On absolute convergence of trigonometrical series.* II.

The following extension of the Denjoy-Lusin theorem is proved: The set for which the ratio (I) $\sum_1^n \rho_n \left| \cos (nx - \alpha_n) \right| / \sum_1^n \rho_n$ has its upper limit less than or equal to $\alpha$ is of measure zero if $\alpha < 2/\pi$. Moreover any perfect set $P$ in which the ratio (I) has its upper limit less than or equal to $\alpha$ less than $2/\pi$ has the property that every bounded function $F$, non-decreasing, constant in every interval contiguous to $P$, but not everywhere, is such that the Fourier-Stieltjes coefficients of $dF$ do not tend to zero. The constant $2/\pi$ is the best possible one; in fact, if $\rho_n = O(1)$, the ratio (I) tends to $2/\pi$ almost everywhere. (Received January 30, 1941.)

### 143. I. J. Schoenberg: *On integral representations of completely monotone and related functions.*

In a recent paper W. Feller proved very elegantly the chief results of S. Bernstein and Widder concerning the nature of functions $f(x)$ allowing a Laplace-Stieltjes representation (1) $f(x) = \int_0^\infty e^{-xt} d\alpha(t)$ with various restrictions as to $\alpha(t)$. In the first part of the paper, Widder's original tools are used for the purpose of an equally short presentation of this theory. In particular, Widder's inversion formula for (1) appears in a very natural way. In the second part a similar theory is developed for the representation (2) $f(x) = \int_0^\infty K_n(xt) d\alpha(t)$ $(x>0$; $n$ a positive integer) where $K_n(x) = (1-x)^n$ if $0 \leqq x \leqq 1$, $K_n(x) = 0$ if $x > 1$. Since $K_n(x/n) \to e^{-x}$ as $n \to \infty$, (1) is seen to be a limiting case of (2). The results may be described very roughly as follows: The role played by the *completely monotone functions* $f(x)$ (that is, satisfying the conditions (3) $(-1)^n f^{(n)}(x) \geqq 0$ $(x>0$; $n=0, 1, 2, \cdots))$ in the theory of the representation (1), is taken over in the case of (2) by the $(n+1)$*-times monotone functions* $f(x)$. Here a function is called $(n+1)$-times monotone if $f(x) \geqq 0$, $-f'(x) \geqq 0$, $f''(x) \geqq 0, \cdots,$ $(-1)^{n-1} f^{(n-1)}(x) \geqq 0$, $(x>0)$, the last function $(-1)^{n-1} f^{(n-1)}(x)$ being also required to be non-increasing and convex for $x>0$. (Received January 27, 1941).

### 144. W. T. Scott and H. S. Wall: *The transformation of series and sequences.*

The transformation $S_n = \sum_{p=0}^n (1/2)^{2n+1} C_{2n+3,n-p} \cdot s_p$, $(n=0, 1, 2, \cdots)$, which arises from the power series transformation $\frac{1}{2}(1-x)f(x) = F(z)$, $z = 4x/(1-x)^2$, is shown to define a regular method of summation, that is, $s_p \to s$ implies $S_n \to s$. Let this be called Schur summability and denoted by the letter $S$. It is proved that $S \supset VP$, but $VP \not\supset S$, where $VP$ denotes de la Vallée Poussin summability; $S$ sums the geometrical series in precisely the region in which $VP$ sums this series; $S \approx M$, where $M$ is the summability introduced by Mersman (this Bulletin, vol. 44 (1938), pp. 667–673). It is shown that if $\phi(u) \in BV[0, 1]$, and $g(x) = \int_0^1 d\phi(u)/(1+xu)$, then the sequence $\{g(n)\}$, $(n=0, 1, 2, \cdots)$, is a *C-Folge* in the sense of Hausdorff, and the Hausdorff mean $[H, g(n)]$ is regular if and only if $\phi(1)-\phi(0)=1$. If $\phi(u)$ is real and monotone and $\phi(1)-\phi(0)=1$, then $H, g(n) \subset (C, 1)$, and is equivalent to convergence if and only if the function $g(x)$ is bounded away from zero in the half-plane $R(x) > -\frac{1}{2}$. Let $\theta(u) = \sum_{p=1}^{\infty} q_p u^p$, $q_p \geqq 0$, $\sum q_p < \infty$. Then $a_n = 1 + n\int_0^1 u^n d\theta(u)$, $(n=0, 1, 2, \cdots)$, defines a Hausdorff mean $[H, a_n]$ equivalent to convergence if and only if $\sum nq_n < \infty$. (Received January 24, 1941.)

### 145. J. A. Shohat: *On Bernoulli numbers and polynomials.*

Certain relations are established between Bernoulli polynomials $B_n(x)$ and those of Legendre and symmetric polynomials of Jacobi; and a new proof is given of Jacobi's theorem concerning the expression of $B_{2n}(x)$ in powers of $x(1-x)$. A reasoning of Mandl is utilized and made rigorous in order to obtain in a new way the well known remarkable relation between the Bernoulli number $B_{2n}$ and the zeta-function $\zeta(2n)$. Applications are made to $S_n(p) = 1^p + 2^p + \cdots + (n-1)^p$. (Received January 8, 1941.)

### 146. S. E. Warschawski: *On conformal mapping of infinite strips.*

Let $S$ denote the strip $\phi_-(u) < v < \phi_+(u)$, $-\infty < u < +\infty$ ($\phi_+(u)$, $\phi_-(u)$ continuous), in the $w$-plane, $w = u + iv$. Let $z = Z(w)$ ($\lim_{u\to+\infty} \Re Z(w) = +\infty$) map $S$ conformally onto the strip $|y| < \pi/2$ of the $z$-plane, $z = x + iy$. The principal object of this paper is to obtain asymptotic expressions for $Z(w)$ and its derivative $Z'(w)$ as $u \to +\infty$. For this purpose two inequalities are established, which are similar to those of L. Ahlfors (Acta Societatis Scientiarum Fennicae, new series A, vol. 1 (1930) p. 10 and p. 16), but which, due to some assumptions regarding the smoothness of the boundary of $S$, yield sharper estimates for large values of $\Re w_1$ and $\Re w_2$. The asymptotic expressions for $Z(w)$ and $Z'(w)$ are then applied, after suitable transformations, to the study of the order of magnitude of the mapping function of a region $R$ onto a circle in a neighborhood of a finite boundary point $P$ of $R$. These applications include the cases where $P$ is the vertex of a corner, or of a cusp, or is the asymptotic point of two "concurrent" spirals, and contain as special cases the results presented by the writer in two previous abstracts (42-5-219 and 42-11-409). (Received December 9, 1940.)

## APPLIED MATHEMATICS

### 147. E. S. Allen and Harvey Diehl: *The enumeration of glycols.*

Recently the authors extended the method of Henze and Blair for enumerating certain organic compounds; in particular, the alcohols. This work is used as a basis for a recursive method of enumerating isomeric glycols, that is, compounds whose molecules have two oxygen atoms. The basic numbers which result are the number of structurally asymmetric glycols possessing $n$ carbon atoms and having $\alpha$ enantiomorphically distinct forms, and the number of structurally symmetric glycols possessing $n$ carbon atoms and having $\alpha$ distinct forms, of which $\beta$ are completely symmetric. In all cases symmetry indicates identity of aspect of the molecule when viewed from the two hydroxyl radicals. (Received December 31, 1940.)

### 148. Stefan Bergman: *Numerical methods for conformal mapping of polygonal domains.*

The problem of the approximate determination of a conformal mapping can be reduced in certain cases to numerical operations which can be carried out with the aid of devices for computation already in existence or by electrical calculating machines combined, perhaps, with punch machines. In the present paper the calculation of the constants (that is, the branch points of the integrand) in the Schwarz-Christoffel formula (which transforms the half-plane into a polygonal domain) is reduced to the determination of the Fourier coefficients of $[R(\phi)]^n$, and to the solution of a system of linear equations. If the domain is a star domain, $R=R(\phi)$ is the equation of its boundary curve in polar coordinates. Further, methods for the calculation of the resulting integrals are discussed. The method is applied to a technical problem, namely to the problem of torsion in a beam with a polygonal section. The analogous method can be applied for the solution of boundary problems and has important applications in certain problems of aeronautics. (Received January 22, 1941.)

### 149. F. H. Clauser: *Exact solutions of the equations for the flow of a compressible fluid.* Preliminary report.

Several solutions of the equations given by Tschaplygin for the flow of a compressible fluid are discussed and a method presented for easily accomplishing the transformation of the solutions in the hodograph plane back to the physical plane. (Received December 18, 1940.)

## GEOMETRY

### 150. L. M. Blumenthal and G. E. Wahlin: *On the spherical surface of smallest radius enclosing a bounded subset of n-dimensional euclidean space.*

A short elementary proof is given for the theorem: *If $M$ is any bounded subset of $n$-dimensional euclidean space $E_n$ with positive diameter $d$, then there is a unique $(n-1)$-dimensional spherical surface of smallest radius $r$ enclosing $M$, and* $r \leqq [n/2(n+1)]^{1/2}\cdot d$. In a proof abounding with algebraic difficulties, H. W. E. Jung established these results in 1901 for the case of finite point sets and indicated their extension to infinite sets at the end of his long paper (Journal für die reine und angewandte Mathematik, vol. 123 (1901), pp. 241–257). The simplification offered by the present proof is afforded in large measure by a lemma which shows that *if each $n+1$ points of a subset $M$ in $E_n$ may be enclosed by an $(n-1)$-dimensional spherical surface of radius $r$ then $M$ itself has this property.* The proof exhibits the geometrically simple nature of the theorem. (Received January 24, 1941.)

### 151. J. J. DeCicco: *Equilong geometry of differential equations of first order.*

With this paper the study of the equilong geometry of a field of lineal elements is begun. This may be considered to be an analogue of a preceding paper by Kasner and the author in which the conformal geometry of a field is developed. As defined by Kasner, a dual-isothermal family consists of $\infty^1$ curves which are equilongly equivalent to a pencil of circles (all those tangent to two fixed lines, distinct or coincident). Obviously all dual-isothermal families are equilongly equivalent. It is found that any non-dual-isothermal family possesses two absolute differential covariants of the third order. Conversely if two fields possess the same two covariants, then there exist $2\infty^2$ equilong correspondences which will carry one field into the other. For any two nearby nonparallel elements of the field $w=w(u, v)$, the general equilong distance is defined by $ds^2=w_{vv}du^2$, whereas for two nearby parallel elements, this distance is $ds^2=w_{vv}dv^2$. The final result is that in equilong geometry any absolute differential covariant of a field is a function of two fundamental covariants and their derivatives with respect to the equilong arc lengths of the unions and the equiparallel series of the field. (Received January 20, 1941.)

### 152. J. J. DeCicco: *Isodeviate systems of geodesic series.*

This paper is concerned with the derivation of some further results in the differential geometry of a field of lineal elements in the plane. A system of $\infty^1$ series of a given field, which corresponds by an equideviate transformation to a linear pencil of turbines of a flat field, is called an isodeviate system. If a field possesses at least one isodeviate system of geodesic series, then it must possess $\infty^2$ such systems. The condition necessary and sufficient for this is that the gaussian curvature $K$ be the same at all parallel elements of a field. To compare this with the corresponding theory in the geometry of surfaces in euclidean three-space, see Kasner, *Isothermal systems of geodesics*, Transactions of this Society, vol. 5 (1904), pp. 56–60. (Received December 31, 1940.)

### 153. J. J. DeCicco: *Lineal element transformations which preserve dual-isothermal families.*

This paper is analogous to the one by Kasner and the author in which the group of all lineal element transformations which preserve isothermal families is determined. Kasner has defined a dual-isothermal family to consist of $\infty^1$ curves which are equilongly equivalent to a pencil of circles (in line geometry). In this work all lineal element transformations are found which preserve dual-isothermal families. In hessian or equilong coordinates this group is $U=\phi$, $V=(a_1v+b_1w+c_1)/(av+bw+c)$, $W=(a_2v+b_2w+c_2)/(av+bw+c)$, where $\phi$, $a$, $b$, $c$, $a_1$, $b_1$, $c_1$, $a_2$, $b_2$, $c_2$ are functions of $u$ only. The subgroup of contact transformations is $U=\phi(u)$, $V=v\psi(u)+\chi(u)$, obviously larger than the equilong group. Earlier the author had found that a non-dual-isothermal family possesses two absolute differential covariants of the third order. In the final part of the present paper, it is proved that any lineal element transformation which preserves both of these two covariants must be an equilong transformation. (Received January 21, 1941.)

### 154. J. J. DeCicco: *The differential geometry of the Laguerre group $G_6$.*

In the Laguerre inversive geometry of the plane the length of arc of a curve is defined in equilong coordinates $(x, y)$ by $s=\int(y''')^{1/2}dx$. The Laguerre inversive curvature is $K=[4y'''y^{v}-5(y^{iv})^2]/4(y''')^2$. The fundamental result of the paper is that two curves are equivalent under the Laguerre group $G_6$ if and only if their curvatures are the same functions of the arc length. Thus $K=F(s)$ is the intrinsic equation of any curve. As the Laguerre group of the complex plane is isomorphic to the group of rigid motions of complex euclidean three-space with the curves of the plane corresponding to the minimal curves of space, the fundamental result also furnishes the intrinsic equations of the minimal curves of space. (Received December 31, 1940.)

### 155. J. M. Feld: *The geometry of whirls and whirl-motions in space.*

The geometry of the six-parameter whirl-motion group $G_6$ in the plane had its origin in Kasner's fundamental paper, *The geometry of turns and slides, and the geometry of turbines* (American Journal of Mathematics, vol. 33 (1911), p. 193). In a recent paper by Feld, *Whirl-similitudes, euclidean kinematics, and non-euclidean geometry* (this Bulletin, abstract 46-5-270), Kasner's $G_6$ was extended first to a mixed six-parameter group composed of eight distinct continuous families, one of which is $G_6$, and second to a seven-parameter mixed group of eight families—the *group of whirl-similitudes*. The latter group was shown to be isomorphic with the group of automorphisms of *pseudo-elliptic three-space*. Feld also presented a (1, 1) mapping of lineal elements, flat fields, and turbines on planar euclidean displacements, symmetries, and ordered pairs of points respectively. The subject of this paper is the development of an analogous theory for three-space, where a mixed eight-family twelve-parameter continuous group of whirl-motions plays the principal role. Quaternion geometry is employed. (Received January 14, 1941.)

### 156. Aaron Fialkow: *The foundations of the conformal differential geometry of a subspace.*

For an arbitrary subspace $V_n$ in any Riemann space $V_m$ $(0<n<m;\ m>2)$, the author shows that it is possible to define a system of differential forms, termed the *conformal fundamental forms* (c.f.f.'s) of $V_n$, which enjoy the following properties: (1) If $V_n \subset V_m$, $\bar{V}_n \subset \bar{V}_m$ and $V_m \leftrightarrow \bar{V}_m$, $V_n \leftrightarrow \bar{V}_n$ by a conformal map, then the c.f.f.'s of $V_n$ and $\bar{V}_n$ are equal. (2) Conversely, if $V_m$ and $\bar{V}_m$ are conformally euclidean spaces and the c.f.f.'s of $V_n$ and $\bar{V}_n$ are equal, then a conformal transformation exists so that $V_m \leftrightarrow \bar{V}_m$, $V_n \leftrightarrow \bar{V}_n$. (3) A $V_n$ exists with any preassigned c.f.f. whose coefficients satisfy certain conformally invariant partial differential equations analogous to the classical generalized Gauss-Codazzi equations. The results for $n \geqq 4$ are typical while the cases $n = 3$, 2, and 1 respectively exhibit increasing degrees of deviation from the normal situation. These theorems serve as the basis for the development of a conformal differertial geometry in many ways similar to classical differential geometry but also exhibiting a number of essential differences. The principal tool is a new simple type of differentiation (with respect to the subspace) which enjoys all the usual properties of covariant differentiation as well as a number of others which give it its distinctive conformal character. (Received January 25, 1941.)

### 157. M. C. Foster: *Note on autopolar surfaces.*

This paper is concerned with surfaces autopolar with respect to the paraboloid $2z = x^2 + y^2$. Such autopolar surfaces are considered as special solutions of those partial differential equations which are invariant under the dual transformation for which the above paraboloid is the quadric of reference. Various metric properties are considered. (Received January 22, 1941.)

### 158. Edward Kasner and J. J. DeCicco: *Conformal geometry of differential equations of first order.*

In this paper the conformal geometry of nonisothermal fields of lineal elements in the plane is studied. A field $F$ possesses two absolute conformal covariants of the third order. Conversely, if two fields $F$ and $G$ possess these same covariants at corresponding elements, then there exist $2\infty^2$ conformal transformations which will carry one into the other. These results may be interpreted geometrically as follows. If the equation of the field $F$ is $\theta=\theta(x, y)$, where $\theta$ is the inclination, the conformal distance $dS$ between two lineal elements of $F$ is defined by $dS^2=\frac{1}{4}(\theta_{xx}+\theta_{yy})(dx^2+dy^2)$. With this definition of distance and the usual notion of angle, it results that the two covariants are the geodesic curvatures of the unions and the orthogonal series of $F$. All other covariants of $F$ are functions of these two geodesic curvatures and their partial derivatives with respect to the conformal arc lengths of the unions and the orthogonal series. In the final part of the paper, many interesting conformal properties of $n$-webs of curves are obtained. (Received December 31, 1940.)

### 159. Edward Kasner and J. J. DeCicco: *Conformal geometry of velocity systems.*

In this paper the conformal geometry of velocity systems is studied. A set of $\infty^2$ curves is a velocity system if and only if the $\infty^1$ osculating circles of the $\infty^1$ curves of the set passing through a fixed point, constructed at this point, form a pencil. Any arbitrary point transformation for which neither of the two families of minimal lines is preserved carries exactly one velocity system into a velocity system. Any point transformation which preserves only one family of minimal lines converts $\infty^1$ velocity systems into velocity systems. Finally, the conformal group preserves all velocity systems. A velocity system may contain exactly $\infty^2$, $\infty^1$, one, or no isothermal systems. A velocity system is called a $\Gamma$ family if it is conformally equivalent to the $\infty^2$ circles orthogonal to a fixed circle. A velocity system is a $\Gamma$ family if and only if it possesses $\infty^2$ isothermal families. All the conformal covariants of a velocity system are obtained. Finally, the reciprocal system of any $\Gamma$ system is treated. (Received December 31, 1940.)

### 160. Edward Kasner and J. J. DeCicco: *Infinite groups generated by equilong involutions and symmetries.*

In equilong geometry, the set of all equilong transformations of period two may be classified into three distinct types: equilong involutions, $K$ symmetries, and $D$ inversions. (Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 471–476.) In the present paper the infinite groups generated by these transformations are determined. The group $K'_{sym}$ formed by all $K$ symmetries consists of $K$ symmetries and $K$ translations (products of two $K$ symmetries). Any transformation of the group $G'_{invol}$ ($D'_{invers}$) formed by all equilong involutions ($D$ inversions) can be factored into involutions ($D$ inversions) in an infinitude of ways of which at least one can be factored into four or fewer involutions ($D$ inversions). The fundamental result is that the group generated by all equilong transformations of period two is identical with the group generated by $K$ symmetries and $D$ inversions. To contrast this with the conformal theory, see Kasner, American Journal of Mathematics, vol. 38 (1916), pp. 177–184. (Received December 31, 1940.)

### 161. Edward Kasner and J. J. DeCicco: *Lineal element transformations which preserve isothermal families.*

In this paper the authors seek to generalize the well known result that the conformal transformations are the only *point correspondences* which carry every isothermal family into an isothermal family of curves. They find the group of all *lineal element transformations* of the plane (not necessarily of the contact type) which preserves all isothermal families. This group is given in $(x, y, \theta)$, where $(x, y)$ are the cartesian coordinates of the point, and $\theta$ is the inclination of the element, by $X=\phi(x, y)$, $Y=\psi(x, y)$, $\Theta=k\theta+h(x, y)$, where $k$ is a nonzero constant, $\phi$ and $\psi$ satisfy the Cauchy-Riemann equations (direct or reverse), and $h$ is any harmonic function. The only contact correspondences are the conformal ones. In the authors' previous work, it was shown that any non-isothermal field possesses two absolute differential covariants of the third order under the conformal group. In the final part of the present paper it is proved that a lineal element transformation which preserves either one of these two covariants must be a conformal transformation. (Received January 20, 1941.)

### 162. Edward Kasner and J. J. DeCicco: *The classification of analytic arcs or elements under the group of arbitrary point transformations.*

This paper begins the study of the invariant theory of a single irregular analytic arc or element based on the infinite group of arbitrary point transformations in the plane. The correspondences are (regular) analytic in $x$ and $y$. Then $y$ may be written as a power series which proceeds according to positive integral powers of the $p$th root of $x$. Let $q$ be the first power of the $p$th root of $x$ which is not a multiple of $p$. The index $p$ and the rank $q$ are arithmetic invariants; and all elements with the same $p$ and $q$ form the single species $(p, q)$. Absolute differential invariants exist for all irregular species except in the cases (4, 5), (4, 6), (4, 7), $(3, q)$, and $(2, q)$. The species (4, 5) and (4, 7) may be separated into two and three distinct sets respectively. The species (4, 6) and $(3, q)$ possess an additional arithmetic invariant. All the elements of the species $(2, q)$ are equivalent. The paper will appear in the Proceedings of the National Academy of Sciences. (Received December 12, 1940.)

### 163. Don Mittleman: *Theory of ortho-family: A generalization of natural family.*

A surface will be said to be mapped orthogonally onto a plane if the image of an orthogonal net on the surface is the rectangular cartesian net of the plane. An ortho-family is defined as the image of the geodesics of the surface under the particular orthogonal mapping. The necessary and sufficient conditions that a two-parameter family of curves in the plane be an ortho-family are given analytically. It is a simple consequence of these conditions that an ortho-family which is a velocity family must be a natural family. Finally, if an ortho-family which is obtained by an orthogonal, non-conformal mapping is a natural family, then the surface whose geodesics are the pre-image of the given natural family is a surface of Liouville referred to the coordinate system for which $ds^2=(1/f-1/g)(dx^2/f+dy^2/g)$, where $f$ is a function of $x$ alone, and $g$ a function of $y$ alone. (Received January 24, 1941.)

### 164. Nelson Robinson: *On the contact of a quartic surface with a general analytic surface.*

Using the differential equations of a general analytic surface $S$ referred to its asymptotic net, the power series development of $S$ is computed to terms of the sixth degree. By means of this expansion conditions are determined in order that a general quartic surface have various orders of contact with the original surface $S$ at a point $P$. Special cases of the quartics $Q$ are discussed with particular attention given to the cases where $Q$ is composite, the components being quadric surfaces. Necessary and sufficient conditions are found in order that $Q$ be composed of quadrics of Darboux, or quadrics of Lie. (Received December 30, 1940.)

### 165. Peter Scherk: *On real closed curves of order $n+1$ in projective $n$-space.* II. Preliminary report.

In the first part of this paper (abstract 46-11-502) the author discussed differentiable closed curves $K^{n+1}$ of real order $n+1$ in $R_n$ by means of a certain single-valued correspondence of the $K^{n+1}$. He proved that $S\leqq n+1$, $S\equiv n+1 \pmod 2$ if $S$ is the sum of the multiplicities of the singular points, and he characterized the case $S=n+1$. Extending a simple remark on rotation numbers to multi-valued correspondences, the author discusses a two-valued and a three-valued correspondence defined on certain arcs of the $K^{n+1}$ and on the whole $K^{n+1}$ respectively, and connected with the projections of the $K^{n+1}$ from its osculating $(n-2)$-spaces and $(n-3)$-spaces respectively. The study of these two correspondences yields: (1) the first estimates of the number of osculating $(n-2)$-spaces which meet the $K^{n+1}$ again; (2) the classification of the $K^{n+1}$ with $S=n-1$; (3) the classification of the $K^5$; (4) a more systematic access to the classification of the $K^4$ (previously obtained by the author). (Received January 24, 1941.)

### 166. Alexander Wundheiler: *Abstract algebraic definition of an affine vector space.* Preliminary report.

A linear set over the field of real numbers will be called a simple vector space, and its elements, simple vectors. Two simple vector spaces $A$ and $B$ are cogrediently coupled if for any $a$ in $A$ and $b$ in $B$ a real number $f(a, b)$ is defined, such that $f(ka, b)=f(a, kb)=kf(a, b)$; $f(a, b'+b'')=f(a, b')+f(a, b'')$; $f(a'+a'', b)=f(a', b)+f(a'', b)$. The $a$'s and $b$'s are then contragredient vectors. If $A$ and $B$ are of the same dimension, the set $A+B$ is called an affine vector space, $a$ is a contravariant affine vector, $b$ a covariant one, or vice versa. Various illustrations are given, as electrical networks, the space of fruit juice cocktail cans, and so on. (Received January 24, 1941.)

### 167. Oscar Zariski: *Pencils on an algebraic variety and a new proof of a theorem of Bertini.*

The theorem of Bertini-Enriques states that if a linear system of $W_{r-1}$'s on a $V_r$ is reducible (that is, every $W_{r-1}$ of the system is reducible) and is free from fixed components, then the system is composite with a pencil. In this paper a new proof of this theorem is given, together with an extension to irrational pencils. With every pencil $\{W\}$ there is associated a field $P$ of algebraic functions of one variable, a subfield of the field $\Sigma$ of rational functions on $V_r$. The essential point of the proof is the remark that $\{W\}$ is composite if and only if $P$ is not maximally algebraic in $\Sigma$. The rest of the proof, in the case of pencils, follows from the fact that an irreducible algebraic variety $V_r$ over a ground field $K$ is absolutely irreducible if $K$ is maximally algebraic in $\Sigma$. In the case of linear systems of dimension $>1$, the proof is based on the following lemma: if $K$ is maximally algebraic in $\Sigma$ and if $x_1$, $x_2$ are algebraically independent elements of $\Sigma$, then for all but a finite number of elements $c$ in $K$ the field $K(x_1+cx_2)$ is maximally algebraic in $\Sigma$. (Received December 12, 1940.)

## LOGIC AND FOUNDATIONS

### 168. Alvin Sugar: *Postulates for the calculus of binary relations in terms of a single operation.*

In a recent paper (*Postulates for the calculus of binary relations*, Journal of Symbolic Logic, vol. 5 (1940), pp. 85–97) J. C. C. McKinsey gave a set of postulates for the calculus of binary relations in terms of the two operations $|$ and $<$. In his paper McKinsey shows that $<$ is definable in terms of $|$ but not conversely. In this paper the author develops a set of independent postulates for the calculus of binary relations in terms of the single operation $|$. (Received January 25, 1941.)

## STATISTICS AND PROBABILITY

### 169. Archie Blake: *The exploratory determination of statistical distributions.*

The most powerful statistics are not always the most "efficient" or those whose distributions have already been tabled, but their distributions can be computed in small samples without inordinate labor. To find the distribution of $g(X=x_1, \cdots, x_n)$ subject to the condition $P(X)$, compute requisite values of $g^{-1}$ (multiple-valued) and require $\int^{P, g^{-1}} f(X)dX$, $f$ being the given distribution function. The chief task is the computation of many values of the functions involved; this is alleviated by modern machine methods (especially punched cards). Tables of $f$, $g$, and so on, with their derivatives or the required fractions of the latter are prepared once for all on cards; thereafter the work consists only of interpolation. (Taylor's series recommends itself in this problem, as it converges more rapidly than ordinary interpolation formulae, and in the case of multivariate interpolation is much less complicated.) For a statistic whose asymptotic distribution is known, we can interpolate approximately between this and the results of the computation for small $n$ to obtain an estimated distribution for any $n$. (Received January 23, 1941.)

### 170. W. G. Madow: *The distribution of the general quadratic form in normally distributed random variables.*

The distribution of the general quadratic form in normally distributed random variables is obtained. This distribution is used to obtain the distribution of Neyman's estimate in the theory of the representative method of sampling, and it is also used to obtain a generalization of P. L. Hsu's distribution of Student's ratio when the true means and variances are unequal. The distribution is also used in tests occurring in the analysis of variance with non-orthogonal data, and the study of differences of various orders. In the latter use, a test for periodicity is obtained. (Received January 25, 1941.)

### 171. Henry Scheffé: *An inverse problem in correlation theory.*

An $m\times n$ matrix $Y$ may be used to represent $m$ sets of measurements on $n$ variables. The $n\times n$ matrix $R$ of correlation coefficients $r_{ij}$ is a function of the matrix $Y$, $R=F(Y)$. Necessary conditions (C) that $R=F(Y)$ are that $R$ be real symmetric with diagonal elements unity, and positive (rank=index). Given any matrix $R$ satisfying the conditions (C), does a "statistics problem $Y$" exist such that $R=F(Y)$? It is proved by matrix methods that there are no solutions $Y$ with $m\leqq \text{rank } R$, but $\infty$ solutions for each $m>\text{rank } R$. Particular solutions are constructed and the most general solution is characterized. Some corollaries are drawn. (Received January 8, 1941.)

### 172. Jacob Wolfowitz: *Tests of statistical hypotheses where the distribution forms are unknown.*

The likelihood ratio criterion for testing composite statistical hypotheses, discovered by Neyman and Pearson and recently proved by Wald to be asymptotically most powerful, is extended to testing composite hypotheses where the forms of the distribution functions are entirely unknown (continuity is assumed) and where tests must be based on the order relations among the observations. Thus a general method for treating problems of this character is obtained. For the problem of two samples (Wald and Wolfowitz, Annals of Mathematical Statistics, June, 1940) the resultant statistic is $\prod_j l_j^{l_j}/(l_j)!$, where $l_j$ is the length of the $j$th run. Its logarithm is asymptotically normally distributed. The result is immediately extensible to the problem of $k$ samples. For the problem of independence (Hotelling and Pabst, Annals of Mathematical Statistics, March, 1936) a similar statistic is obtainable which differs from the commonly used rank correlation coefficient. The method used to prove the logarithms of these statistics asymptotically normally distributed is applicable to proving the asymptotic normality of a large class of functions of partitions of an integer, of functions of sequences where the subsequences of odd and even numbered elements are themselves partitions of different integers of fixed ratio, and to similar problems (Received January 13, 1941.)

## THEORY OF NUMBERS

### 173. Paul Erdös and Joseph Lehner: *On the distribution of the number of summands in the partitions of a positive integer.*

Let $p_k(n)$ denote the number of partitions of $n$ into not more than $k$ summands. Then for $k=n^{1/2}(\log n/C)+xn^{1/2}$, $C=\pi(2/3)^{1/2}$, $p_k(n)/p(n)\sim\exp\{-2\exp(-Cx/2)/C\}$, where $p(n)$ is the number of unrestricted partitions of $n$. For $k=o(n^{1/3})$, $p_k(n)\sim C_{n-1,k-1}/k!$, uniformly in $k$. Let $P(n)$ be the number of partitions of $n$ into different summands. Then for "almost all" partitions, the number of summands in a given partition not exceeding $xn^{1/2}$ lies between $2n^{1/2}\{\log 2/(1+\exp(-Dx))\}/D\pm\epsilon n^{1/2}$, $D=\pi(1/3)^{1/2}$. The methods used are elementary in character. (Received January 23, 1941.)

### 174. Gordon Pall: *The construction of positive ternary quadratic forms.*

A method is developed of writing down quickly the reduced integral positive ternary quadratic forms of a given determinant, order, or genus. (Received January 30, 1941.)

### 175. H. A. Rademacher: *Ramanujan's identities under modular transformations.*

If the Ramanujan identities which exhibit the divisibility of $p(5n+4)$ and $p(7n+5)$ by 5 and 7, respectively, are expressed in terms of the Dedekind function $\eta(\tau)$, they can be subjected to modular transformations. Each of the two identities goes over into a new one, which is noteworthy because of the occurrence of the Legendre residue symbol. These new identities lead to a direct proof of the Ramanujan identities. The same procedure can be applied to an identity given by Zuckerman, involving $p(13n+6)$. Other identities, proved by Watson and Zuckerman, lead to modular equations of "level" (Stufe) 5 and 7. (Received January 24, 1941.)

## TOPOLOGY

### 176. F. B. Jones: *Monotonic collections of peripherally-separable connected domains.*

It is shown that: *In a locally connected metric space, every monotonic collection of* *peripherally-separable connected domains contains a countable subcollection running through it.* Weakening the hypothesis of the theorem in any one respect results in a false proposition. A few simple applications are indicated. (Received January 24, 1941.)

### 177. Shizuo Kakutani: *A generalization of Brouwer's fixed-point theorem.*

A generalization of Brouwer's fixed-point theorem (concerning a continuous point-to-point mapping $\phi(x)$ of a finite dimensional simplex $S$ into itself) is obtained for the case of a point-to-set mapping $\Phi(x)$, when the image $\Phi(x)$ of each point is a closed, convex set contained in or equal to $S$. Instead of the continuity of $\phi(x)$, the upper semi-continuity of $\Phi(x)$ is assumed. The fixed-point theorems of J. von Neumann in his theory of games (Mathematische Annalen, vol. 100 (1928)) and mathematical economics (Ergebnisse eines Mathematischen Kolloquiums, vol. 8 (1937)) are easy consequences of this result. In order to prove the theorem, the fixed-point theorem of Brouwer must be assumed. (Received January 22, 1941.)

### 178. J. L. Kelley: *A theorem on transformations.*

Let $f(X)=Y$ be a monotone interior transformation of a compact metric space and suppose dim $Y<+\infty$. Further suppose diam $f^{-1}(y)\geqq d>0$ for all $y\in Y$. Then there exists a closed, totally disconnected subset $X_0$ of $X$ such that $f(X_0)=Y$. As a result it is shown that if there exists a Knaster continuum (that is, an indecomposable continuum every subcontinuum of which is decomposable) of dimension greater than 1, then there exists a Knaster continuum of infinite dimension. (Received December 30, 1940.)

### 179. R. C. Stephens: *On abstract mappings.*

Following the notation of Chittenden, a space $(P, \phi)$ denotes an abstract set $P$ and a relation between subsets of $P$. A mapping $T$ of $(P, \phi)$ onto $(Q, \phi)$ will be called a $\phi$-mapping provided that for every subset $E$ of $Q$, the inverse transformation $T^{-1}$ satisfies the relation $\phi T^{-1}(E)=T^{-1}\phi(E)$. In this paper, theorems are established for $\phi$-mappings with $\phi$ having various simple properties. These results are then applied to standard operators. The relationship of $\phi$-mappings to interior transformations is shown. (Received January 20, 1941.)

### 180. Hassler Whitney: *Geometric methods in combinatorial topology.* Preliminary report.

Singular cycles in a polyhedron $P$ give a geometric content to the theory of cycles in a corresponding complex $K$. "Hypercycles" in $P$ are defined corresponding to cocycles in $K$. A hyperchain $\rho$ is, essentially, a homomorphism, into a coefficient group $G$, of singular cycles $C$ whose boundaries do not touch the "nucleus" $N_\rho$ of $\rho$, and which themselves do not touch the "nuclear boundary" $N_\rho'$ $(N_\rho'\subset N_\rho)$. The "hyperboundary" $\rho'$ of $\rho$ is easily defined: $N_{\rho'}=N_\rho'$, $N_{\rho'}'=0$. Any $\rho^r\cdot C^r$ is an element of $G$. If $K'$ is the first derived of $K$, to each oriented $\sigma$ corresponds a $\rho_\sigma$, whose nucleus is the set of closed cells of $K'$ touching $\sigma$ only at its center, and with "orientation" that of $\sigma$. Then $\rho_\sigma\cdot\sigma=1$. In an orientable manifold $M^n=K$, a cycle $C^r$ of $K$ corresponds to a $\rho^{n-r}$ with nucleus $C^r$, and a chosen orientation of cells $\sigma^{n-r}$ normal to $C^r$. If $\rho^r$ and $\rho^s$ are in "general position," their "product" gives a $\rho^{r+s}=\rho^r\smile\rho^s$; $N_{\rho^{r+s}}=N_{\rho^r}\cap N_{\rho^s}$. Also $\rho^s\frown C^{r+s}=C^r$. In $M^n$, passing to cycles, both products give ordinary intersections. (Received January 25, 1941.)

### 181. G. T. Whyburn: *On mappings into the circle.*

If $f(x)$ is a continuous mapping of a metric space $X$ into the unit circle of the complex plane, the property of a subset $Y$ of $X$ to have $f$ not equivalent to 1 (in the Eilenberg sense) on $Y$ is inducible. Thus if $X$ is compact and $f$ is not equivalent to 1 on $X$, there exists a compact subset $Y$ of $X$ on which $f$ is irreducibly not equivalent to 1. Further, any such set $Y$ is a continuum not separated by any proper subcontinuum. Hence the property of having $f \sim 1$ is $E_0$-extensible for compact sets $X$; that is, if $f \sim 1$ on each $E_0$-set of every component of $X$, then $f \sim 1$ on $X$. In particular, property (b): to have every continuous mapping into the circle equivalent to 1, is $E_0$-extensible and reducible. Using this result it is shown directly that the number $b(Y)$ is cyclicly additive for compact sets $X$, that is, $b(X) = \sum b(E_\alpha)$ where the $E_\alpha$ are the simple links of the components of $X$ and where, in general, $b(Y)$ denotes the rank of the factor group $S^Y/P(Y)$ where $S^Y$ is the group of all mappings into $S$ and $P(Y)$ is the subgroup of all elements of $S^Y$ which are equivalent to 1 on $Y$. (Received January 21, 1941.)

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# NEW PUBLICATIONS

ALBERT, A. A. Introduction to Algebraic Theories. Chicago University Press, 1941. 8+137 pp. $1.75.

Bibliography on Orthogonal Polynomials. (Bulletin of the National Research Council, no. 103.) Washington, National Research Council, 1940. 9+204 pp. $3.00.

BRITISH ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE. Mathematical Tables. Vol. 9. Tables of Powers Giving Integral Powers of Integers. Cambridge, University Press; New York, Macmillan; 1940. 12+132 pp. $4.25.

COHEN, A. Elements of Calculus. Boston, Heath, 1940. 5+583 pp. $3.50.

COWLES COMMISSION FOR RESEARCH IN ECONOMICS. Report of Sixth Annual Research Conference on Economics and Statistics at Colorado Springs, July 1 to 26, 1940. Chicago, Cowles Commission, 1940. 99 pp.

DEMING, W. E. Facsimiles of Two Papers by Bayes. Washington, Department of Agriculture, 1940. 12+59 pp. $1.00.

GONZALEZ, M. O. Introducción al Análisis Matemático. Cuba, Matanzas, 1940. 206 pp.

GUPTA, H. Symmetric Functions in the Theory of Integral Numbers. (Lucknow University Studies, no. 14.) Lucknow University, 1940. 7+105 pp.

HARDY, G. H. A Mathematician's Apology. Cambridge, University Press, 1940. 7+94 pp. 3s.6d.

—— Ramanujan; Twelve Lectures on Subjects Suggested by His Life and Work. Cambridge, University Press; New York, Macmillan, 1940. 236 pp. $6.00.

HUNTINGTON, E. V. A Survey of Methods of Apportionment in Congress. (Senate Document, no. 304.) Washington, Government Printing Office, 1940. 41 pp. $0.10.

KOLLER, S. Graphische Tafeln zur Beurteilung statistischer Zahlen. Dresden and Leipzig, Steinkopff, 1940. 7+73 pp. RM 10.

LEVINSON, N. Gap and Density Theorems. (American Mathematical Society Colloquium Publications, vol. 26.) New York, American Mathematical Society, 1940. 8+246 pp. $4.00.

LOCHER-ERNST, L. Projektive Geometrie. Zürich and Leipzig, Füssli, 1940. 151 pp. Fr. 12.50, RM 7.50.

PHILLIPS, E. G. Functions of a Complex Variable; With Applications. New York, Interscience Publishers; Edinburgh, Oliver and Boyd; 1940. 140 pp. $1.50.

ROBINSON, G. DE B. The Foundations of Geometry. (Mathematical Expositions, no. 1.) University of Toronto Press, 1940. 11+167 pp. $2.00.

TERRY, G. S. The Dozen System. An Easier Method of Arithmetic. London, New York, and Toronto, Longmans Green, 1941. 53 pp. $0.50.

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# SOME ASPECTS OF THE PROBLEM OF MATHEMATICAL RIGOR[1]

HASKELL B. CURRY

**1. Introduction.** The cue for the title to this address is taken from that of one by Pierpont before the Nashville meeting of this Society several years ago.[2] This is typical of a number of expository treatments of this topic which have been presented to the mathematical public in recent years.[3] In the present paper I shall discuss the same theme in a somewhat different manner. Relying upon these expository addresses for the historical background, I propose to treat certain aspects of the subject which have been rather neglected in them. The discussion is frankly from a single point of view, which is a species of formalism. I shall try, in the first place, to explain the fundamental concepts of formalism, and, in the second place, to add some new suggestions and criticisms in matters of detail.[4]

The problem of mathematical rigor is that of giving an objective definition of a rigorous proof. If you will examine your ideas on this subject I think you will agree that there is something vague and subjective about them. This does not mean, of course, that they are unsatisfactory for the needs of working mathematicians. In daily life, when we say that a piece of cloth is a yard wide, we really mean that its width is a certain legally defined fraction of the distance between two scratches on a metal bar located in Paris; nevertheless we do not rush to Paris when we wish to verify that a piece of cloth has this property. Secondary standards of varying degrees of accuracy suffice for the needs of daily life and of science; but neither science nor business would be possible without exact primary standards. Even so we need a primary standard of rigor in mathematics. The definition of such a standard, and the elaboration of practical secondary standards in relation to it, constitutes the outstanding problem of the foundations of mathematics.

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[1] An address delivered before the meeting of the Society in New York City on October 26, 1940, by invitation of the Program Committee.

[2] J. Pierpont, *Mathematical rigor, past and present*, this Bulletin, vol. 34 (1928), pp. 23–53.

[3] See for example A. Dresden, *Some philosophical aspects of mathematics*, this Bulletin, vol. 34 (1928), pp. 438–452; G. H. Hardy, *Mathematical proof*, Mind, vol. 38 (1929), pp. 1–25; E. R. Hedrick, *Tendencies in the logic of mathematics*, Science, vol. 77 (1933), pp. 335–343.

[4] For views related to those here presented, see my paper *Remarks on the definition and nature of mathematics*, Journal of Unified Science, vol. 9, pp. 164–169. This is an abstract of an address delivered before the Fifth International Congress of Unified Science at Cambridge, Massachusetts, September 5, 1939.

This problem is evidently tantamount to finding exact criteria for the truth of a mathematical proposition. For we should regard a rigorous proof as a process of making manifest the truth of its conclusion; and the problem of rigor is inseparable from that of truth.

2. **Critique of the non-formal theories of mathematics.** There are three main types of view as to the nature of mathematical truth; these I shall call realism, idealism, and formalism. The first of these is the view that mathematical propositions concern the real world (in the sense of our physical environment); the second holds that mathematics deals with mental objects of some sort. These two views have in common the notion that mathematical propositions have an essential subject-matter or content, with respect to which truth and rigor are definable; they are what the Germans would call "inhaltlich." Since there is no satisfactory translation of this word into English,[5] I propose that we call them contensive, where 'contensive' is derived from 'content' in the same way as 'intensive' from 'intent.' The formalist view will form the main subject of the present paper; but before we take it up it will be appropriate to consider a few remarks about the others.

The realist view evidently does not need to be taken seriously at the present time. Of course it is the original view—the mathematics of primitive peoples is essentially empirical—and it is tenable for simple arithmetic propositions such as a $2+2=4$. But that the infinitistic conceptions of modern mathematics have no counterpart in the external environment is a point which needs no elaboration.

The idealist view has many varieties according to the nature of the mental objects on which it is based. On the one hand there is the view, called by Bernays[6] Platonism, which ascribes an independent existence to all the infinitistic conceptions of mathematics; at the other extreme there is intuitionism, which denies this existence, but bases everything on an a priori intuition. Whatever the nature of these mental objects, all forms of idealism are subject to the same fundamental criticism, namely, that the criterion of truth is vague at best, and the existence of the mental objects is a metaphysical hypothesis from which mathematics should be free, if it is to be an objective science.

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[5] Translations sometimes used are 'material' and 'intuitive'; but these have undesirable connotations.

[6] P. Bernays, *Sur le platonisme dans les mathématiques*, L'Enseignement Mathématique, vol. 34 (1935–1936), pp. 52–69. There are naturally many different varieties of Platonism and also, for that matter, of intuitionism.

That this criticism applies to Platonism has been shown by its intuitionist critics, and is now fairly generally admitted. But it also applies to more moderate forms of idealism, and even to intuitionism itself. For the vagueness of the fundamental "oerintuitie der wiskunde" has been noticed by several persons; and Heyting himself, in his Ergebnisse report of 1934, explicitly denies the possibility of describing it exactly.[7] On the other hand it is clear from the intuitionists' writings that they conceive their fundamental intuition to be (1) essentially a thinking activity,[8] (2) a priori,[9] (3) independent of language,[10] and (4) objective, in that it is the same in all thinking beings.[11] The existence of an intuition having these properties—or even the first three of them—is an outright assumption. It is doubtful, indeed, if there is any a priori knowledge; and it has been rationally maintained that thinking of any kind is impossible without language. Furthermore the fourth property, although it is absolutely vital if intuitionism is to give any account of mathematical truth at all, seems patently incompatible with the other three. In short, the intuitionist definition of mathematics has meaning only for one who postulates an a priori intuition which is both objective and pre-linguistic. Although such a postulate is agreeable to certain types of philosophy, yet it is an assumption for all that; and one which, from other points of view, is highly dubious and metaphysical.

The intuitionist school has made contributions of great value to the foundations of mathematics. Not only have they cleared away much of the rubbish of old-fashioned Platonism, but they have helped to sharpen the ideas of the formalists. We shall see later that most of the constructive suggestions of the intuitionists, when cleared of their metaphysics, find their justification in modern formalism. But the fundamental basis of their theory is a metaphysical doctrine.

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[7] A. Heyting, *Mathematische Grundlagenforschung, Intuitionismus, Beweistheorie*, Ergebnisse der Mathematik und ihre Grenzgebiete, vol. 3, no. 4, Berlin, Springer, 1934. See for example, page 12 where he writes "Überdies ist es an sich widersinnig, die Möglichkeiten des Denkens in das Mieder bestimmter zuvor angebener Konstruktionsprinzipien zwängen zu wollen. Man muss sich also darauf beschränken, durch mehr oder weniger vage Umschreibungen in dem Hörer die mathematische Geisteshaltung hervorzurufen."

[8] "Eine konstruktive Tätigkeit unseres Verstandes" (Heyting, loc. cit., p. 2).

[9] "Die mathematischen Gegenstände werden von dem denkenden Geist unmittelbar erfasst; die mathematische Erkenntnis ist daher von der Erfahrung unabhängig." (Heyting, loc. cit., p. 3.)

[10] Cf. Heyting, loc. cit., p. 13.

[11] A fifth characteristic, its relation to time, is irrelevant for us.

Before leaving the subject of contensive definitions it will be necessary to consider the term *logicism* (or logisticism). It is sometimes maintained that there are three principal views as to the foundations of mathematics: intuitionism, formalism, and logicism, where the last is the view that mathematics is reducible to logic. If we had a satisfactory definition of logical truth and of logical rigor, logicism would solve our problem very nicely. But the paradoxes have shown that this is not the case; and in fact logic is as much in need of definition as mathematics. Hence to say that mathematics is logic is merely to replace one undefined term by another. Practically all competent writers realize this; and those who maintain that mathematics is logic generally couple this contention with some explanation of what logic is. But when we take these definitions of logic into account, the resulting definitions of mathematics are not a unified view according to the classification here adopted. Ramsey was a Platonist; Frege was essentially a formalist, and most of the modern writers who are popularly classified as logicists are definitely so. 'Logicism,' then, is not the name of a distinct view in regard to the foundations of mathematics.

From these considerations we see that a satisfactory contensive definition of mathematical truth and rigor has not been made; and indeed there is reasonable doubt as to whether such a definition is possible. Although such conceptions are current among mathematicians, they are useful only as secondary standards.

3. **The notion of formal system.** This brings us to formalism. According to this view the emphasis is not on the content of mathematical propositions, but on the method of derivation. We have an exact definition of a rigorous proof, and identify truth with the existence of such a proof.

The precise analysis of this rather vague definition requires preliminary consideration of some basic concepts. The first of these concepts is the notion of formal system.[12] This is a refinement of the naive notion of a postulate system, and it will be convenient to introduce the new notion by taking an example of the latter notion and formalizing it. The example selected is Dickson's postulate set for group theory,[13] as follows:

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[12] This statement is meant in the sense that the notion of formal system is taken as basic in the present paper, not that it is characteristic of formalism as such. Cf. below, §5.

[13] L. E. Dickson, *Definition of a group and a field by independent postulates*, Transactions of this Society, vol. 6 (1905), pp. 198–204.

### DICKSON'S POSTULATES FOR GROUPS

A. *Primitive ideas.* $G$, $\circ$.

B. *Postulates.*

1. If $a$ and $b$ are elements of $G$, then $a \circ b$ is a uniquely determined element of $G$.
2. If $a$, $b$, $c$ are any elements of $G$, then
$$a \circ (b \circ c) = (a \circ b) \circ c.$$
3. There exists an element $e$ of $G$, such that for any element $a$ of $G$
$$a \circ e = a.$$
4. For each element $a$ of $G$ there exists an element $a'$ such that
$$a \circ a' = e.$$

This set of postulates determines a set of theorems which are derived from them by the processes of logical deduction. But, as we have seen in the discussion of logicism, this is not an objective characterization. It is necessary to specify in detail the way in which theorems are to be derived.

A *formal system* may be naively described as what such a postulate system becomes when we put in these refinements. More precisely, such a system is defined by a set of conventions, which I shall call its *primitive frame.* These conventions are of three kinds, as follows: first, those which specify the objects of the system, which I shall call its *terms*—in the case of group theory these are the elements of the group; second, those determining a set of propositions, which I shall call the *elementary propositions*, concerning these terms—in the case of group theory these are the equations; and third, those which specify which of the elementary propositions are true, that is, are theorems.[14] These specifications have the form of recursive definitions. Thus the term-specifications give a list of primitive terms, or *tokens*, together with *operations* and *rules of term formation* which describe how further terms are to be constructed from the tokens. The specifications for elementary propositions consist of a list of *predicates* (properties, relations, and so on) together with *rules of proposition formation* for constructing elementary propositions by means of them. Finally, the specifications for theorems consist of a set of elementary propositions, called *axioms*, which are stated outright to be true; together with *rules of deduction* showing how further theorems are to be derived.

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[14] Throughout this paper I shall use 'theorem' in the sense of 'true proposition.' Thus an axiom is a theorem by definition.

As in recursive definitions generally a property of closure is understood, namely, that the specifications give all the entities recursively —for example, *all* the terms are obtained by the processes described in the term specifications, and so on.

For a system G, related to group theory, the primitive frame is as follows:

### SYSTEM G

I. TERMS.

  A. *Primitive terms, or tokens.* $e$, $a_1, \cdots, a_n$.

  B. *Operations.* $\circ$ (binary), $'$ (unary).

  C. *Rules of term formation.*

    1. If $A$ and $B$ are terms, $A \circ B$ is a term.

    2. If $A$ is a term, $A'$ is a term.

II. ELEMENTARY PROPOSITIONS.

  A. *Predicates.* $=$ (binary).

  B. *Rule.* If $A$ and $B$ are terms, then $A = B$ is an elementary proposition.[15]

III. THEOREMS.

  A. *Axioms.* If $A$, $B$, $C$ are terms, then—

    1. $A \circ (B \circ C) = (A \circ B) \circ C$,

    2. $A \circ e = A$,

    3. $A \circ A' = e$.

  B. *Rules of deduction.* For any terms $A$, $B$, $C$—

    1. If $A = B$, then $B = A$.

    2. If $A = B$ and $B = C$, then $A = C$.

    3. If $A = B$, then $C \circ A = C \circ B$.

    4. If $A = B$, then $A \circ C = B \circ C$.

    5. If $A = B$, then $A' = B'$.

The following remarks are intended to clear up certain points in regard to the primitive frame. (What a formal system essentially is will concern us later; for the present we are interested only in the specifications by which a formal system is defined.)

First, as to the symbols used. In the above statement of the primitive frame the symbols '$e$,' '$a_1$,' $\cdots$, '$a_n$,' '$\circ$,' '$'$,' '$=$' are used to designate definite constituents of the system; on the other hand the capital letters, parentheses, and also words such as 'proposition' are to be taken as part of ordinary language.[16] The capital letters, in particular, are intuitive variables whose values are unspecified terms. Thus the specifications III A are to be understood in the sense that for each particular determination of '$A$,' '$B$,' '$C$,' each of the three propositions stated is an axiom; there are, therefore, infinitely many axioms subsumed under three *axiom-schemes*.[17]

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[15] On the omission of quotation marks here see Footnote 20.

[16] Ordinary language must here be understood as including the use of variables and parentheses in the way in which they are employed in mathematics.

Again, the conventions I, II, have a different character from III, and it is convenient to have a name for them collectively. I shall call these specifications and considerations based on them *morphological*, as opposed to the conventions III which will be called *theoretical*. The morphology of the system G has evidently a quite trivial character; in more sophisticated systems we may have a much more complicated morphology—involving different categories of terms, or even a whole hierarchy of types, relations between terms and definitions of complicated derived notions such as substitution. But we must suppose in all cases that the morphological specifications enable us to determine whether a given combination of symbols denotes a term (or an elementary proposition).[18]

These notions may be made somewhat clearer if I add a word or two concerning the interpretation. In the system G the $a_1, \cdots, a_n$ may be thought of as the generators of the group. On the other hand it may be shown that if in any theorem involving any $a_i$ we substitute an arbitrary term $A$ for $a_i$, the result is again an elementary theorem. It follows that the $a_i$ may also be thought of as variables, and that the system contains those general equations of group theory which do not contain more than $n$ variables. To include all equations we must, naturally, postulate infinitely many primitive terms. If a substitution rule is added to the rules as Rule 6, then the axiom schemes may be replaced by three single axioms as follows:

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[17] This term is due to J. von Neumann, *Zur Hilbertschen Beweistheorie*, Mathematische Zeitschrift, vol. 26 (1927), pp. 1–46.

[18] It is to some extent arbitrary what considerations are taken as morphological; for in case of systems with a complicated morphology—say with different types of terms—we can take these types of terms as new predicates and transfer the rules concerning them into the theory. If this is carried to the limit we should arrive at a formal system, like the system G, in which there is only one category of terms, and it is not really necessary to state any morphological rules beyond giving the number of arguments for each operation and predicate. The natural formulation of the primitive frame on such a basis would be—

I. Primitive ideas.
  A. Tokens (primitive terms).
  B. Operations (classified as to number of arguments).
  C. Predicates (do.).
II. Primitive theorems.
  A. Axioms (that is, elementary primitive theorems).
  B. Rules.

Such a system let us call a *completely formal system*. Evidently it is the more definite concept, and for a primary standard of mathematical rigor it forms the proper basis. G is an example of such a system. In most theories, however, there is a class of theorems, of a relatively trivial nature (from the point of view of that theory) which it is convenient to take unformalized. The concept of a formal system, with the possibility of some complication in the morphology, is thus more easily fitted to existing theories than that of a completely formal one.

1. $a_1 \circ (a_2 \circ a_3) = (a_1 \circ a_2) \circ a_3$,
2. $a_1 \circ e = a_1$,
3. $a_1 \circ a_1' = e$.

The above formal system has of course a quite trivial character. (I do not claim that it is an adequate formalization of all of group theory.) On account of limitations of time, however, it will have to do as an example.

**4. Discussion of the nature of a formal system.** Let us now turn to the discussion of some points about the nature of a formal system. We shall inquire in what sense we may think of a formal system as an object, and whether we are justified in the use of the words 'proposition' and 'true' in connection with it. Also, on account of the prominence given to the question by Hilbert, it is necessary to say something about the relation of a formal system to symbolism; in this section and the next I shall defend the thesis that the extreme nominalism, with which formalism is now popularly associated, is no essential part of the doctrine.

As a preliminary to this discussion we must take account of the distinction between the use of a symbol as denoting something and the consideration of that symbol as an object. This is illustrated by the two sentences[19]

1. $\omega$ is an order type.
2. $\omega$ is a Greek letter.

From these it would be false to conclude that a Greek letter is an order type. Of course, under certain circumstances it is legitimate to use a symbol to designate itself. But, for maximum definiteness, I shall, when talking about a symbol (or expression), use a specimen of that symbol (or expression) enclosed in single quotation marks as a name for it.[20]

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[19] These are due to R. Carnap, *Logical Syntax of Language*, London and New York, 1937 (German edition, 1934), p. 156. Some changes in wording have been made.

[20] This notation is due to Frege (see, e.g., *Grundgesetze der Arithmetik*, vol. 1, Jena, 1893, p. 4) and is now quite generally used by logicians. However, the distinction itself is made in some manner by linguists as a matter of course (and doubtless was before Frege's day). The distinction is important wherever the discussion has

In setting up the primitive frame of a formal system we are, of course, introducing certain symbols; and, although we do not define these symbols in the ordinary sense, yet we do specify how they are to be used in connection with words of ordinary language (such as 'proposition' and 'true'). Thus, for G, in I A we say in effect that '$e$,' '$a_1$,' $\cdots$ , '$a_n$' are nouns; we do not say what they are the names of, but they are names of some objects or other, which objects we call terms. Let us call these symbols the ***primitive term nouns***. Then in I C we state that certain further expressions, formed from the primitive term nouns, the *operators* 'o' and '′,' and parentheses, are also term nouns. II says that '$=$' is a verb, and that elementary sentences are formed by placing it between two term-nouns. It is evident that

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specific reference to the relation between symbols and their referents; but I think the confusion caused by non-observance of the distinction in ordinary mathematics is somewhat exaggerated. For discussion of this question cf. R. Carnap, *Logical Syntax*, pp. 156–160, or W. V. Quine, *Mathematical Logic*, New York, 1940, §4.

The discussion in the text has primary reference to symbols and expressions which are nouns. With respect to other types of expressions there should be at least a threefold, rather than a twofold, distinction. Thus, in connection with sentences we need to distinguish (a) the sentence as asserting something, (b) a noun naming the proposition asserted, and (c) a name for the sentence as a linguistic phenomenon. In the present paper I use the same expression in senses (a) and (b), relying upon the context to make the distinction clear. There is abundant precedent for this uage, both in ordinary and technical discourse, and it avoids overburdening the discussion with excessive symbolism. Nevertheless, a systematic method of making the distinction would be desirable. Such a method would be to use "corners," as in Quine's "quasi-quotation" (loc. cit., pp. 33–37, or Journal of Symbolic Logic, vol. 2, p. 146). This should be subject to the same rules as his quasi-quotation in regard to substitution for intuitive variables, so that, for example, we might write the conventions II of the primitive frame for G as—

There is one binary predicate, $\ulcorner = . \urcorner$
If $A$ and $B$ are terms, then $\ulcorner A = B \urcorner$ is an elementary proposition.

If this is coupled with an understanding of what symbols are intuitive variables we should have a satisfactory notation.

The distinction between (a) and (b) may be what the Principia Mathematica (and also Frege) had in mind in introducing the assertion sign '$\vdash$.' However their explanations in this connection are somewhat vague. All that is clear is that '$\vdash$' is placed before a noun to make a sentence—that is, it is an intransitive verb. It is therefore natural to use this symbol for the single unary predicate in formal systems which have such; it then corresponds to Hilbert's 'ist beweisbar.'

The neglect of the distinction between (b) and (c) seems to me to cause as much confusion as the neglect of that between (a) and (c). On this point compare the discussion of 'presentation' below.

It should be understood that single quotation marks, and these only, are used in the technical sense stated. Double quotation marks are used as in ordinary language.

what has been said could be expanded into a way of stating the primitive frame, and that the new way of statement would be essentially equivalent to the preceding one.

The new form of statement, however—aside from the disadvantages inherent in its greater cumbersomeness—is a decided misplacement of the emphasis. To a mathematician the older form of statement indicates that the particular symbols used are irrelevant; whereas the newer form appears to be tied to a particular choice of them. Let us call a particular statement of the primitive frame, with a particular choice of symbols, a *presentation* of the system. Then the newer mode of statement connotes that we have to do with a particular presentation. On the other hand, a formal system is to be understood as an abstraction from its presentations. This does not mean that a formal system is a mystic entity subsisting apart from its presentations. Indeed the question of the philosophic nature of such an abstraction is irrelevant to the present discussion. All we need to know is that when we think of a presentation as that of a formal system we regard certain features as essential and others as irrelevant, and that what we have to say applies equally well to any other presentation agreeing in the essential features. A similar remark applies to the words 'proposition,' 'predicate,' and so on, as opposed to the corresponding linguistic words 'sentence,' 'verb' (or 'sentential function'), and so on.[21]

The next question is the indeterminacy which we have already noticed in regard to the terms. The primitive frame does not assign any definite objects to the term nouns. We are then free to make what determinations we please. Let us call any determination of such objects, which assigns a unique term-object to each term noun and conversely, a *representation* of the system. Then, given any presentation of the system, we can always supply a representation. In fact we can construct a mechanical model in which the tokens are different kinds of buttons and the operations are different ways of tying them together. A scheme of doing this for the system G using auxiliary gadgets for the operations is shown by the diagrams. (Here it is to be understood that the holes are for the attachment of strings from the constituents $A, B$; while the rings are for attachment of a string connecting to an operation when the term $-A \circ B$ or $A'$ is itself a constituent of a more complex term.)

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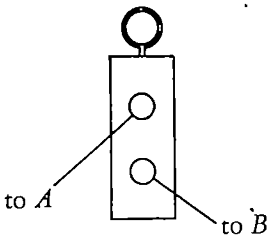
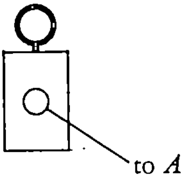
[21] It is not even necessary that the essential features be precisely formulated. As ordinarily used 'proposition' is a vague word, in that no exact criteria are given for deciding when two sentences express the same proposition. This vagueness is harmless because we seldom or never have occasion to ask the question indicated. The word is useful because it indicates in a rough way the level of abstraction at which we are operating; which is better than no indication at all, or a misleading one.

When this is done the term nouns have been given a contensive meaning. Now the system does not admit of any further determination of such meaning. The indeterminacy which was noted for the terms does not extend to the predicates; for the rules II enumerate all of the significant instances of these predicates and III all the true instances, so that the predicates are defined completely by the primitive frame. Thus in the above representation the pairs of artifacts between which the relation of equality holds are defined by the primitive frame and nothing else. Consequently although the terms are unspecified, yet for any formal system it is always possible to construct a representation supplying this deficiency. The elementary theorems hold without regard to how this is done. The representation is therefore an accident; and a formal system is an abstraction from its representations just as it is from its presentations.



$A \circ B$



$A'$

The notion of representation just discussed must be distinguished from another notion, that of *interpretation*. We speak of an interpretation of a formal system when the elementary propositions are put into relation with certain contensive propositions independently defined. Thus we can interpret G in a well known manner as a group of transformations, where two transformations are regarded as equal when they associate the same image with each point. This interpretation is a *valid* one, because the transformation processes associated with (formally) equal terms are also equal contensively; but the converse is not true. In the case of a representation such questions of validity cannot arise; all representations are isomorphic by definition.

A formal system, then, is something which is abstract in two ways; first, as to the symbols used to present it, and second, as to the objects used to represent it. The truth of the elementary propositions is independent of these contingencies. Moreover by virtue of the conventions III (and the closure property mentioned at the beginning) an elementary proposition is true when and only when there exists a sequence of elementary propositions, of which the given proposition is the last, such that every proposition in the sequence is either an axiom or is derived from certain of its predecessors by a rule of deduction. What an elementary proposition asserts is, therefore, precisely the existence of such a proof. Assuming that the specifications for axioms and rules have a sufficiently definite character—which is to be taken as part of the definition of a formal system—the checking of an alleged proof is an objective process.[22] An elementary proposition therefore states a question or problem such that an affirmative answer can be verified objectively, without any indeterminacy whatever.

5. **The notion of a calculus.** Although the notion of formal system is here taken as fundamental, yet other related notions are now more popular among specialists. As a typical such notion let us discuss here the notion of *calculus* as defined by Scholz.[23]

In a calculus it is explicitly stated that the objects we are dealing with are symbols. We start with a certain stock of symbols and with two kinds of rules for manipulating them. The first kind of rules, called *formation rules*, specifies recursively a certain set of *expressions* —that is, linear sets of symbols—which set I shall call *formulas*. The second set of rules, called *transformation rules*, specifies a class of formulas which I shall call *assertible formulas*; the rules consist of first a definite list of formulas, called here *axiom formulas*, which are assertible, and second, rules determining recursively how further assertible formulas are to be constructed.

Now it is evident that the rules of a calculus must be stated in a language such as English; and since the subject matter is symbols and expressions, the rules have somewhat the same character as the rules of syntax in grammar. For that reason the language is called the *syntax language*. On the other hand the expressions being talked about have also, in virtue of their being ordered by the rules, some of the properties of a language; and it is customary to refer to them as constituting the *object language*. There are thus two languages associated with a calculus, and the calculus may be thought of as a statement of the syntax of the object language in the syntax language. However the object language is a language only in a highly technical sense; it may or may not be a language in the usual sense.

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[22] We may require of a proof that not only is the sequence of propositions stated, but the reasons for the inference at each stage.

[23] H. Scholz, *Was ist ein Kalkül und was hat Frege für eine pünktliche Beantwortung dieser Frage geleistet*, Semester Berichte (Münster), 7th Semester, Summer, 1935, pp. 16–22. The notion is similar to that of R. Carnap, *Logical Syntax*, pp. 4–8, 167 ff., but is somewhat more explicit.

Let us now consider the relations between a calculus and a formal system. We shall see that a formal system can be converted into a calculus, or something essentially similar, in two ways, either of which can be carried out mechanically. Conversely, by passing to a higher level of abstraction, a calculus can be converted into a formal system.

The process indicated in the discussion of presentation gives a method of forming a calculus from the symbols of the presentation. Thus, for the above presentation of the system G, take as symbols '$e$,' '$a_1$,' $\cdots$, '$a_n$,' '$\circ$,' '$'$,' '$=$,' '(,' ')'; take as formulas the elementary sentences, and as assertible formulas those which express true propositions. Then, reinterpreting the rules, we have a calculus. This process I shall call *calculization* of the presentation. It is subject to the objections which were mentioned in the discussion of presentation.

Another way of reducing a formal system to a calculus is to represent the system symbolically. A uniform process for doing this is illustrated by the following procedure for G; Let $e$, $a_1$, $\cdots$, $a_n$ be respectively '$\epsilon$,' '$\alpha_1$,' $\cdots$, '$\alpha_n$,' and let '$\phi$,' '$\psi$,' be two new symbols; for given terms $A$ and $B$, let $A \circ B$ be the expression got by writing in order first '$\phi$,' then $A$, then $B$; and let $A'$ be obtained by prefixing '$\psi$' to $A$. That this is a representation follows by certain results of Łukasiewicz.[24] Then the terms will be a certain class of expressions which we can take as formulas. The resulting representation is not a calculus, because a relation between formulas and not a class of formulas is defined, but it has the essential nature of a calculus, and may be called a *generalized calculus*.[25] The reduction would lead precisely to a calculus if the original system had a single unary predicate; and by a rather trivial change every formal system can be reduced to one of that character.[26] By this mode of reduction—as in any representation—the symbols of the primitive frame belong to the syntax language, and the object language is the invented representation.[27]

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[24] J. Łukasiewicz and A. Tarski, *Untersuchungen über den Aussagenkalkul*, Comptes Rendus de la Societé des Sciences et des Lettres de Varsovie, class III, vol. 23 (1930), p. 31. where references to earlier papers by Łukasiewicz (in Polish) may be found.

[25] The Church theory of conversion is an example of such a generalized calculus which occurs in the literature. See A. Church, *Mathematical Logic*, Princeton University (mimeographed), 1936, chap. 1.

[26] For the system G this can be done by introducing a new binary operation $*$ and unary predicate $\vdash$, then replacing $A = B$ by $\vdash (A * B)$.

In a calculus we do not consider all properties of the symbols, but only such as follow from the syntactical rules. Thus a calculus is to a certain extent abstract. We can make this abstractness explicit by converting to a formal system. In fact, using methods due to Tarski and Hermes[28] we can find a (represented) formal system such that the tokens are the object symbols, the terms are the expressions, there is a single unary predicate, and the elementary theorems ascribe this predicate to those and only those expressions which are assertible formulas. We then have a formal system of a special kind with a symbolic representation. The object language can now be abolished altogether. Other processes of formalization—in which the terms are formulas or other special classes of expressions—may be possible; and if the calculus is obtained by calculization of a formal system, a process of formalization can be found which will bring us back to another presentation of the original formal system.

These considerations show that a formal system and a calculus are essentially equivalent notions. A calculus is, in fact, merely a formal system tied to a special representation. This representation is accidental. A formal system is just as exact a notion; it does not force emphasis on extraneous considerations; it requires only one set of symbols, and these are adjuncts to the language we use, not that which we talk about; it leaves the way open for representations in terms of some subject matter more suggestive than symbols; and a symbolic representation can be manufactured for it when needed. For these reasons the notion of formal system is here preferred as the fundamental notion of formalism.

So much for the notions of formal system and calculus. The important point is that for elementary propositions we have an objective criterion of rigor. We turn now to the consideration of propositions which are not elementary.

6. **Metatheory.** In the actual study of a formal system we do not confine ourselves to deducing elementary propositions step by step. Rather we take the formal system as datum, and, having defined the

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[27] In the case of a completely formal system (see Footnote 16) with one unary predicate, this object language would consist of formulas and coordinating conjunctions only.

[28] E.g., see A. Tarski, *Einige Betrachtungen über die Begriffe der ω-Widerspruchsfreiheit und der ω-Vollständigkeit*, Monatshefte für Mathematik und Physik, vol. 40 (1930), pp. 97–112, and H. Hermes, *Semiotik*, Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften, new series, vol. 5, 1938.

object of study by setting up the primitive frame, we investigate it by any means available. In so doing we may formulate further propositions. I shall call these non-elementary propositions *metapropositions* (when true, *metatheorems*), and the method of study which gives rise to them the metatheoretic method.

We have already had an example of a metatheorem, namely, the substitution rule for the system G. It will be instructive to consider this theorem a little more carefully. First we need a precise definition of what we mean by the result of substituting the term $C$ for the token $a_i$ in the term $A$. Let us call this result $A^*$. Ordinarily we define it with reference to the symbols used to designate $A$ and $C$; but this apparent dependence on symbols can be avoided by defining $A^*$ recursively thus (here '$\equiv$' denotes definitional identity):

1. $a_i^* \equiv C$;
2. If $A$ is a token distinct from $a_i$, $A^* \equiv A$;
3. $(A \circ B)^* \equiv A^* \circ B^*$;
4. $(A')^* \equiv (A^*)'$.

The theorem, now, is the following: Suppose that $A$ and $B$ are terms such that

$$A = B, \tag{1}$$

and $C$ is a term, then

$$A^* = B^*. \tag{2}$$

Without going into details we may notice that the proof contains two essential steps. First, the conclusion is true if the hypothesis is an axiom; in fact then the conclusion is an axiom. Next, suppose the hypothesis is derived by a rule of procedure from certain premises, and that we have already shown that the theorem holds for these premises; then the same rule of procedure allows us to derive the conclusion from the transforms of the premises. At this point it is customary to say that the theorem follows by induction. However, what we have actually done is to exhibit a process whereby, given a proof of (1), we can convert it step by step into a proof of (2).

The variety of metatheorems is immense. Some further examples are the following. In the first place we may have simple combinations of the elementary propositions, namely,

$$a_1 \circ e = e \circ a_1 = a_1,$$

which may be regarded as a conjunction of three elementary theorems. Then there are general theorems conjoining whole classes of elementary propositions such as

$$A' \circ A = e$$

(the axiom schemes, considered as single propositions, are of this character); or derived rules, such as—

$$\text{If} \quad A \circ B = e, \quad \text{then} \quad B = A'.$$

Again we may introduce additional operations and predicates by recursive definitions, and consider general theorems concerning them, like the one just treated or the generalized associative law. Moreover there are properties of the system as a whole; such as consistency, completeness, and resolvability. We may also consider the relations of a system to other systems, in particular to its own subsystems and supersystems, its extensions of various kinds and so on. Here we tap the resources of modern algebra, with its extensions and ideals, including the four morph brothers—homo-, iso-, auto-, and holo-. Such metatheorems form the very life and soul of mathematics. Finally we may have considerations relating a system to extraneous (non-mathematical) considerations, idealistic hypotheses, or what not, such as the semantical investigations of Tarski and Carnap.

If the notion of formal system is taken as a basis, these metatheorems are expressed in the same language as the elementary theorems. Metatheory is thus not quite the same notion as Hilbert's metamathematics, although it is related to it. Hilbert sets up a calculus which may be regarded as the calculization of a portion of mathematics; and then calls the study of this calculus metamathematics. This is, to say the least, an unfortunate terminology. Hilbert, presumably, does not mean to imply that mathematics is merely a meaningless-playing with the counters of his calculus, but rather that mathematics is an activity carried out in an interpreted object language from which meaning has been abstracted in the process of calculization; on the other hand his calculization does seem to imply, insofar as mathematics is formal, that the only legitimate mathematics is the explicit derivation of elementary theorems. But metatheorems have as good a right to be called mathematical as elementary ones; moreover from the general point of view which I shall mention later, Hilbert's metamathematics is a branch of mathematics rather than something which transcends it. Furthermore, we have already seen that if Hilbert's calculus is formalized we come back to another presentation of the original mathematics in the syntax language. The absolute separation of mathematics and metamathematics is fictitious. Hilbert's calculization had, I think, a psychological purpose, which it served well; but there is no sound basis for affirming the necessity of any such process.

As to the criteria of rigor for metatheoretic proofs, it is evident that we cannot handle the question as definitely as for the elementary ones. Many metatheoretic propositions have a vague character and require analysis. Such analysis can be given only by considering the specific kinds of metapropositions concerned. However, supposing this analysis made, we can divide metatheoretic proofs into two kinds; constructive and non-constructive. A constructive proof is one which, like the above proof of the substitution theorem, exhibits a process which can actually be carried through in any particular case arising under the hypotheses. For such a proof the criterion of rigor is reduced in the last analysis to that for the elementary propositions, and it has the same objective character. On the other hand the non-constructive proof depends on idealistic or other extraneous assumptions; and therefore we cannot regard them as purely formal.

An especially important class of metatheorems is constituted by certain theorems of incompleteness which have played a prominent rôle in recent years. These started with the Löwenheim[29] theorem of 1915, which was later refined by Skolem.[30] The upshot of these theorems was that any system of a certain kind would have an enumerable model in a sense which cannot be gone into here; since these systems have also interpretations which are non-enumerable, it follows that the interpretation cannot be unique as to structure. This, of course, is a non-constructive theorem. Sixteen years later Gödel[31] proved constructively that in systems strong enough for the usual mathematical purposes there are elementary propositions such that, if the system is consistent, neither these propositions nor their formal negatives within the system can be proved. A similar theorem of Kleene and Rosser, published in 1935,[32] shows that a formal system cannot have a kind of completeness as to term-formation and at the same time be capable of formalizing—in a sense which cannot be explained here—its own deductive processes.

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[29] L. Löwenheim, *Über Möglichkeiten im Relativkalkul*, Mathematische Annalen, vol. 76 (1915), pp. 447–470.

[30] Th. Skolem, *Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theoreme über dichte Mengen*, Videnskapsselskapet Skrifter, class I, 1920, no. 6.

[31] K. Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme*, Monatshefte für Mathematik und Physik, vol. 38, pp. 173–198. For expositions of this famous theorem see also D. Hilbert and P. Bernays, *Grundlagen der Mathematik*, vol. 2, 1939, pp. 269–289, also J. B. Rosser, *An informal exposition of proofs of Gödel's theorem and Church's theorem*, Journal of Symbolic Logic, vol. 4 (1939), pp. 53–60.

[32] S. C. Kleene and J. B. Rosser, *The inconsistency of certain formal logics*, Annals of Mathematics, (2), vol. 36 (1935), pp. 630–636. A revised proof of this theorem is con-

7. **Definition of mathematics.** We are now in a position to discuss the formalist definition of mathematics. In this connection the incompleteness theorems just discussed have important consequences for the formalist conception of mathematics. Indeed they show that it is not feasible to consider mathematics as the development of a single formal system. There are other reasons, too, which support such a view; for the arbitrary nature of the definitions constituting the primitive frame of a formal system shows that, in principle at least, all formal systems stand on a par.[33] The essence of mathematics lies, therefore, not in any particular kind of formal system, but in formal structure as such. *Mathematics*, then, *should be defined as the science of formal systems in general*; it should include all propositions elementary or metatheoretic, relating to one system or several, or to formal systems in general, so long as their criteria of truth depend on formal considerations alone.

This last proviso would exclude from mathematics proper all theorems depending on non-constructive proofs. This does not mean that these theorems are without mathematical interest. They are simply mixtures of mathematics and something else. As the example of the Löwenheim-Skolem theorem shows, these theorems are often of great significance for us; moreover they may lead to purely mathematical propositions at a higher level of formalization.

Mathematics, so conceived, has the following characteristics. First, it includes everything we know as mathematics. Classical analysis, for example, although not constructive when regarded as a metatheory of arithmetic, nevertheless can be embodied in a more inclusive formal system.[34] Second, mathematics is a science, in that it consists of propositions—not formulas but real propositions, with a definite criterion of truth. This criterion of truth is capable of fully

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tained in my paper *The paradox of Kleene and Rosser* (not yet published), and a considerable simplification in a paper, under the same title as that of Kleene and Rosser, which is now in preparation.

[33] This statement is not to be confused with the naive view that mathematics consists of more or less autonomous postulate systems. For if we could formulate a completely adequate logic, then these postulates could be added to the hypotheses of the theorems, and so all the postulate systems could be subsumed under one master-system—logic. But the point is that we can not have any such logic.

[34] Such formal systems, insofar as they are known at present, are not wholly satisfactory. Cf. Church's appendix to F. Gonseth, *Philosophie Mathématique*, Paris, 1940.

objective verification or proof—the primary standard of rigor for such a proof being the definition of derivability of an elementary proposition. On the other hand the gist of mathematics is that we make definitions by recursion, and then draw particular consequences by applying the definitions and general consequences by mathematical induction. There is thus a certain amount of justice in the view that mathematical theorems are consequences of definitions; but since the definitions are recursive, mathematics does not have the trivial character which that seems to imply. Likewise there is justice in the view that mathematical induction is a process of cardinal importance. However, it is not a mysterious power of the human mind, as Poincaré seemed to think, but is simply a corollary of recursive definitions; for if we demonstrate by mathematical induction that all members of a recursively defined class have a certain property, that demonstration is constructively valid because *by definition* every member of the class can be reached by the inductive process.

At an earlier stage I stated that most of the positive criticisms of the intuitionists find their justification in modern formalism. We have seen that this is so for the Kronecker demand for constructivity; indeed we are more stringent in this particular than the intuitionists themselves.[35] In the preceding paragraph we have also seen that their emphasis on mathematical induction has some justification. As for the law of excluded middle, it is necessary first to explain what it means. If we interpret it as a metaproposition concerning a formal system $S$, to the effect that every elementary proposition of $S$ is true or false, then we need to know the meaning of 'false' (not to mention 'or'). Now although we know what it means to verify an elementary proposition of $S$, yet we have no constructive definition of what it means to say it is false. We might define constructive falsity in any one of a number of different ways; but for any of the suggested definitions the law of excluded middle holds only if the system has a relatively trivial character.[36]

There is not time to go further with this. We can say, however, that it is useless to deny that intuition is involved in mathematics—if one defines intuition properly the statement is a tautology. However, we do not have to postulate a metaphysical character for it;

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[35] This is shown by Gödel's reduction of classical arithmetic to that of intuitionism (*Zur intuitionistischen Arithmetik und Zahlentheorie*, Ergebnisse eines mathematischen Kolloquiums, vol. 4 (1932), pp. 34–38). The consistency of classical arithmetic cannot be proved *constructively* in any such simple manner.

[36] These matters I plan to discuss more fully in a forthcoming paper *Some properties of formal deducibility.*

indeed to many of us it seems that this intuition is an empirical, linguistic phenomenon. On the other hand if we subtract from intuitionism its metaphysics, the remaining differences between it and formalism are relatively superficial as regards mathematical rigor. The main point is that the intuitionist prefers to confine attention to certain special types of formal systems. This leads to considerations, going beyond the definition of rigor, to which we must now devote some attention.

8. **Acceptability.** It is obvious that we are not interested in all formal systems. Those considerations which lead us to choose one system rather than another need to be distinguished from those relating to the truth of propositions concerning a formal system once chosen. To make this distinction I shall call that property of a formal system which leads us to adopt it for consideration its *acceptability*. This is evidently a quasi-truth concept which applies to a system as a whole.

In strictness acceptability is irrelevant to the problem of mathematical rigor. A proof of a proposition relating to a formal system is rigorous if and only if it satisfies certain objective criteria which are independent of acceptability. But it is necessary to give a brief account of acceptability for two reasons: first, because there has been some confusion in regard to it, and second because it sheds some light on the significance of mathematical rigor from the point of view of science.

The first point is that acceptability is relative to a purpose. It usually means that we are interested in some interpretation of the formal system; and the validity of the interpretation in relation to the subject matter is the prime consideration. Without statement of the purpose any discussion of acceptability is futile. Moreover the various schools of thought are not necessarily in conflict with one another. There is no one absolutely acceptable system; for different purposes entirely different systems may be acceptable.

Again if the purpose is an empirical one, as it is in physics, the question of acceptability is empirical also. Neither intuitive evidence, which is stressed by the intuitionists, nor demonstrable consistency, which is insisted on by Hilbert, is of more than secondary importance. From this point of view I think we must agree that, at the present state of our knowledge, some system of classical analysis is acceptable, however meaningless it may be to the intuitionists, and however far we may be from a formulation with a consistency proof. Of course if an inconsistency should be found, we should have to modify the system; but at the present time the acceptability of analysis is an empirical fact which requires no justification.

It is perhaps worth mentioning that the demand for intuitive evidence and the demand for consistency proof amount, in the last analysis, to nearly the same thing. For a thoroughly satisfactory consistency proof would exhibit a process whereby, given a classical proof with an intuitionistically significant conclusion, we could transform it into an intuitionistic proof of the same conclusion. Ackermann's recent proof of the consistency of arithmetic[37] does just this for a certain class of propositions. The proof is accomplished by a series of total replacements ("Gesamtersetzungen"); if the conclusion of the original proof is a particular numerical formula, the last total replacement gives an intuitive proof of the same formula. A complete solution of the consistency problem for analysis would show that any classical proof of a proposition with a direct intuitive meaning could be transformed into a purely constructive proof. In view of the Löwenheim-Skolem theorem there is reasonable doubt as to the possibility of such a transformation. And even if we succeed in finding it, it is not improbable that we shall always have uses for systems whose consistency is unknown.

The upshot of this is that we should cultivate a tolerant attitude in matters of acceptability.[38] Acceptability is not a question of right and wrong but of choice of subject matter. Such a choice should be free; and some difference of opinion is not only allowable but desirable. As mathematicians we should know to what sorts of system our theorems—if formalized—belong; but to exclude systems which fail to satisfy this or that criterion of acceptability is pedantry.

From the point of view of formalism, then, the question of truth of a naive mathematical proposition can be split into two parts. The first is the discovery of the formal system to which the proposition is to be referred, and the proof of the corresponding formal proposition; this is the problem of mathematical rigor, and we have seen that it has an objective character. The second is the acceptability of that system for the purpose in hand; this is a problem of applied mathematics.

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[37] W. Ackermann, *Zur Widerspruchsfreiheit der Zahlentheorie*, Mathematische Annalen, vol. 117 (1940), pp. 162–194.

[38] Cf. Carnap's "Principle of Tolerance" (*Logical Syntax*, p. 51 ff.); also Weyl, H., *Die heutige Erkenntnislage in der Mathematik*, Sonderdrucke des Symposion, vol. 3, Erlangen, 1926, pp. 31 ff.

# SPACE CREMONA TRANSFORMATIONS OF ORDER $m+n-1$[1]

EDWIN J. PURCELL

1. **Introduction.** This paper discusses a space Cremona transformation of order $m+n-1$ ($m$, $n$ any integers) generated by two rational twisted curves. One special position of the defining curves gives rise to an involution recently described,[2] while another special position results in an involution somewhat similar to one which was defined in a different manner by Montesano.[3]

2. **Cremona transformation.** Consider a curve $C_n$ of order $n$ having $n-1$ points on each of two skew lines $d$ and $d'$, and a curve $C_m'$ of order $m$ having $m-1$ points on each of $d$ and $d'$ ($m$, $n$, any integers). A generic point $P$ determines a ray through it intersecting $C_n$ once in $\alpha$ and $d$ once in $\beta$. $P$ also determines a ray through it intersecting $C_m'$ once in $\gamma$ and $d$ once in $\delta$. We define $P'$, the correspondent of $P$, to be the intersection of lines $\alpha\delta$ and $\beta\gamma$.

It is to be noted that if $C_n$ should become identical with $C_m'$ but $d$ and $d'$ remain distinct, there would result the Cremona involution we discussed in a recent paper (loc. cit.).

Let the equations of $d$ be $x_1=0$, $x_2=0$, and those of $d'$ be $x_3=0$, $x_4=0$. Let $C_n$ be

$$x_1 = (as+bt)\prod_1^{n-1}(t_is-s_it), \qquad x_2 = (cs+dt)\prod_1^{n-1}(t_is-s_it),$$

$$x_3 = (es+ft)\prod_n^{2n-2}(t_is-s_it), \qquad x_4 = (gs+ht)\prod_n^{2n-2}(t_is-s_it),$$

where $s_i$, $t_i$ for $i=1, 2, \cdots, n-1$ are values of the parameters of $C_n$ for points on $d$, and for $i=n, n+1, \cdots, 2n-2$, for points on $d'$.

Let the equations of $C_m'$ be

$$x_1 = (AS+BT)\prod_1^{m-1}(T_iS-S_iT), \quad x_2 = (CS+DT)\prod_1^{m-1}(T_iS-S_iT),$$

$$x_3 = (ES+FT)\prod_m^{2m-2}(T_iS-S_iT), \quad x_4 = (GS+HT)\prod_m^{2m-2}(T_iS-S_iT),$$

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[1] Presented to the Society, September 10, 1940.

[2] E. J. Purcell, *A multiple null-correspondence and a space Cremona involution of order* $2n-1$, this Bulletin, vol. 46 (1940), pp. 339–444.

[3] D. Montesano, *Su una classe di trasformazioni involutorie dello spazio*, Rendiconti del' Istituto Lombardo di Scienze e Lettere, (2), vol. 21 (1888), pp. 688–690.

where $S_i$, $T_i$ for $i=1, 2, \cdots, m-1$ are values of the parameters of $C_m'$ for points on $d$, and for $i=m, m+1, \cdots, 2m-2$, for points on $d'$. Then the equations of the transformation are

$$\begin{aligned}
x_1' &= k(Q_1x_3 + Q_2x_4)\left(\prod_1^{n-1} \theta_i\right)\left(\prod_1^{m-1} \Phi_i\right),\\
x_2' &= k(R_1x_3 + R_2x_4)\left(\prod_1^{n-1} \theta_i\right)\left(\prod_1^{m-1} \Phi_i\right),\\
x_3' &= K'(r_2x_1 - q_2x_2)\left(\prod_n^{2n-2} \theta_i\right)\left(\prod_m^{2m-2} \Phi_i\right),\\
x_4' &= K'(q_1x_2 - r_1x_1)\left(\prod_n^{2n-2} \theta_i\right)\left(\prod_m^{2m-2} \Phi_i\right),
\end{aligned}$$

where $k\equiv(bc-ad)$, $K'\equiv(FG-EH)$, and

$$\begin{aligned}
Q_1 &\equiv (AH - BG), & Q_2 &\equiv (BE - AF),\\
R_1 &\equiv (CH - DG), & R_2 &\equiv (DE - CF),\\
q_1 &\equiv (ah - bg), & q_2 &\equiv (be - af),\\
r_1 &\equiv (ch - dg), & r_2 &\equiv (de - cf),
\end{aligned}$$

$$\begin{aligned}
\theta_i &\equiv \{t_i(bx_2 - dx_1) - s_i(cx_1 - ax_2)\},\\
\Phi_i &\equiv \{T_i(Hx_3 - Fx_4) - S_i(Ex_4 - Gx_3)\}.
\end{aligned}$$

The inverse transformation is

$$\begin{aligned}
x_1 &= K(q_1x_3' + q_2x_4')\left(\prod_1^{n-1} \phi_i'\right)\left(\prod_1^{m-1}\Theta_i'\right),\\
x_2 &= K(r_1x_3' + r_2x_4')\left(\prod_1^{n-1} \phi_i'\right)\left(\prod_1^{m-1}\Theta_i'\right),\\
x_3 &= k'(R_2x_1' - Q_2x_2')\left(\prod_n^{2n-2} \phi_i'\right)\left(\prod_m^{2m-2}\Theta_i'\right),\\
x_4 &= k'(Q_1x_2' - R_1x_1')\left(\prod_n^{2n-2} \phi_i'\right)\left(\prod_m^{2m-2}\Theta_i'\right),
\end{aligned}$$

where $K\equiv(BC-AD)$, $k'\equiv(fg-eh)$,

$$\begin{aligned}
\phi_i' &\equiv \{t_i(hx_3' - fx_4') - s_i(ex_4' - gx_3')\},\\
\Theta_i' &\equiv \{T_i(Dx_1' - Bx_2') - S_i(Ax_2' - Cx_1')\}.
\end{aligned}$$

Both the direct and inverse transformations are of order $m+n-1$, where $m$ and $n$ are any integers.

The fundamental system and its images for the direct transformation are as follows.

$d$ is an $(n-1)$-fold $F$-line of simple contact. The fixed tangent planes are $\theta_i=0$, where $i=1, 2, \cdots, n-1$. It is of the first species and its $P$-surface consists in the planes $\phi_i'=0$, where $i=1, 2, \cdots, n-1$, which pass through $d'$.

$d'$ is an $(m-1)$-fold $F$-line of simple contact. The fixed tangent planes are $\Phi_i=0$, where $i=m, m+1, \cdots, 2m-2$. It is of the first species and its $P$-surface consists in the $m-1$ planes $\Theta_i'=0$ through $d$, where $i=m, m+1, \cdots, 2m-2$.

Each of the $m-1$ intersections of $C_m'$ and $d$ is an $n$-fold isolated $F$-point. Their $P$-surfaces are $\Theta_i'=0$, where $i=1, 2, \cdots, m-1$, respectively.

Each of the $n-1$ intersections of $C_n$ and $d'$ is an $m$-fold isolated $F$-point. Their $P$-surfaces are $\phi_i'=0$ $(i=n, n+1, \cdots, 2n-2)$ respectively.

The $(n-1)(m-1)$ lines of intersection of the $n-1$ fixed tangent planes through $d$ with the $m-1$ fixed tangent planes through $d'$ are simple $F$-lines without contact. They are of the second species.

The $(m-1)(n-1)$ lines joining the $m-1$ $n$-fold isolated $F$-points on $d$ with the $n-1$ $m$-fold isolated $F$-points on $d'$ are simple $F$-lines without contact. They are of the second species.

We may obtain a description of the fundamental system of the inverse transformation by interchanging $m$ and $n$, $C_n$ and $C_m'$, $\theta_i$ and $\Theta_i'$, $\Phi_i$ and $\phi_i'$, wherever they appear in the foregoing.

$C_n$, $d$, and $d'$ lie on the same quadric surface $Q$, and $C_m'$, $d$, and $d'$ lie on a quadric surface $Q'$. These quadrics may be the same or distinct and, while this does not affect the preceding discussion, the invariant systems for the two cases are different.

When $Q$ and $Q'$ are distinct, they intersect in $d$, $d'$, and two transversals $l_1$ and $l_2$. The $d$ and $d'$ are common generators of the $\mu$-systems of the two quadrics, while $l_1$ and $l_2$ are common generators of their $\lambda$-systems. The transformation sends each $\lambda$-generator of $Q$ over into a $\lambda$-generator of $Q'$, and each $\lambda$-generator of $Q'$ over into a $\lambda$-generator of $Q$. Thus $Q$ as a whole corresponds to $Q'$ and vice versa. Each $\lambda$-generator of either quadric belongs to a cycle of index four—that is, four applications of the transformation leave every $\lambda$-generator invariant. The transformation interchanges $C_n$ and $C_m'$. The points of $l_1$ are in involution; thus $l_1$ is an invariant line and the two fixed points of the involution are invariant points. Similarly for $l_2$. These four invariant points are the only invariant points that are not also $F$-points.

Let us now consider the case where $C_n$, $C_m'$, $d$, and $d'$ all lie on the same quadric $Q\equiv x_1x_4-x_2x_3=0$. The transformation causes $C_n$ and $C_m'$ to interchange. The pencil of planes $x_4-\lambda x_3=0$ is in involution with the pencil $x_2-\lambda x_1=0$ and this makes each $\lambda$-generator of $Q$ invariant. Consequently $Q$ is invariant. The locus of invariant points is a curve $K_{m+n}$ of order $m+n$ lying on $Q$. $K_{m+n}$ passes through the $m+n-2$ points of intersection of $C_n$ and $C_m'$ and intersects $d$ and $d'$ in the $m+n-2$ isolated $F$-points on each of them. It intersects every $\mu$-generator of $Q$ in $m+n-2$ points and intersects every $\lambda$-generator in two points.

3. **Involution.** Consider a twisted curve $C_n$ having $n-1$ points $\Delta_i$ on a straight line $d$, and a curve $C_m'$ having $m-1$ points $\Sigma_i$ on the same straight line $d$ ($m$, $n$ any integers). A generic point $P$ determines a ray through it intersecting $C_n$ in $\alpha$ and $d$ in $\beta$, and also a ray through it intersecting $C_m'$ in $\gamma$ and $d$ in $\delta$. We define $P'$, the correspondent of $P$ in the involution, to be the intersection of lines $\alpha\delta$ and $\beta\gamma$.

If, in §2, we make $d$ and $d'$ identical, we obtain an involution of this kind. However, the curves $C_n$ and $C_m'$ of the present section do not necessarily lie on quadric surfaces.

The involution is of order $m+n-1$.

The fundamental system and its principal images follow.

$d$ is an $(n+m-2)$-fold $F$-line of simple contact. The fixed tangent planes are $\theta_i=0$, where $i=1, 2, \cdots, n-1$, and $\Theta_i=0$, where $i=1, 2, \cdots, m-1$. It is of the second species and counts $(n+m-1)\cdot(n+m-2)$ times in the intersection of any two homaloids.

Points $\Delta_i$ are isolated $F$-points. Their $P$-surfaces are the planes $\theta_i=0$ $(i=1, 2, \cdots, n-1)$ respectively.

Points $\Sigma_i$ are isolated $F$-points. The $P$-element of each is $\Theta_i=0$ $(i=1, 2, \cdots, m-1)$ respectively.

As we have seen, a general point $P$ determines with $d$ a plane $\pi$ intersecting $C_n$ in $\alpha$ and $C_m'$ in $\gamma$. Call $L$ the intersection of lines $\alpha\gamma$ and $d$. Then $J$, the harmonic conjugate of $L$ with respect to $\alpha$ and $\gamma$, will be the only invariant point of $\pi$ other than points of $d$. As $\pi$ makes one revolution about $d$, $\alpha$ moves in $\pi$ and crosses $d$ $n-1$ times; $\gamma$ also moves in $\pi$, crossing $d$ $m-1$ times. As $\alpha$ approaches $d$, $J$ approaches the same point on $d$, and the locus of $J$ intersects $d$ in all the points $d$ has in common with $C_n$ and $C_m'$. The locus of $J$ is a rational curve $K_{m+n-1}$ of order $m+n-1$ having $m+n-2$ points on $d$. $K_{m+n-1}$ is the locus of invariant points.

It is clear that the line $PP'$ intersects $K_{m+n-1}$ in $J$ and $d$ in $L$, and that $P$ and $P'$ are harmonic conjugates with respect to[4] $J$ and $L$.

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[4] Compare with Montesano, loc. cit.

**4. Lower order for particular positions of the defining elements.** Each of the fixed tangent planes $\theta_i=0$ mentioned in the contact conditions for the involution passes through $d$ and is tangent to $C_n$ at the corresponding $\Delta_i$. The fixed tangent planes $\Theta_i=0$ are similarly related to the curve $C_m'$.

If $C_n$ and $C_m'$ are so situated that a plane of $\theta_i=0$ $(i=1, 2, \cdots, n-1)$ coincides with a plane of $\Theta_i=0$ $(i=1, 2, \cdots, m-1)$, the order of the involution is reduced by one. In this way we may reduce the order by any integer up to, and including, the smaller of the two numbers $n-1$ and $m-1$.

UNIVERSITY OF ARIZONA

# NOTE ON AUTOPOLAR CURVES[1]

MALCOLM FOSTER

1. **Introduction.** The aim of this paper is to study those curves which are autopolar with respect to the parabola $2\eta=\xi^2$. The method, which is believed to be new, is to consider these curves as special solutions of those differential equations which are invariant under the dual substitutions for the above conic of reference.[2] It will be obvious that this method may be readily modified for the study of curves which are autopolar with respect to any conic. The parabola $2\eta=\xi^2$ has been chosen for the sake of the simplicity of the substitutions.

2. **Dual substitutions for the conic of reference.** In the ordinary differential equation,

$$f(x, y, y', y'', \cdots, y^{(n)}) = 0, \tag{1}$$

let us make the well known dual substitutions[3]

$$\begin{gathered} x = P = Y', \qquad y = Y'X - Y, \qquad p = y' = X, \\ y'' = 1/Y'',\ y''' = -Y'''/Y''^3, \cdots \end{gathered} \tag{2}$$

for which the conic of reference is the parabola $2\eta=\xi^2$. We obtain a new differential equation,

$$f(Y', Y'X - Y, X, 1/Y'', \cdots) = 0, \tag{3}$$

whose solution is, let us say,

$$\phi(X, Y, c_1, c_2, \cdots, c_n) = 0. \tag{4}$$

If we eliminate $X$, $Y$ from equation (4) and the following two equations,[4]

$$x\frac{\partial\phi}{\partial Y}+\frac{\partial\phi}{\partial X}=0, \qquad -y\frac{\partial\phi}{\partial Y}=Y\frac{\partial\phi}{\partial Y}+X\frac{\partial\phi}{\partial X}, \tag{5}$$

we shall have the solution of the original differential equation (1), which we shall denote by

$$F(x, y, c_1, c_2, \cdots, c_n) = 0. \tag{6}$$

3. **Geometrical interpretation.** Let $C$ be any curve of the family

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[1] Presented to the Society, October 28, 1939.

[2] A. R. Forsyth, *A Treatise on Differential Equations*, 3d edition, 1903, pp. 45–47.

[3] Forsyth, op. cit.

[4] Forsyth, loc. cit.

(6). As a point $P$ traverses $C$, the polar of $P$ relative to the conic of reference will envelop some member $C'$ of the family (4), and vice versa. That is, $C$ and $C'$ are polar reciprocals.

If (1) is of the first order, equations (1) and (3) may have singular solutions. If $E$ and $E'$ denote the envelopes of the families (4) and (6), it is evident from the above that $E$ and $E'$ are also polar reciprocals.

In addition to these relations between the families (4) and (6), there are several well known relations between the extraneous loci which may exist in connection with the integral curves. For example, a cuspidal locus for the integral curves (4), [(6)], will correspond to the locus of points of inflexion for the integral curves (6), [(4)].[5]

4. **Condition that $y=f(x)$ be autopolar.** If a curve $C$, $y=f(x)$, be autopolar, that is, its own polar reciprocal, the polar of any point $(x_1, y_1)$ on $C$ will be tangent to the curve at some other point $(x_2, y_2)$. Relative to the above conic of reference, the polar of any point $(x, y)$ on $C$ is $x\xi-\eta-y=0$, which we may write

$$x\xi - \eta - f(x) = 0. \tag{7}$$

This is a one-parameter family of lines with $x$ as the parameter, and their envelope will be found by the elimination of $x$ from (7) and the following equation,

$$\xi - f'(x) = 0. \tag{8}$$

The necessary and sufficient condition that $C$ be autopolar is that on eliminating $x$ we shall get $\eta=f(\xi)$ as the envelope. From (7) and (8) we have $xf'(x)-\eta-f(x)=0$; and on replacing $\eta$ by $f(\xi)$, or $f[f'(x)]$, we have

$$f[f'(x)] + f(x) - xf'(x) = 0. \tag{9}$$

We have, therefore, the following theorem:

THEOREM 1. *A necessary and sufficient condition that a curve $y=f(x)$ be autopolar with respect to the conic $2\eta=\xi^2$ is that the relation* (9) *be satisfied.*

It is interesting to note that (9) is of the Clairaut type.

5. **Relations between conjugate pairs, $(x_1, y_1)$ and $(x_2, y_2)$.** Consider any point $P_1(x_1, y_1)$ on an autopolar curve $C$. The polar of $P_1$ is $x_1\xi-\eta-y_1=0$; and since this line is tangent to $C$ at some point

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[5] Sophus Lie and Georg Scheffers, *Geometrie der Berührungstransformationen*, vol. 1, 1896, pp. 24–27.

$P_2(x_2, y_2)$, the equation of the polar of $P_1$ must be satisfied by the coördinates of $P_2$. Consequently,[6]

$$y_1 + y_2 = x_1x_2. \tag{10}$$

From the theory of polar reciprocals we know that the polar of $P_2$ is tangent to $C$ at $P_1$, and since the slopes of the polars of $P_1$ and $P_2$ are respectively $x_1$ and $x_2$, we have

$$x_1 = f'(x_2), \qquad x_2 = f'(x_1). \tag{11}$$

Let us call such a pair of points, $P_1$ and $P_2$, a conjugate pair, and say that each is the conjugate of the other. The relations (11) are also evident from the set of substitutions (2) in which $x = P$, $X = p$.

From (10) and (11) we have the following theorem:

THEOREM 2. *On any autopolar curve with respect to the conic* $2\eta = \xi^2$, *the product of the slopes at a conjugate pair is equal to the sum of the ordinates at these points.*

6. **Self-conjugate points.** For any real conjugate pair, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, let us suppose $x_2 > x_1$. Let us also assume that $f(x)$ is defined for all values of $x$ between $x_1$ and $x_2$. Now let $x_2$ decrease, that is, let $P_2$ approach the original position of $P_1$. When $P_2$ arrives at this position. $P_1$ will have arrived at the original position of $P_2$. Hence $x_1$ must increase as $x_2$ decreases. Consequently, between each real conjugate pair there exists a point $P_3(x_3, y_3)$ which is self-conjugate. From (10) we shall have $2y_3 = x_3^2$, and therefore $P_3$ lies on the conic of reference. Since the polar of $P_3$ is tangent to the conic of reference at this point, we see that every autopolar curve has a common tangent with this conic.

Also from (11), and the law of the mean, we have

$$\frac{f'(x_2) - f'(x_1)}{x_2 - x_1} = -1 = f''(x_4), \qquad x_1 < x_4 < x_2, \tag{12}$$

and since this relation is satisfied by every conjugate pair, no matter how near they may be to $P_3$, we see that

$$f''(x_3) = -1. \tag{13}$$

From (11) and (13) we readily find the curvature of any autopolar curve at $P_3$ to be

$$K = \frac{-1}{[1 + f'^2(x_3)]^{3/2}} = \frac{-1}{(1 + x_3^2)^{3/2}}.$$

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[6] It is obvious from (2) that (10) is simply a restatement of (9).

It is readily verified that, except for sign, this is the curvature of the conic of reference at $P_3$. Hence we have:

THEOREM 3. *For every autopolar curve with respect to the conic* $2\eta=\xi^2$, *the curvature at the self-conjugate points* $P_3$ *is the same, except for sign, as the curvature of the conic at* $P_3$.

We note that this theorem applies also to the case of the general conic by virtue of the fact that under any collineation the ratio of the curvatures of two curves at a point at which they are tangent is invariant.[7]

From (12) it also follows that between real conjugate pairs $f'(x)$ is a decreasing function.

7. **Loci associated with a conjugate pair.** Consider the locus of $Q$, the intersection of the polars for any conjugate pair, $P_1$ and $P_2$. The coördinates $(\xi, \eta)$ of $Q$ are readily found from the equations of the polars of $P_1$ and $P_2$; we find

$$\xi = \frac{y_1 - y_2}{x_1 - x_2}, \qquad \eta = \frac{x_2y_1 - x_1y_2}{x_1 - x_2}. \tag{14}$$

By means of (11) these equations of the locus of $Q$ are readily given in terms of one parameter, say $x_1$.

Since the polar of $Q$ is the line $P_1P_2$, we see that the locus of $Q$ and the envelope of the lines $P_1P_2$ are also polar reciprocals.

It will be of interest to consider also the locus of $S$, the mid-point of the segment $P_1P_2$.

We shall also consider the locus of $R$, the intersection of normals at conjugate pairs. The coördinates of $R$ are readily found to be

$$\xi = \frac{x_1^2 + x_1x_2y_1 - x_2^2 - x_1x_2y_2}{x_1 - x_2}, \qquad \eta = \frac{x_2 + x_1y_2 - x_1 - x_2y_1}{x_1 - x_2}. \tag{15}$$

From §6 it is evident that the loci of $Q$ and $S$ must pass through $P_3$.

8. **Differential equations invariant under the above transformation.** Any differential equation of the form

$$f(x)y'' + f(y') = 0 \tag{16}$$

becomes, on using (2), $f(X)Y''+f(Y')=0$, which is identical with (16). Let us denote the solution of (16) by

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[7] G. Fubini and E. Čech, *Introduction à la Géométrie Projective Différentielle des Surfaces*, 1931, pp. 17–20.

$$\phi(x, y, c_1, c_2) = 0. \tag{17}$$

As a point $P$ traverses some member $C$ of (17), the polar of $P$ will, in general, envelop some other member $C'$ of the same family. We wish to determine if there are any members of the two-parameter family (17), which are autopolar. The method for the determination of these curves will be illustrated by particular examples.

EXAMPLE 1. $(x+1)y''+y'+1=0$. The solution of this is

$$y = c_1 \log (x + 1) - x + c_2. \tag{18}$$

Here $f(x)=c_1 \log (x+1)-x+c_2$, and $f'(x)=c_1/(x+1)-1$. If we put these expressions in (9), we get $(c_1-c_1 \log c_1-1-2c_2)x+c_1-c_1 \log c_1 -1-2c_2=0$, which is satisfied only if $c_2=\frac{1}{2}c_1-\frac{1}{2}c_1 \log c_1-\frac{1}{2}$. Consequently, of the two-parameter family of curves (18), we have the following one-parameter family whose members are autopolar:

$$y = c_1 \log (x + 1) - x + \tfrac{1}{2}c_1 - \tfrac{1}{2}c_1 \log c_1 - \tfrac{1}{2}. \tag{19}$$

The particular member of (19) for which $c_1=1$ has the property that the locus of $S$ (the mid-point of $P_1P_2$) is the line $x+y=0$.

It may be readily verified that, as indicated in §6, the conic of reference, $2\eta=\xi^2$, is the envelope of the one-parameter family of autopolar curves (19).

EXAMPLE 2. $x^2y''+y'^3=0$. The solution is $c_1^2(y-c_2)^2=c_1x^2-1$; and on putting this in (9) we have $2c_2(c_1x^2-1)^{1/2}=0$. Hence $c_2=0$, and of this two-parameter family of hyperbolas there exists the one-parameter autopolar family $c_1x^2-c_1^2y^2=1$. It is readily found from (14) that the locus of $Q$ for these curves is the line $\eta=1/c_1$. And from (15) the locus of $R$ is the line $\eta=-(c_1+1)/c_1$.

EXAMPLE 3. $y''+1=0$. The solution is $y=-\frac{1}{2}x^2+c_1x+c_2$, and this in (9) gives $c_2=-\frac{1}{4}c_1^2$. Hence the parabolas $y=-\frac{1}{2}x^2+c_1x-\frac{1}{4}c_1^2$ are autopolar. It is readily found that for these curves the locus of $Q$ is the line $\xi=\frac{1}{2}c_1$, and that this line is also the locus of $S$. We also find that the coördinates of the mid-point of the segment $QS$, $(\frac{1}{2}c_1, \frac{1}{8}c_1^2)$, are identical with the coördinates of $P_3$, the point of contact of the parabolas with the conic of reference.

EXAMPLE 4. $x+y'=xy'$. The solution of this first-order equation is $y=x+\log (x-1)+c$, and this in (9) gives $c=0$. Hence there is but one member of the family which is autopolar.

EXAMPLE 5. $y'y''+x=0$. From this equation we derive the following one-parameter family of autopolar curves:

$$y = \tfrac{1}{2}c_1^2 \operatorname{arc\,sin} x/c_1 + \tfrac{1}{2}x(c_1^2 - x^2)^{1/2} - \tfrac{1}{8}c_1^2\pi.$$

There are, of course, many other types of differential equations which are invariant under (2). We mention a few of these: $f(x)+f(y')=F(xy')$, $2y-xy'=f(y')-f(x)$, and $f(y)y''+f(xy'-y)=0$.

The above examples suggest that, in general, we have the following distinction between the autopolar curves obtained from invariant differential equations of the first and second orders. When the solution of such a first-order equation is put in (9), this relation will be satisfied by only a finite number of values of the one arbitrary constant; and hence of this single infinity of curves we shall have but a finite number which are autopolar. When the solution of an invariant second-order equation is put in (9), the relation will be satisfied in many cases by some relation between the two arbitrary constants; that is, from the double infinity of solutions we select a single infinity of autopolar curves. Moreover, in every case, this one-parameter family of autopolar curves envelopes the conic of reference $2\eta=\xi^2$.

There are, however, exceptions to the above. For example, consider the differential equation $(y-x^2)y'+xy=0$, invariant under (2), whose solution is $cx^2+y^2-2cy=0$. On using (9) we find there is no value of $c$ which satisfies this relation. That is, among these conics there are no autopolar curves; but for each conic of the family the polar reciprocal is some other conic of the same family. It may readily be shown that of the above conics, those for which the values of $c$ are negatives of one another are polar reciprocals.

9. **Other dual substitutions.** The dual substitutions for any given conic of reference are easily derived after the manner outlined in Forsyth. And any differential equation which is invariant under these substitutions will have among its solutions certain curves which are autopolar with respect to the given conic.

For example, if the conic of reference be the unit circle, these dual substitutions are[8]

$$(20)\qquad y' = -\frac{X}{Y},\quad x = \frac{-Y'}{Y-XY'},\quad y = \frac{1}{Y-XY'},\ \cdots .$$

Without repeating the argument of §4, the necessary and sufficient condition that a curve $y=F(x)$ be autopolar with respect to the unit circle is

$$(21)\qquad \frac{1}{f(x)-xf'(x)} = F\left[\frac{f'(x)}{xf'(x)-f(x)}\right].$$

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[8] Lie und Scheffers, loc. cit., p. 23.

EXAMPLE 6. $xy'+y=0$. This equation is invariant under (20); and its solution, $y=c/x$, when put in (21), satisfies this relation when $c=\pm\frac{1}{2}$. Consequently, the equilateral hyperbolas, $2xy=\pm 1$, are autopolar with respect to the unit circle.

For a complete bibliography on the subject of autopolar curves the reader is asked to consult H. Brocard and T. Lemoyne, *Courbes Géométriques Remarquables*, vol. 1, 1919, pp. 430–432.

WESLEYAN UNIVERSITY

# ON THE EQUATION $dy/dx=f(x, y)$[1]

ANDRÉ GLEYZAL

We consider here the differential equation $dy/dx=f(x, y)$, where $f(x, y)$ is a one-valued function defined on an open region $R$ of the $xy$-plane. By a solution curve of this equation we mean a curve $y=y(x)$ which has a derivative at every point and which satisfies everywhere the differential equation. There are known sufficiency conditions on $f$ for the existence of a one parameter family of solution curves simply covering $R$. But, as Professor Bamforth mentioned to me orally, there seems to be in the literature no corresponding necessary conditions. We shall prove that one necessary condition is that $f$ be the limit of a sequence of continuous functions.[2]

A curve is the $xy$-plane will be termed a *continuous function curve* if the points of the curve are the points $(x, \phi(x))$, $a<x<b$, where $\phi(x)$ is a one-valued continuous function defined on an open interval $(a, b)$. An open region $R$ of the $xy$-plane will be said to be simply covered by a set $\mathcal{J}$ of such curves if:

(1) Every point of $R$ is on one and only one curve of $\mathcal{J}$.

(2) Every curve of $\mathcal{J}$ stretches from boundary to boundary of $R$; that is, if $S$ is any set of points on a curve $C$ of $\mathcal{J}$, each limit point of $S$ is either itself a point of $C$ or a boundary point of $R$.

THEOREM. *If an open region $R$ of the $xy$-plane is simply covered by a set $\mathcal{J}$ of continuous function curves $y=\phi(x)$, then for every point $(x_0, y_0)$ of $R$ there exists an open subregion $R_0$ of $R$ containing $(x_0, y_0)$ such that the family of curves constituted by the portions of the curves of $\mathcal{J}$ in $R_0$ is representable by the equation $y=\phi(x, a)$, where $\phi$ is a continuous function of $x$ and the parameter $a$.*

PROOF. Let $(x_0, y_0)$ be a point of such a given region $R$; $R_1$ a rectangle interior to $R$ with $(x_0, y_0)$ as center; and $h$ a positive number such that the points $(x_0, y_0-h)$ and $(x_0, y_0+h)$ are inside $R_1$. Since $\mathcal{J}$ simply covers $R$, there exist curves $y=\phi_1(x)$ and $y=\phi_2(x)$ of $\mathcal{J}$ containing the points $(x_0, y_0-h)$ and $(x_0, y_0+h)$ respectively. Also, the continuity of $\phi_1(x)$ and $\phi_2(x)$ insures the existence of an open interval $I$ containing $x_0$ such that the points of the curves $y=\phi_1(x)$ and

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[1] I wish to express my thanks to Professor Henry Blumberg for his aid in the preparation of this paper.

[2] Of course, as is well known, the derivative $f'(x)$ of a function $f(x)$ has this property. It may be expected that $f(x, y)$ necessarily has also other properties corresponding to known properties of $f'(x)$.

$y=\phi_2(x)$, with $x$ in $I$, lie in $R_1$. If $(x_1, x_2)$ is a closed interval interior to $I$ and containing $x_0$, we have $\phi_1(x)<\phi_2(x)$ for $x_1\leqq x\leqq x_2$, since $\phi_1(x_0)<\phi_2(x_0)$ and no two curves of $\mathcal{J}$ cross each other. We denote by $R_0$ the open region of points $(x, y)$ bounded by the parallels $x=x_1$, $x=x_2$, and the curves $y=\phi_1(x)$, $y=\phi_2(x)$. In view of conditions (1) and (2) it readily follows that the interval of definition of any curve $y=\phi(x)$ of $\mathcal{J}$ having points in $R_0$ contains the entire closed interval $(x_1, x_2)$.

We now define the function $\phi(x, a)$, mentioned in the theorem, by letting $y=\phi(x, a)$ be the equation of the curve of $\mathcal{J}$ passing through the point $(x_1, a)$ on the boundary of $R_0$. Thus $\phi(x, a)$ is defined in the region $\overline{R}_0$ of the $xa$-plane, $x_1<x<x_2$, $\phi_1(x_1)<a<\phi_2(x_1)$. The family of curves $y=\phi(x, a)$, where $x$ and the parameter $a$ take on values in $\overline{R}_0$, is identical with the set consisting of the portions of the curves of $\mathcal{J}$ which are interior to $R_0$. $\phi(x, a)$ is a continuous function of $x$ and $a$ in $\overline{R}_0$. For let $(\xi, \alpha)$ be a point of this region, $\epsilon$ any positive number, and $\alpha_1$, $\alpha_2$ numbers such that $y=\phi(x, \alpha_1)$, $y=\phi(x, \alpha_2)$ are respectively equations of the curves of $\mathcal{J}$ which pass through the points $(\xi, \phi(\xi, \alpha)-\epsilon')$, $(\xi, \phi(\xi, \alpha)+\epsilon')$ of $R_0$, $\epsilon'$ being a positive number less than $\epsilon/4$. Due to the continuity of $\phi(x, \alpha)$, considered as a function of $x$, we may find an interval $I_1$, containing $\xi$, such that $\phi(x, \alpha_1)$ differs from $\phi(\xi, \alpha_1)$ by an amount less than $\epsilon/4$ for values of $x$ in $I_1$. Also, we may choose an interval $I_2$ such that $\phi(x, \alpha_2)$ differs from $\phi(\xi, \alpha_2)$ by less than $\epsilon/4$ for $x$ in $I_2$. Thus $\phi(x, a)$ differs from $\phi(\xi, \alpha)$ by less than $\epsilon$ if $\alpha_1<a<\alpha_2$ and $x$ is in $I_1I_2$—the interval consisting of the points common to both $I_1$ and $I_2$. Consequently, $\phi(x, a)$ is continuous in $\overline{R}_0$, and the theorem is proved.

LEMMA. *The function $\phi(x, a)$ has a continuous inverse function $a(x, y)$ defined on $R_0$, which satisfies the identity $\phi(x, a(x, y))\equiv y$.*

PROOF. We associate with every point $(x, y)$ of $R_0$ that value $a=a(x, y)$ such that the curve $y=\phi(x, a)$ passes through the point $(x, y)$. In short, $\phi(x, a(x, y))=y$. The function $a(x, y)$ thus defined is continuous in $R_0$. For suppose it were discontinuous at a point $(\xi, \eta)$ of $R_0$. Since $\phi(x, a)$ is continuous at $(\xi, a(\xi, \eta))$, and properly monotone in $a$, the composite function $\phi(x, a(x, y))$, considered as a function of $x$ and $y$, is discontinuous at $(\xi, \eta)$. But $\phi(x, a(x, y))\equiv y$ for points $(x, y)$ in $R_0$ and is therefore continuous at the point $(\xi, \eta)$, contrary to assumption. Therefore $a(x, y)$ is continuous in $R_0$.

THEOREM. *For the existence of a family $\mathcal{J}$ of continuous function curves simply covering $R$ which are solution curves of the equation*

*$dy/dx=f(x, y)$, where $R$ is an open region on which $f$ is defined, it is necessary that $f$ be the limit of a sequence of continuous functions.*

PROOF. We consider $f(x, y)$ given, and assume that a family $\mathcal{J}$, as described, exists. If $(x_0, y_0)$ is a point of $R$ there may be determined, as we have shown, an open region $R_0$ containing the point such that the portions of the curves of $\mathcal{J}$ in $R_0$ are the curves $y=\phi(x, a)$, $x_1<x<x_2, a_1<a<a_2$. $\phi(x, a)$ has, according to the above lemma, a continuous inverse function $a(x, y)$ defined on $R_0$ for which $\phi(x, a(x, y))=y$. Let $R_0'$ be the open subregion consisting of the points $(x, y)$ of $R_0$ where $x_1<x<x_2-k$, $k$ being a positive number less than $x_2-x_0$. If $(\xi, \eta)$ is a point of $R_0'$, the equation of the curve of $\mathcal{J}$ which passes through this point is $y=\phi(x, \alpha)$, where $\alpha=a(\xi, \eta)$. By hypothesis, $y=\phi(x, \alpha)$ is a solution curve of the differential equation $dy/dx=f(x, y)$. At $(\xi, \eta)$ this equation may be written:

$$f(\xi, \eta) = \lim_{n\to\infty} \{\phi(\xi + k/n, a(\xi, \eta) - \phi(\xi, a(\xi, \eta))\}/(k/n)$$

where $n$ is a positive integer. For convenience, we denote the difference quotient present in the right-hand member of this equation by $\psi_n(\xi, \eta)$. Inasmuch as $(\xi, \eta)$ is a general element of $R_0'$, we have $f(x, y)=\lim_{n\to\infty} \psi_n(x, y)$ in this region. Moreover, $\psi_n(x, y)$ is continuous since $a(x, y)$ is continuous in $R_0'$ and $\phi(x, a)$ is continuous at points $(x, a(x, y))$ and $(x+k/n, a(x, y))$, where $(x, y)$ is in $R_0'$. We conclude that for every point $(x_0, y_0)$ of $R$, there exists an open region $R_0'$ containing $(x_0, y_0)$ such that in $R_0'$ the function $f(x, y)$ is the limit of a sequence of continuous functions.

It follows that $f$ is the limit of a sequence of continuous functions in its entire region of definition $R$. For let $P$ be any perfect subset of the points of the $xy$-plane containing points of $R$, and $(x_0, y_0)$ any point of $PR$. As we have seen, there exists an open region $R_0'$ containing $(x_0, y_0)$ such that, in this region, $f$ is the limit of a sequence of continuous functions. The latter statement implies, it may be shown, that $PR_0'$ has a point of continuity of $f$ with respect to $PR_0'$.[3] Clearly, this point is a point of continuity of $f$ with respect, not merely to the subset $PR_0'$, but with respect to $P$, and by Baire's theorem $f$ is the limit of a sequence of continuous functions.

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[3] For, a necessary and sufficient condition that $f(x, y)$, defined on an open region $R$ of the $xy$-plane, be the limit of a sequence of continuous functions is that every set $PR_0$, where $P$ is a perfect set and $R_0$ is an open subregion of $R$, have a point of continuity of $f$ with respect to $PR_0$. This is a slight modification of Baire's theorem and is proved in a paper *On interval functions* which the author is preparing for publication.

# NOTE ON INTERPOLATION

G. GRÜNWALD

Let

$$(1) \qquad A_n: x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)}, \qquad -1 \leqq x_1^{(n)}, \ 1 \geqq x_n^{(n)},$$

denote a set of $n$ distinct points of the interval $-1 \leqq x \leqq +1$. If $f(x)$ is a given function defined in $-1 \leqq x \leqq +1$ we call the polynomial

$$(2) \qquad I_n(x; f) = \sum_{k=1}^{n} f(x_k^{(n)}) q_k^{(n)}(x)$$

an interpolation polynomial of $f(x)$ corresponding to the abscissas (1), where for the polynomials $q_1^{(n)}(x), q_2^{(n)}(x), \cdots, q_n^{(n)}(x)$

$$(3) \qquad q_k^{(n)}(x_k^{(n)}) = 1, \qquad q_k^{(n)}(x_i^{(n)}) = 0, \qquad i \neq k.$$

Then the polynomial (2) represents a polynomial which assumes the value $f(x_k^{(n)})$ at $x = x_k^{(n)}$ $(k = 1, 2, \cdots, n)$. The polynomials $q_k^{(n)}(x)$ $(k = 1, 2, \cdots, n)$ are called the fundamental polynomials of the interpolation corresponding to the set $A_n$. We consider the sequence $I_n(x; f)$ $(n = 1, 2, \cdots)$ under the condition of continuity of $f(x)$.

Let $\omega_n(x)$ be a polynomial of degree $n$, not identically zero, vanishing at $x = x_k^{(n)}$ $(k = 1, 2, \cdots, n)$ and let

$$(4) \qquad q_k^{(n)}(x) = \frac{\omega_n(x)}{\omega_n'(x_k^{(n)})(x - x_k^{(n)})} \equiv l_k^{(n)}(x);$$

then (2) represents the $n$th Lagrange polynomial of $f(x)$ corresponding to the abscissas (1), which is the uniquely determined polynomial of degree $n-1$ which assumes the value $f(x_k^{(n)})$ at $x = x_k^{(n)}$ $(k = 1, 2, \cdots, n)$.

It is known that for a given arbitrary sequence $\{A_n\}$:

$$(5) \quad x_1^{(1)};\ x_1^{(2)}, x_2^{(2)};\ x_1^{(3)}, x_2^{(3)}, x_3^{(3)};\ \cdots;\ x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)};\ \cdots$$

there exists a continuous function $f(x)$ such that the sequence of the Lagrange interpolation polynomials is not uniformly convergent, even divergent at a preassigned point.[1]

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[1] G. Faber, *Über die interpolarische Darstellung stetiger Funktionen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 389–408. S. Bernstein, *Sur la limitation des valeurs d'une polynome $P_n(x)$ de degré $n$ sur tout un segment par ses valeurs en $(n+1)$ points du segment*, Bulletin de l'Académie des Sciences de l'URSS, 1931, pp. 1025–1050. In the important special case $x_k^{(n)} = \cos (2k-1)\pi/2n$, that is, for the zeros $\omega_n(x) = \cos n(\text{arc} \cos x)$ ($\omega_n(x)$ is the $n$th Tschebycheff polynomial) much more is known. I have proved the existence of a continuous function $f(x)$ for which

Let

$$v_k^{(n)}(x) = 1 - (x - x_k^{(n)}) \frac{\omega_n''(x_k^{(n)})}{\omega_n'(x_k^{(n)})}, \tag{6}$$

$$q_k^{(n)}(x) = v_k^{(n)}(x)\{l_k^{(n)}(x)\}^2; \tag{7}$$

then $I_n(x; f)$, the $n$th Hermite interpolation polynomial of $f(x)$, represents the uniquely determined polynomial of degree $2n-1$ for which $I_n(x_k^{(n)}; f) = f(x_k^{(n)})$ $(k = 1, 2, \cdots, n)$ and $I_n'(x_k^{(n)}; f) = 0$ $(k = 1, \cdots, n)$. There are sequences $\{A_n\}$ for which the sequence of Hermite interpolation polynomials is uniformly convergent for an arbitrary continuous function.[2]

The question arises whether it is possible to determine, for a given arbitrary sequence $\{A_n\}$, the fundamental polynomials

$$\begin{gathered} q_1^{(1)}(x);\quad q_1^{(2)}(x),\ q_2^{(2)}(x);\quad q_1^{(3)}(x),\ q_2^{(3)}(x),\ q_3^{(3)}(x);\ \cdots;\\ q_1^{(n)}(x),\ q_2^{(n)}(x), \cdots, q_n^{(n)}(x);\ \cdots \end{gathered} \tag{8}$$

so that for an arbitrary continuous function $f(x)$

$$\lim_{n=\infty} I_n(x; f) = \lim_{n=\infty} \sum_{k=1}^{n} f(x_k^{(n)}) q_k^{(n)}(x) = f(x) \tag{9}$$

holds uniformly in $-1 \leqq x \leqq 1$.

For the sequence $\{A_n\}$ we must suppose that the set (5) is everywhere dense in $-1 \leqq x \leqq +1$, that is, $\max_{(i)} (x_{i+1}^{(n)} - x_i^{(n)}) \to 0$ when $n \to \infty$. We prove that this trivial necessary condition is however sufficient to construct the fundamental polynomials (8) so that (9) holds for every continuous $f(x)$.

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the sequence of Lagrange polynomials corresponding to these abscissas is everywhere divergent, even everywhere unbounded. G. Grünwald, *Über Divergenzerscheinungen der Lagrange'schen Interpolationspolynome stetiger Funktionen*, Annals of Mathematics, (2), vol. 37 (1936), pp. 908–918. See also J. Marcinkiewicz, *Sur la divergence des polynomes d'interpolation*, Acta Litterarum ac Scientiarum, Szeged, vol. 8 (1937), pp. 131–135.

[2] Such a sequence is, for example, $x_k^{(n)} = \cos(2k-1)\pi/2n$. See L. Fejér, *Über Weierstrass'sche Approximation, besonders durch Hermite'sche Interpolation*, Mathematische Annalen, vol. 102 (1930), pp. 707–725. It was Fejér who began the investigation of the Hermite interpolation polynomials instead of and beside the Lagrange polynomials, and the fundamental methods and theorems of the theory are due to him. See, for example, L. Fejér, *On the characterisation of some remarkable systems of points of interpolation by means of conjugate points*, American Mathematical Monthly, vol. 41, (1939), pp. 1–14.

Let $\phi_k^{(n)}(x)$ be the continuous function defined in the following way: $\phi_1^{(1)}(x)\equiv 1$; for $n > 1$, and $k \neq 1, n$,

$$\phi_k^{(n)}(x) = \begin{cases} 0 \text{ in } -1 \leqq x \leqq x_{k-1}^{(n)}, \\ \text{linear in } x_{k-1}^{(n)} \leqq x \leqq \tfrac{1}{2}(x_{k-1}^{(n)} + x_k^{(n)}), \\ 1 \text{ in } \tfrac{1}{2}(x_{k-1}^{(n)} + x_k^{(n)}) \leqq x \leqq x_k^{(n)}, \\ \text{linear in } x_k^{(n)} \leqq x \leqq \tfrac{1}{2}(x_k^{(n)} + x_{k+1}^{(n)}), \\ 0 \text{ in } \tfrac{1}{2}(x_k^{(n)} + x_{k+1}^{(n)}) \leqq x \leqq +1; \end{cases}$$

for $k=1, n$

$$\phi_1^{(n)}(x) = \begin{cases} 1 \text{ in } -1 \leqq x \leqq x_1^{(n)}, \\ \text{linear in } x_1^{(n)} \leqq x \leqq \tfrac{1}{2}(x_1^{(n)} + x_2^{(n)}), \\ 0 \text{ in } \tfrac{1}{2}(x_1^{(n)} + x_2^{(n)}) \leqq x \leqq +1; \end{cases}$$

$$\phi_n^{(n)}(x) = \begin{cases} 0 \text{ in } -1 \leqq x \leqq x_{n-1}^{(n)}, \\ \text{linear in } x_{n-1}^{(n)} \leqq x \leqq \tfrac{1}{2}(x_{n-1}^{(n)} + x_n^{(n)}), \\ 1 \text{ in } \tfrac{1}{2}(x_{n-1}^{(n)} + x_n^{(n)}) \leqq x \leqq +1. \end{cases}$$

We have evidently

$$\sum_{k=1}^{n} \phi_k^{(n)}(x) \equiv 1. \tag{10}$$

Let $q_k^{(n)}(x)$ be a polynomial for which

$$q_k^{(n)}(x_i^{(n)}) = \phi_k^{(n)}(x_i^{(n)}), \qquad i = 1, 2, \cdots, n, \tag{11}$$

$$\left| q_k^{(n)}(x) - \phi_k^{(n)}(x) \right| < 1/n^2, \qquad -1 \leqq x \leqq +1. \tag{12}$$

The existence of such a polynomial follows from a slight modification of the Weierstrass approximation theorem.[3] Equation (3) follows from (11), and from (12) and (10) we have

$$\begin{aligned} \left| \sum_{k=1}^{n} q_k^{(n)}(x) - 1 \right| &= \left| \sum_{k=1}^{n} (q_k^{(n)}(x) - \phi_k^{(n)}(x)) \right| \\ &\leqq \sum_{k=1}^{n} \left| q_k^{(n)}(x) - \phi_k^{(n)}(x) \right| \leqq \frac{1}{n} \cdot \end{aligned} \tag{13}$$

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[3] This modification is as follows: Let $f(x)$ be continuous in $-1 \leqq x \leqq +1$ and $x_1, x_2, \cdots, x_n$ given numbers in $-1 \leqq x \leqq +1$. Then for any $\epsilon > 0$ there exists a polynomial $P(x)$ so that $|P(x) - f(x)| < \epsilon$ and $P(x_i) = f(x_i)$, $i = 1, 2, \cdots, n$.

Let $x$ be a fixed number in the interval $-1 \leqq x \leqq +1$ and $\epsilon > 0$. Because of the continuity of the function $f(x)$ if $n$ is large enough and $|x - x_k^{(n)}| < \delta(\epsilon) = \delta$ we have

$$|f(x) - f(x_k^{(n)})| < \epsilon/2;$$

such $x_k^{(n)}$ exist if $n$ is large enough because the sequence is everywhere dense. Then we have from (13)

$$\begin{aligned}
|I_n(x;f) - f(x)| &= \left| \sum_{k=1}^{n} f(x_k^{(n)}) q_k^{(n)}(x) - f(x) \right| \\
&= \left| \sum_{k=1}^{n} (f(x_k^{(n)}) - f(x)) q_k^{(n)}(x) - f(x) + f(x) \sum_{k=1}^{n} q_k^{(n)}(x) \right| \\
&\leqq \sum_{k=1}^{n} | f(x) - f(x_k^{(n)}) | \, | q_k^{(n)}(x) | \\
&\quad + | f(x) | \left| \sum_{k=1}^{n} q_k^{(n)}(x) - 1 \right| \qquad (14) \\
&\leqq \sum_{|x - x_k^{(n)}| > \delta} | f(x) - f(x_k^{(n)}) | \, | q_k^{(n)}(x) | \\
&\quad + \frac{\epsilon}{2} \sum_{|x - x_k^{(n)}| < \delta} | q_k^{(n)}(x) | + | f(x) | / n \\
&\leqq 2M \sum_{|x - x_k| > \delta} | q_k^{(n)}(x) | + \frac{\epsilon}{2} (1 + 1/n) + M/n,
\end{aligned}$$

where $M = \max_{-1 \leqq x \leqq +1} |f(x)|$. For large $n$ and fixed $\delta$, $\phi_k^{(n)}(x) = 0$ if $|x - x_k^{(n)}| > \delta$, so it follows from (12) that $|q_k^{(n)}(x)| < 1/n^2$ if $|x - x_k^{(n)}| > \delta$, (14) gives for large $n$

$$| I_n(x; f) - f(x) | < 2M/n + \tfrac{1}{2}\epsilon(1 + 1/n) + M/n < \epsilon,$$

which was to be proved.

BUDAPEST, HUNGARY

# ON DERIVATIVES OF ORTHOGONAL POLYNOMIALS. II[1]

H. L. KRALL

It has been shown [1, 2, 7][2] that if $\{y_n(x)\}$ is a set of orthogonal polynomials whose derivatives $\{y_n'(x)\}$ also form an orthogonal set, then the original set $\{y_n(x)\}$ are either Jacobi, Laguerre, or Hermite polynomials. In the papers referred to, the ordinary definition of orthogonal polynomials was used, so of course the result was based on this definition. Since then, however, there has appeared [3, 5][3] a definition of generalized orthogonal polynomials[4] and it is our purpose here to find what G.O.P. sets have G.O.P. derivatives. Besides the classical polynomials, we shall find another class of polynomials which have this property. Of course, this new class will be G.O.P., but not ordinary orthogonal polynomials.

DEFINITION. *Given a set of constants* $\{c_n\}$ *subject to the condition*

$$\Delta_{n+1} = \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdots & \cdot \\ c_n & c_{n+1} & \cdots & c_{2n} \end{vmatrix} \neq 0, \qquad n = 0, 1, \cdots,$$

*the polynomials*

$$y_n(x) = \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdots & \cdot \\ c_{n-1} & c_n & \cdots & c_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}$$

*form a set of G.O.P.*

It has been shown [5, 4] that there exists a function of bounded variation $\psi(x)$ (hence infinitely many), such that

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[1] Presented to the Society, April 26, 1940, under the title *Orthogonal polynomial solutions of a certain fourth order differential equation.*

[2] The numbers in square brackets refer to the bibliography at the end.

[3] So far as this author knows, the idea of generalized orthogonal polynomials originated with I. M. Sheffer, who suggested its use in [3].

[4] Hereafter we shall use the abbreviation G.O.P. for generalized orthogonal polynomials.

$$c_n = \int_{-\infty}^{+\infty} x^n d\psi(x).$$

If the constants $\{c_n\}$ are real and $\Delta_n > 0$ for all values of $n$, this definition becomes the usual definition of orthogonal polynomials. In this case one of the $\psi(x)$ will be monotonically non-decreasing.

In this paper we follow the method of Hahn [1] who showed that if $\{y_n(x)\}$ and $\{y_n'(x)\}$ are sets of orthogonal polynomials, the set $\{y_n(x)\}$ satisfies a differential equation of the form[5]

$$(l_{22}x^2 + l_{21}x + l_{20})y_n''(x) + (l_{11}x + l_{10})y_n'(x) = \lambda_n y_n(x) \tag{1}$$

where the $\{l_{ij}\}$ are constants and $\lambda_n = l_{11}n + l_{22}n(n-1)$. Hahn divided his discussion of (1) according to the roots of the coefficient of $y_n''(x)$. Following his example and making suitable linear transformations, we find that there are four distinct cases of (1) to be considered:

$$\begin{aligned} x(1-x)y_n''(x) + \{\alpha - (\alpha+\beta)x\}y_n'(x) \qquad & \\ + n(\alpha+\beta+n-1)y_n(x) &= 0, \end{aligned} \tag{2}$$

$$xy_n''(x) + (\alpha - hx)y_n'(x) + hny_n(x) = 0, \tag{3}$$

$$y_n''(x) - 2xy_n'(x) + 2ny_n(x) = 0, \tag{4}$$

$$x^2y_n''(x) + (kx + l)y_n'(x) - n(k+n-1)y_n(x) = 0. \tag{5}$$

By using the ordinary definition, it readily follows that the only orthogonal polynomial solutions of these equations are the Jacobi polynomials (with $\alpha, \beta > 0$) from (2), the Laguerre polynomials (with $\alpha > 0$) from (3), and the Hermite polynomials from (4).

We now examine these differential equations for G.O.P. solutions. For this discussion, we need a result from [3]:

In order that there exists a set of G.O.P. as solutions of (1), it is necessary and sufficient that the "moments" $\{c_n\}$ satisfy the conditions

$$\begin{aligned} \{l_{11} + l_{22}(n-1)\}c_n + \{l_{10} + l_{21}(n-1)\}c_{n-1} + l_{20}(n-1)c_{n-2} &= 0, \\ n &= 1, 2, \cdots, \\ \Delta_{n+1} \neq 0, \qquad\qquad n &= 0, 1, 2, \cdots. \end{aligned} \tag{6}$$

**Discussion of** (2). Here, the recurrence relation (6) becomes $(n+\alpha+\beta-1)c_n - (n+\alpha-1)c_{n-1} = 0$, $n = 1, 2, \cdots$. From this we get

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[5] This part of Hahn's work applies to G.O.P. His notation has been changed to conform to [3].

$$c_n = \frac{\Gamma(\alpha+n)\Gamma(\alpha+\beta)c_0}{\Gamma(\alpha+\beta+n)\Gamma(\alpha)},$$

$$\Delta_{n+1} = \prod_{i=0}^{n} \frac{\Gamma(i+1)\Gamma(\alpha+i)\Gamma(\beta+i)\Gamma(\alpha+\beta)c_0}{\Gamma(\alpha+\beta+i+n)\Gamma(\alpha)\Gamma(\beta)}.$$

Accordingly, if $\alpha$, $\beta$, $\alpha+\beta \neq 0, -1, -2, \cdots$, the differential equation (3) has a set of G.O.P. as solutions. However, it does not seem to be easy to write out the weight function when either $\alpha$ or $\beta$ is less than zero. These polynomials have been discussed by Szegö [6].

**Discussion of** (3). In this case we have

$$hc_n - (n+\alpha-1)c_{n-1} = 0,$$

$$c_n = \frac{\Gamma(\alpha+n)c_0}{h^n\Gamma(\alpha)}, \qquad \Delta_{n+1} = \prod_{i=0}^{n} \frac{\Gamma(i+1)\Gamma(\alpha+i)c_0}{h^{2i}\Gamma(\alpha)}.$$

Accordingly, if $h \neq 0$, and $\alpha \neq 0, -1, -2, \cdots$, the differential equation (3) has a set of G.O.P. as solutions. These polynomials have also been discussed by Szegö [6].

**Discussion of** (4). There are no exceptional cases here. The solutions are Hermite polynomials.

**Discussion of** (5). In this case we have

$$(k+n-1)c_n + lc_{n-1} = 0,$$

$$c_n = \frac{l^n\Gamma(k)c_0}{\Gamma(k+n)}, \qquad \Delta_{n+1} = \prod_{i=0}^{n} \frac{(-l^2)^i\Gamma(k)\Gamma(i+1)c_0}{\Gamma(k+i+n)}.$$

Accordingly, if $l \neq 0$, and $k \neq 0, -1, -2, \cdots$, the differential equation (5) has a set of G.O.P. as solutions. For no value of $k$ will $\Delta_{n+1}$ be greater than zero for all values of $n$. These polynomials were discussed by Hahn [1] to the extent of showing that they are not orthogonal polynomials in the ordinary sense. They are not mentioned by any of the other authors.

Returning to (1) and differentiating, we get

$$(7) \quad (l_{22}x^2 + l_{21}x + l_{20})y_n'''(x) + \{(l_{11} + 2l_{22})x + l_{10} + l_{21}\}y_n''(x) = (\lambda_n - l_{11})y_n'(x).$$

Thus, if a set of polynomials $\{y_n(x)\}$ satisfies (1), the set of derivatives $\{y_n'(x)\}$ satisfies the equation

$$(8) \quad (l_{22}x^2 + l_{21}x + l_{20})\phi''(x) + \{(l_{11} + 2l_{22})x + l_{10} + l_{21}\}\phi'(x) = (\lambda_n - l_{11})\phi(x).$$

Since (7) is the same type as (1) and since (7) will not be an exceptional case if (1) is not, it follows that the only sets of G.O.P. whose derivatives are again G.O.P. are:

(i) the Jacobi polynomials with $\alpha$, $\beta$, $\alpha+\beta\neq 0, -1, -2, \cdots$,

(ii) the Laguerre polynomials with $h\neq 0$ and $\alpha\neq 0, -1, -2, \cdots$,

(iii) the Hermite polynomials,

(iv) the polynomial solutions of (5) with $l\neq 0$, and $k\neq 0, -1, -2, \cdots$.

## BIBLIOGRAPHY

1. W. Hahn, *Über die Jacobischen Polynome und zwei verwandte Polynomklassen*, Mathematische Zeitschrift, vol. 39 (1935), pp. 634–638.

2. H. L. Krall, *On derivatives of orthogonal polynomials*, this Bulletin, vol. 42 (1936), pp. 423–428.

3. ———, *Certain differential equations for Tchebycheff polynomials*, Duke Mathematical Journal, vol. 4 (1938), pp. 705–718.

4. G. Pólya, *Sur l'indétermination d'un problème voisin du problème des moments*, Comptes Rendus, de l'Académie des Sciences, Paris, vol. 207 (1938), pp. 708–711.

5. J. Shohat, *Sur les polynomes orthogonaux généralisés*, ibid., vol. 207 (1938), pp. 556–558.

6. G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, 1939.

7. M. S. Webster, *Orthogonal polynomials with orthogonal derivatives*, this Bulletin, vol. 44 (1938), pp. 880–887.

PENNSYLVANIA STATE COLLEGE

# NOTE ON PROBABILITY IMPLICATION

HANS REICHENBACH

In a recently published paper[1] J. C. C. McKinsey has pointed out some difficulties which arise from Axiom I of my theory of probability implication.[2] This axiom states the unambiguity of the degree $p$ of a given probability implication $(O \ni_p P)$ for the case that the class $O$ is not empty, a condition formulated by $(\bar{\bar{O}})$, but postulates ambiguity of $p$ in case of an empty class $O$, this condition being formulated by $(\bar{O})$. The latter ambiguity is necessary for probability implication because of the relation to Russell's material implication.[3] From the proof published by McKinsey we can infer that this ambiguity has to be restricted to values of $p$ between 0 and 1, limits included, in correspondence with the same restriction holding for the unambiguous degree $p$ of probability in cases of a non-empty class $O$, formulated by me in (8, §13).[4] That this general restriction is derivable from Axiom II, 2 is obvious as this axiom contains $O$ and $p$ as free variables and therefore states the restriction for all classes $O$ and all values $p$.

A further objection, which was already indicated in a footnote of McKinsey's paper, has been presented to me in a letter by the referee of this journal, Mr. S. C. Kleene. This objection shows that if the ambiguity of degrees of probability for empty classes $O$ is assumed, it can be proved that this ambiguity cannot be restricted to the limits 0 to 1.

This proof is connected with the theorem of addition (Axiom III) which reads[5]

III. $(O \ni_p P).(O \ni_q Q).(O.P \supset \bar{Q}) \supset (\exists r)(O \ni_r P \vee Q).(r=p+q)$.

The condition $r \leqq 1$ implies that $p+q \leqq 1$. If we demand $r \leqq 1$ only for non-empty classes $O$, the mentioned restriction for $p$ and $q$, which

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[1] This Bulletin, vol. 45 (1939), pp. 799–800.

[2] Published in *Wahrscheinlichkeitslehre*, Leiden, 1935, §§12–14. My further quotations refer to this book.

[3] Page 66.

[4] To avoid misunderstandings let me add here the remark that this relation is meant only for the case that the probability $W(O, P)$ *exists*, and would be written in the implicational writing

$$[(\exists x)\,(O \ni_x P)] \supset [(\exists y)\,(O \ni_y P).(0 \leqq y \leqq 1)].$$

If $O$ is not empty and therefore the probability has only one value, this means that this value is restricted to the limits 0 to 1, limits included.

[5] I write here the existential operator on the right-hand side because the abbreviation introduced on page 62, according to which the existential operator is omitted in the corresponding formula of my book, may be misleading.

is stronger than (8, §13), need not be stated in the implicans of the axiom because we then rather infer from III that the two conditions

$$(1) \qquad p + q > 1, \qquad (O.P \supset \bar{Q})$$

are incompatible. Following the principle that numerical values leading to contradictions are to be excluded, we can thus deduce all numerical restrictions for the values of probabilities; in (15, §19) I have developed a formula which states these restrictions for all possible cases. The incompatibility of the two formulae (1) for non-empty classes $O$ is included in (15, §19) because this relation shows that if $p+q>1$ we have $W(O.P, Q)>0$, which in consideration of (9, §13) contradicts $(O.P\supset\bar{Q})$ in case $O$ is not empty. However in applying the explained principle we have to be sure that the excluded numerical values cannot be introduced by means of other ways of deduction. It turns out that this is the case for an empty class $O$. In this case, that is for $(\bar{O})$, we can for instance assume in III the numerical values $p=1$ and $q=1$ whereas the condition $(O.P\supset\bar{Q})$ is also satisfied; the latter follows from the properties of material implication according to which a false proposition implies every proposition. It follows that in this case $r$ can be greater than 1.

These difficulties are eliminated if we introduce in Axiom II,2 the condition that $O$ is not empty, and write this axiom:

II,2. $\overline{(\bar{O})}.(O\ni_p P)\supset(p\geqq 0)$.

With the qualification $\overline{(\bar{O})}$ in the implicans we renounce the unnecessary extension of numerical restrictions to probability values referring to empty classes. It is without danger to assume that in case $O$ is empty the probability $p$ of $(O\ni_p P)$ can be greater than 1 or smaller than 0, and we shall make use of this liberty if it helps us to escape contradictions. The range of the variables expressed by small letters "$p$," "$q$," and so on, extends therefore through all real numbers, the necessary restrictions for non-empty classes $O$ being expressed by the Axiom II,2.

As far as I see we need not introduce any further qualifications within the system of probability implication. I should like however to add some remarks concerning the use of the functor $W(\ )$. A functor is usually conceived as being unambiguous; thus $W(O, P)$ would mean "*the* probability from $O$ to $P$." Our symbol has this meaning only in case $O$ is not empty. To include the case of $O$ being empty we have to translate every formula of the functor-writing into an existence statement such as explained on pages 61–62, or in the examples on pages 69–70 and page 73. The formula then means, in accordance with the rule of existence: if $n-1$ of the $n$ probabilities of the formula exist and have among their values the values $p, q, \cdots$, then also the $n$th probability exists and has among its values a value $w$ such that the $p, q, \cdots, w$ satisfy the equation expressed in the formula. It is obvious that for non-empty classes $O$ this interpretation is identical with the meaning of unambiguous functors. For the case of empty classes $O$ however it also will lead to consistent interpretations although of course in this case the formulae actually state nothing, as all numerical values of the probabilities then are possible. It follows that we need not add the condition $\overline{(\overline{O})}$ to any of the formulae of the functor-writing.

For all practical purposes it is convenient to interpret the functor formulae as referring to a non-empty class $O$, and then to interpret the functor in the usual way as meaning "*the* probability from . . . to . . . ." Here the term $O$ is defined as that term which occurs in the first place of every probability expression of a formula. It is interesting that this interpretation can also be carried through if some of the other classes are empty, even if they appear in the first place of the probability functor; it turns out that each formula can be written in a form in which the indeterminate probability values are multiplied by 0 such as in (3), p. 73, if we assume $P$ to be empty.

I am much indebted to Mr. J. C. C. McKinsey and to Mr. S. C. Kleene for having pointed out the necessity of the correction of my axioms of probability implication with respect to empty classes $O$.

University of California at Los Angeles

# ON THE ANALOGUE FOR DIFFERENTIAL EQUATIONS OF THE HILBERT-NETTO THEOREM

RICHARD COHN

If

$$F_1, \cdots, F_r \tag{1}$$

is a finite system of differential polynomials in the unknowns $y_1, \cdots, y_n$, and if $G$ is a differential polynomial which is annulled by every solution of the system (1), some power $G^p$ of $G$ is a linear combination of the $F_i$ and their derivatives of various orders, with differential polynomials for coefficients. This analogue of the Hilbert-Netto theorem was proved by J. F. Ritt[1] for forms with meromorphic coefficients, and by H. W. Raudenbush[2] for the case of coefficients belonging to an abstract differential field. In these proofs it is shown that the denial of the existence of the exponent $p$, above, of $G$ leads to a contradiction; no constructive method for obtaining admissible values of $p$ is given. The object of the present note is to present a new proof of the analogue, for the case of meromorphic coefficients, which is entirely constructive and produces a definite $G^p$ as described above.

Our proof will be based on the considerations in Chapters V and VII of A.D.E. In Chapter VII, the problem of obtaining $G$ is reduced to the problem of determining unity as a linear combination of the $F_i$ in (1) and their derivatives, in the case in which (1) has no solutions. In Chapter V it is shown how to decide, in a finite number of steps, whether or not (1) has solutions. Our problem thus assumes the following form: *Given that* (1) *has no solutions, it is required to express unity as a linear combination of the* $F_i$ *and their derivatives.*

We assume that (1) has no solutions and proceed to examine the algorithm developed in §§65–67 of A.D.E. Adjoining to (1) a finite number of linear combinations of the $F_i$ and their derivatives, we obtain a system $\Sigma$, devoid of solutions, with a basic set

$$A_1, \cdots, A_q \tag{2}$$

which has the property that the remainder of every form in $\Sigma$ with respect to (2) is zero. If (2) consists of a single form $A$ which is an element of $\mathcal{J}$, the coefficient field of our forms, we secure immediately a representation of the type desired for unity. Let us suppose that this is not the case. Then (2), considered as a set of simple forms, cannot be a basic set of a prime system; if it were, (1) would possess solutions (A.D.E., §65). Thus there must exist, for some $j \leqq q$, an identity

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[1] Ritt, J. F., *Differential Equations from the Algebraic Standpoint*, chap. 7, referred to below as A.D.E. American Mathematical Society Colloquium Publications, vol. 14, 1932.

[2] Raudenbush, H. W., *Ideal theory and algebraic differential equations*, Transactions of this Society, vol. 36 (1934), pp. 361–368.

$$(3) \qquad J_1^{\mu_1}\cdots J_{j-1}^{\mu_{j-1}}(SA_j - H_1H_2) - L_1A_1 - \cdots - L_{j-1}A_{j-1} = 0,$$

where $J_i$ is the initial of $A_i$, $i=1, \cdots, j-1$; and where $H_1$ and $H_2$ are reduced with respect to $A_1, \cdots, A_j$. Let $\Lambda_1^{(k)}$, $k=1, \cdots, j+1$, represent the systems $\Sigma+J_1, \cdots, \Sigma+J_{j-1}, \Sigma+H_1, \Sigma+H_2$, respectively. We treat each $\Lambda_1^{(k)}$ as (1) was treated. The adjunction of a finite number of forms to any $\Lambda_1^{(k)}$ produces a system $\Sigma_1^{(k)}$, with no solutions, and with basic sets lower than (2) which furnish zero remainders for the forms in $\Sigma_1^{(k)}$.

Let us suppose that each $\Sigma_1^{(k)}$ contains an element of $\mathcal{J}$ different from zero. We see on examining these systems that there exist relations, procurable by constructive methods,

$$(4) \qquad 1 = P + M_0H_1 + M_1H_1' + \cdots,$$

$$(5) \qquad 1 = Q + N_0H_2 + N_1H_2' + \cdots,$$

$$(6) \qquad 1 = R_i + S_{i0}J_i + S_{i1}J_i' + \cdots, \qquad i = 1, \cdots, j-1,$$

accents indicating differentiation, where $P$, $Q$, and the $R_i$ are linear in the $F_i$ and their derivatives.

We equate to unity the product of the right-hand members of (4), (5), and the equations (6). If both sides of the resulting equation are raised to a sufficiently high power, determinable in advance, we secure, as Raudenbush has shown, a relation of the type

$$1 = L + T_0V + T_1V' + \cdots,$$

accents indicating differentiation, where $L$ is linear in the $F_i$ and their derivatives, and $V = J_1^{\mu_1}\cdots J_{j-1}^{\mu_{j-1}}H_1H_2$.[3] From (3) we see that $V$ can be obtained as a linear expression in the $F_i$ and their derivatives. We thus have such an expression as we are seeking for unity.

If, on the other hand, some system $\Sigma_1^{(j)}$ does not contain a nonzero element in $\mathcal{J}$, we apply to it the entire process applied to $\Sigma$. We form in this way systems with basic sets lower than those of $\Sigma_1^{(j)}$. The systems thus formed for the various $\Sigma_1^{(k)}$ receiving our present treatment will be called, with no attempt to describe their complete history, systems $\Sigma_2$. In each $\Sigma_2$ a basic set yields only zero remainders.

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[3] Ritt, J. F., *Algebraic aspects of the theory of differential equations*, Semicentennial Publications of the American Mathematical Society, vol. 2, p. 44.

Let us suppose that each $\Sigma_2$ contains a nonzero element in $\mathcal{J}$. What precedes shows that, for each $\Sigma_1^{(j)}$ as above, unity is linear in the forms of $\Sigma_1^{(j)}$ and their derivatives; this, again, gives the expression which we are seeking for unity.

If there are $\Sigma_2$ which contain no nonzero element in $\mathcal{J}$, we give them the treatment which is now familiar. By §67 of A.D.E., we know that our process can continue for only a finite number of steps, so that the possibility of determining for unity an expression of the type desired is established.

COLUMBIA UNIVERSITY

# ON A CONVERGENCE THEOREM FOR THE LAGRANGE INTERPOLATION POLYNOMIALS

G. GRÜNWALD

The unique polynomial of degree $(n-1)$ assuming the values $f(x_1), \cdots, f(x_n)$ at the abscissas $x_1, x_2, \cdots, x_n$, respectively, is given by the Lagrange interpolation formula

$$L_n(f) = f(x_1)l_1(x) + f(x_2)l_2(x) + \cdots + f(x_n)l_n(x). \tag{1}$$

Here

$$l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x - x_k)}, \qquad k = 1, 2, \cdots, n \tag{2}$$

(fundamental polynomials of the Lagrange interpolation), and the polynomial $\omega(x)$ is defined by

$$\omega(x) = c(x - x_1)(x - x_2) \cdots (x - x_n), \tag{3}$$

where $c$ denotes an arbitrary constant not equal to zero. It is known and easy to verify that

$$l_1(x) + l_2(x) + \cdots + l_n(x) \equiv 1. \tag{4}$$

In the Lagrange interpolation formula let

$$x_k = x_k^{(n)} = \cos (2k - 1)\pi/2n = \cos \theta_k^{(n)} \tag{5}$$

which implies

$$\omega(x) = T_n(x) = \cos (n \arccos x) = \cos n\theta, \qquad \cos \theta = x \tag{6}$$

(Tchebyscheff polynomial). In this case we have

$$l_k(x) = l_k[\theta] = (-1)^{k+1} \frac{\cos n\theta \sin \theta_k^{(n)}}{n(\cos \theta - \cos \theta_k^{(n)})}, \tag{7}$$

$$k = 1, 2, \cdots, n; \; x = \cos \theta;$$

and

$$L_n(f) = L_n[f; \theta] = \sum_{k=1}^{n} f(\cos \theta_k^{(n)})(-1)^{k+1} \frac{\cos n\theta \sin \theta_k^{(n)}}{n(\cos \theta - \cos \theta_k^{(n)})}, \tag{8}$$

$$x = \cos \theta.$$

Suppose $f(x)$ to be a continuous function; then it is known that the sequence $L_n(f)$, $n=1, 2, \cdots$, is not convergent[1] for all $f(x)$. We may even find a continuous function $f_1(x)$ such that the sequence $L_n(f_1)$, $n=1, 2, \cdots$, is divergent for all points of the interval $-1 \leqq x \leqq +1$.[2]

Therefore it is interesting to prove the following theorem:

THEOREM. *Let* $f(x)$ *be a continuous function in the interval* $-1 \leqq x \leqq +1$; *then*

$$\lim_{n\to\infty} \tfrac{1}{2}\{L_n[f; \theta - \pi/2n] + L_n[f; \theta + \pi/2n]\} = f(x), \qquad x = \cos\theta, \tag{9}$$

*and the convergence is uniform in the whole interval* $-1 \leqq x \leqq +1$.

Between the interpolation polynomials (8) and the partial sums $s_{n-1}(f)$ of the Fourier series of the even function $f(x)$ there is a far reaching analogy. We mention here only the following. On one hand, it is easy to verify that

$$L_n(f) = c_0 + c_1 \cos\theta + \cdots + c_{n-1} \cos(n-1)\theta, \qquad x = \cos\theta, \tag{10}$$

where

$$c_0 = \frac{1}{n}\sum_{k=1}^{n} f(\cos\theta_k^{(n)}), \qquad c_r = \frac{2}{n}\sum_{k=1}^{n} f(\cos\theta_k^{(n)}) \cos r\theta_k^{(n)}, \qquad r = 1, 2, \cdots, n-1. \tag{11}$$

On the other hand,

$$s_{n-1}(f) = a_0 + a_1 \cos\theta + \cdots + a_{n-1} \cos(n-1)\theta, \tag{12}$$

where

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(\cos\theta)d\theta, \qquad a_r = \frac{2}{\pi}\int_0^{\pi} f(\cos\theta)\cos r\theta\, d\theta, \qquad r = 1, 2, \cdots, n-1. \tag{13}$$

Our theorem is analogous with the well known theorem of Rogosinski in the theory of Fourier series.

We first prove the following lemma.

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[1] G. Faber, *Über die interpolatorische Darstellung stetiger Funktionen*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 190–210. S. Bernstein, *Sur la limitation des valeurs* $\cdots$, Bulletin de l'Académie des Sciences de l'URSS, 1931, pp. 1025–1050.

[2] G. Grünwald, *Über Divergenzerscheinungen der Lagrangeschen Interpolationspolynome stetiger Funktionen*, Annals of Mathematics, (2), vol. 37 (1936), pp. 908–918. See also J. Marcinkiewicz, *Sur la divergence des polynomes d'interpolation*, Acta Litterarum ac Scientiarum Regiae Universitatis Hungaricae Francisco-Iosephinae, vol. 8 (1937), pp. 131–135.

LEMMA.

$$\frac{1}{2}\sum_{k=1}^{n}\left| l_k[\theta - \pi/2n] + l_k[\theta + \pi/2n]\right| < c_1,$$

*where* $c_1 > 0$ *is an absolute constant.*

From (7) we have, for $\theta \neq \theta_k^{(n)} \pm \pi/2n$,

$$\begin{aligned}
&\tfrac{1}{2}(l_k[\theta - \pi/2n] + l_k[\theta + \pi/2n]) \\
&= \tfrac{1}{2}(-1)^{k+1}\left(\frac{\cos n(\theta - \pi/2n)\sin\theta_k^{(n)}}{n(\cos(\theta - \pi/2n) - \cos\theta_k^{(n)})}\right. \\
&\qquad\qquad \left. + \frac{\cos n(\theta + \pi/2n)\sin\theta_k^{(n)}}{n(\cos(\theta + \pi/2n) - \cos\theta_k^{(n)})}\right) \\
&= \tfrac{1}{2}(-1)^{k+1}\frac{\sin\theta_k^{(n)}\sin n\theta\sin\theta\sin\pi/2n}{4n\sin\frac{1}{2}(\theta + \theta_k^{(n)} + \pi/2n)\sin\frac{1}{2}(\theta - \theta_k^{(n)} + \pi/2n)} \\
&\quad\cdot\frac{1}{\sin\frac{1}{2}(\theta + \theta_k^{(n)} - \pi/2n)\sin\frac{1}{2}(\theta - \theta_k^{(n)} - \pi/2n)}
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&\left|\tfrac{1}{2}(l_k[\theta - \pi/2n] + l_k[\theta + \pi/2n])\right| \\
&\leqq \frac{1}{2n}\left|\frac{\sin\pi/2n}{\sin\frac{1}{2}(\theta - \theta_k^{(n)} + \pi/2n)\sin\frac{1}{2}(\theta - \theta_k^{(n)} - \pi/2n)}\right| \\
&\leqq \frac{1}{2n}\,\frac{\pi/2n}{(2/\pi)\left|\frac{1}{2}(\theta - \theta_k^{(n)} + \pi/2n)\right|(2/\pi)\left|\frac{1}{2}(\theta - \theta_k^{(n)} - \pi/2n)\right|} \\
&\leqq \frac{\pi^3}{4n^2}\,\frac{1}{(\theta - \theta_k^{(n)} - \pi/2n)^2}.
\end{aligned} \tag{15}$$

Now let $\theta$ be fixed and

$$1 < x_1 < x_2 < \cdots < x_j < x = \cos\theta < x_{j+1} < \cdots < x_n < -1. \tag{16}$$

It is known that[3]

$$|l_k(x)| < 4/\pi, \quad k = 1, 2, \cdots, n;\ n = 1, 2, \cdots;\ -1 \leqq x \leqq +1.$$

Thus

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[3] P. Erdös and G. Grünwald, *Note on an elementary problem of interpolation*, this Bulletin, vol. 44 (1938), pp. 515–578. This bound is the best possible; Fejér proved earlier $|l_n(x)| < 2^{1/2}$. See L. Fejér, *Lagrangesche Interpolation und die zugehörigen konjugierten Punkte*, Mathematische Annalen, vol. 106 (1932), pp. 1–55.

$$\begin{aligned}
&\frac{1}{2}\sum_{k=1}^{n}\left| l_k[\theta - \pi/2n] + l_k[\theta + \pi/2n]\right| \\
&\quad\le \frac{16}{\pi} + \frac{1}{2}\sum_{1\le k\le n,\, k\ne j-2,\, j-1,\, j,\, j+1}\left| l_k[\theta - \pi/2n] + l_k[\theta + \pi 2n]\right| \\
(17)\qquad &\quad\le \frac{16}{\pi} + \sum_{1\le k\le n,\, k\ne j-2,\, j-1,\, j,\, j+1} \frac{\pi^3}{4n_2}\,\frac{1}{(\theta - \theta_k^{(n)} - \pi/2n)^2} \\
&\quad\le \frac{16}{\pi} + \frac{\pi^3}{4n^2}\sum_{k=1}^{\infty}\left(\frac{2n}{\pi}\right)^2\frac{1}{k^2} = c_1.
\end{aligned}$$

If $\theta = 0, \pi$, the same inequality evidently holds.

From (17) we obtain for sufficiently large $n$, $\delta > 0$ fixed,

$$(18)\qquad \sum_{1\le k\le n,\, |\theta-\theta_k^{(n)}|>\delta} \tfrac{1}{2}\left| l_k[\theta - \pi/2n] + l_k[\theta + \pi/2n]\right| = O(1/n).$$

Now we are in the position to prove our theorem. Let $\epsilon > 0$ be a fixed number. The identity (4) gives

$$(19)\qquad \frac{1}{2}\sum_{k=1}^{n}\left(l_k[\theta - \pi/2n] + l_k[\theta + \pi/2n]\right) = 1;$$

hence for a fixed $x = \cos\theta$

$$\begin{aligned}
&\tfrac{1}{2}\left\{L_n[f;\theta - \pi/2n] + L_n[f;\theta + \pi/2n]\right\} - f(\cos\theta) \\
(20)\qquad &\quad = \sum_{k=1}^{n}\tfrac{1}{2}\left(f(\cos\theta_k^{(n)}) - f(\cos\theta)\right)\left(l_k[\theta + \pi/2n] + l_k[\theta + \pi/2n]\right).
\end{aligned}$$

The function $f(\cos\theta)$ is continuous; thus we can find a positive number $\delta$ such that

$$(21)\qquad \left| f(\cos\theta) - f(\cos\theta_k^{(n)})\right| < \epsilon/2c_1$$

whenever $|\theta - \theta_k^{(n)}| < \delta$. From (20) and (21) we have

$$\begin{aligned}
\Delta &= \left| \tfrac{1}{2}\left\{L_n[f;\theta-\pi/2n] + L_n[f;\theta+\pi/2n]\right\} - f(\cos\theta)\right| \\
&= \Bigg| \sum_{1\le k\le n,\, |\theta-\theta_k^{(n)}|<\delta} \tfrac{1}{2}\left(f(\cos\theta_k^{(n)}) - f(\cos\theta)\right)\left(l_k[\theta-\pi/2n] + l_k[\theta+\pi/2n]\right) \\
&\quad + \sum_{1\le k\le n,\, |\theta-\theta_k^{(n)}|>\delta} \tfrac{1}{2}\left(f(\cos\theta_k^{(n)}) - f(\cos\theta)\right)\left(l_k[\theta-\pi/2n] + l_k[\theta+\pi/2n]\right)\Bigg| \\
(22)\qquad &\le \frac{\epsilon}{2c_1}\sum_{1\le k\le n,\, |\theta-\theta_k^{(n)}|<\delta} \tfrac{1}{2}\left| l_k[\theta-\pi/2n] + l_k[\theta+\pi/2n]\right| \\
&\quad + \sum_{1\le k\le n,\, |\theta-\theta_k^{(n)}|>\delta} \tfrac{1}{2}\left| f(\cos\theta_k^{(n)}) - f(\cos\theta)\right| \left| l_k[\theta-\pi/2n] + l_k[\theta+\pi/2n]\right|
\end{aligned}$$

and by the lemma and (18) for sufficiently large $n$

$$\Delta \leqq \frac{\epsilon}{2c_1} c_1 + \sum_{1 \leqq k \leqq n, |\theta - \theta_k^{(n)}| > \delta} \tfrac{1}{2} 2M \, | \, l_k[\theta - \pi/2n] + l_k[\theta + \pi/2n] \, |$$
$$< \epsilon/2 + MO(1/n) < \epsilon,$$

where $M = \max_{-1 \leqq x \leqq +1} |f(x)|$, and this proves our theorem.

BUDAPEST, HUNGARY

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# DISCONTINUOUS CONVEX SOLUTIONS OF DIFFERENCE EQUATIONS[1]

FRITZ JOHN

This paper contains some conditions for continuity of convex solutions of a difference equation.

A function $f(x)$ defined for $a \leqq x \leqq b$ is *convex*, if

$$\left(\frac{x+y}{2}\right) \leqq \frac{f(x)+f(y)}{2}. \tag{1}$$

If $f(x)$ is convex and bounded from above in $a \leqq x \leqq b$, then $f(x)$ is continuous (see Bernstein [1, p. 422]).[2] If $f(x)$ is convex in $a \leqq x \leqq b$ and $y$ a fixed number with $a < y < b$, let the function $\phi_y(x)$ be defined by

$$\phi_y(x) = \lim_{\alpha \to x - y} f(y + \alpha),$$

where $\alpha$ assumes *rational* values only. Then $\phi_y(x)$ is uniquely defined, continuous, and convex for $a < x < b$ (F. Bernstein [1, p. 431, Theorem 7]); moreover $\phi_y(x) = f(x)$ for rational $y - x$.

THEOREM 1. *If there exists at most one continuous convex solution of the difference equation*

$$F(x, f(x), f(x+1), \cdots, f(x+n)) = g(x), \qquad x > 0, \tag{2}$$

*where $F$ and $g$ are continuous functions of their arguments, then there exist no discontinuous convex solutions.*

PROOF. If $f(x)$ is a convex solution, then, for $x - y$ rational,

$$F(x, \phi_y(x), \phi_y(x+1), \cdots, \phi_y(x+n)) = g(x);$$

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[1] Presented to the Society, September 12, 1940.

[2] The numbers in brackets refer to the bibliography.

as both members of this equation are continuous functions of $x$, it holds for *all* $x>0$. As there is at most one continuous convex solution, we have

$$\phi_y(x) = \phi_z(x)$$

for all positive $z$, $y$, $x$. As $\phi_z(z)=f(z)$, we see that $f(z)$ is identical with the continuous function $\phi_y(z)$ for all positive $z$.

THEOREM 2. *If the difference equation* (2) *has at most one solution, which is monotone for sufficiently large $x$, then* (2) *has at most one convex solution, and that solution will be continuous.*

PROOF. Every continuous convex solution is monotone for sufficiently large $x$. Apply Theorem 1.

THEOREM 3. *A difference equation of the form*

$$\prod_{k=0}^{n} (f(x+k))^{a_k} = g(x), \qquad x>0, \tag{3}$$

(*$a_k$ real constants*) *has at most one convex solution, if*

(1) *all roots of the equation*

$$\sum_{k=0}^{n} a_k x^k = 0$$

*are simple and of absolute value* 1,

$$a_n > 0, \qquad \sum_{k=0}^{n} a_k \neq 0,$$

(2) $g(x)\neq 0$ *and continuous*,
(3) $\lim_{x\to\infty} (\log |g(x)|)/x=0$,
(4) $\lim_{x\to\infty} (\log |g(x)|)/\log x \neq \sum_k k a_k$.

PROOF. Assumption (3) implies

$$\prod_{k=0}^{n} | f(x+k) |^{a_k} = | g(x) |.$$

From assumption (1) above and the lemma proved below, it follows that there are *non-negative* constants $b_l$ and $c_r$, such that

$$\prod_{l} | g(x+l) |^{b_l} = \prod_{l} \left[ \prod_{k} | f(x+k+l) |^{a_k} \right]^{b_l}$$
$$= \prod_{r} | f(x+r) |^{c_r}.$$

For every continuous convex solution $f(x)$ of (3), $|f(x)|$ is either monotone non-decreasing or monotone decreasing for sufficiently large $x$. Which of these alternatives takes place is determined by $g(x)$, as in the first case

$$\prod_{l} |g(x+l)|^{b_l}$$

is monotone non-decreasing, and in the second case monotone decreasing for sufficiently large $x$. If $|f(x)|$ is monotone non-decreasing, we have in $F(x) = \log |f(x)|$ a non-decreasing solution of the equation

$$\sum_{k} a_k F(x+k) = \log |g(x)|.$$

It follows from our assumptions that such a solution $F(x)$ is uniquely determined for all sufficiently large $x$, and hence for all $x$ (John [3, p. 183]). If $|f(x)|$ is monotone decreasing for sufficiently large $x$, then $F(x) = -\log |f(x)|$ is an increasing solution of

$$\sum_{k} a_k F(x+k) = -\log |g(x)|,$$

and hence uniquely determined. Thus for any continuous convex solution $f(x)$ of (3), $|f(x)|$ is uniquely determined. Then $f(x)$ is uniquely determined as well, unless $f(x)$ is linear for sufficiently large $x$; but if $f(x)$ is linear for large $x$,

$$\lim_{x\to\infty} \frac{\log |g(x)|}{\log x} = \lim_{x\to\infty} \sum_{k} a_k \frac{\log |f(x+k)|}{\log x} = \sum_{k} a_k$$

contrary to assumption. Thus there exists at most one continuous convex solution, and hence at most one convex solution.

*Example.* The equation

$$f(x+1)\cdot f(x) = x^{-p}, \qquad x > 0,\ p > 0,$$

satisfies the assumptions of Theorem 3 and hence has at most one convex solution (proved by A. E. Meyer [4] for $p=1$, for general $p$ by H. P. Thielman [5]). The convex solution is found to be

$$B(\tfrac{1}{2}x, \tfrac{1}{2})^p/(2\pi)^{p/2}.$$

LEMMA. *If* $\phi(x) = \sum_{k=0}^{n} a_k x^k$ *is a polynomial, such that*

(1) $a_n > 0$,

(2) $\phi(x)$ *has no positive real roots,*

*then there are polynomials* $\psi(x)$ *and* $\sigma(x)$ *with non-negative coefficients, such that*

$$\phi(x)\cdot\psi(x) = \sigma(x).$$

PROOF. $\phi(x)$ can be factored in the form

$$\phi(x) = a_n \prod_l (x + \alpha_l) \prod_k (x^2 + 2\beta_k x + \gamma_k)$$

where $\alpha_l \geqq 0$, $\gamma_k > \beta_k^2$. Hence it is sufficient to prove the lemma for the case that

$$\phi(x) = x^2 + 2\beta x + \gamma$$

and $\beta^2 < \gamma$. We define $\theta$ with $0 < \theta < \pi$ by

$$\cos\theta = -\beta/\gamma^{1/2}.$$

Let the non-negative *integer* $s$ be determined by

$$\frac{\pi}{s+2} \leqq \theta < \frac{\pi}{s+1}\cdot$$

Put

$$\psi(x) = \frac{(\gamma^{1/2})^{s+2}\sin(s+1)\theta - x(\gamma^{1/2})^{s+1}\sin(s+2)\theta + x^{s+2}\sin\theta}{[x^2 + 2\beta x + \gamma]\cdot\sin\theta}$$

$$= \sum_{k=0}^{s} (\gamma^{1/2})^k \frac{\sin(k+1)\theta}{\sin\theta} x^{s-k}.$$

$\psi(x)$ and $(x^2+2\beta x+\gamma)\cdot\psi(x)$ obviously are polynomials with non-negative coefficients.

THEOREM 4. *The difference equation*

$$\sum_{k=0}^{n} a_k f(x+k) = g(x), \qquad x > 0, \tag{4}$$

*has no discontinuous convex solutions, if*

(a) *$g(x)$ is bounded from above in every positive interval,*

(b) $a_n > 0$,

(c) *the equation $\sum_{k=0}^{n} a_k x^k = 0$ has no positive real roots.*

PROOF. Let $\phi(x) = \sum_{k=0}^{n} a_k x^k$. Let $\psi(x)$ be a polynomial, such that $\psi(x)$ and $\phi(x)\cdot\psi(x)$ have no negative coefficients. Let

$$\phi(x)\cdot\psi(x) = x^s\cdot\sigma(x),$$

where $\sigma(x)$ is a polynomial of degree $m$ with $\sigma(0) \neq 0$. Then

$$\sigma(x)\cdot x^m \sigma\left(\frac{1}{x}\right) = \sum_{k=0}^{2m} c_k x^k$$

is a polynomial of degree $2m$ with $c_k = c_{2m-k} \geqq 0$. Put

$$\tau(x) = \psi(x) \cdot x^m \sigma\left(\frac{1}{x}\right) = \sum_r b_r x^r.$$

Then

$$\phi(x) \cdot \tau(x) = x^s \sum_{k=0}^{2m} c_k x^k,$$

where $\tau(x)$ is a polynomial with non-negative coefficients $b_r$ and $c_k = c_{2m-k} \geqq 0$. For a convex solution $f(x)$ of (4)

$$\begin{aligned} \sum_r b_r g(x+r) &= \sum_{k=0}^{2m} c_k f(x+k+s) \\ &= \frac{1}{2} \sum_{k=0}^{2m} c_k [f(x+k+s) + f(x+2m-k+s)] \\ &\geqq \frac{1}{2} \sum_{k=0}^{2m} c_k f(x+s+m), \end{aligned}$$

or

$$f(x+s+m) \leqq \frac{2\sum_r b_r g(x+r)}{\sum_k c_k} \cdot$$

As $g(x)$ is bounded above, it follows that $f(x)$ is bounded above, and hence continuous.

THEOREM 5. *If the difference equation*

$$\sum_{0=k}^{n} a_k f(x+k) = g(x) \tag{5}$$

*has a continuous convex solution, and if the equation* $\sum_{k=0}^{n} a_k x^k = 0$ *has a positive real root, then the difference equation has discontinuous convex solutions as well.*

PROOF. It is sufficient to prove that the equation

$$\sum_{k=0}^{n} a_k f(x+k) = 0$$

has a discontinuous convex solution, as the sum of two convex functions is again convex. Let

$$\sum_{k=0}^{n} a_k x^k = (x - \lambda) \sum_{k=0}^{n-1} b_k x^k,$$

where $\lambda > 0$. Then

$$\sum_k a_k f(x+k) = \sum_k b_k[f(x+k+1) - \lambda f(x+k)].$$

It is sufficient to show that the equation

$$f(x+1) - \lambda f(x) = 0 \tag{6}$$

has a discontinuous convex solution.

Let $\Sigma$ be a basis for all real numbers; that is, every real $x$ may be represented in one and only one way in the form

$$x = \alpha a + \beta b + \cdots + \gamma c,$$

where $a, b, \cdots, c$ are in $\Sigma$, and $\alpha, \beta, \cdots, \gamma$ are rational numbers. (The existence of such a base is proved by Hamel [2].) Without restriction of generality we may assume 1 to be an element of $\Sigma$ (this comes back to assuming a normal ordering of the set of real numbers with 1 as first element). For every real $x$, there is then uniquely determined a number $\alpha$, such that

$$x = \alpha\cdot 1 + \beta b + \cdots + \gamma c,$$

where $\alpha, \beta, \cdots, \gamma$ are rational, and $1, b, \cdots, c$ are in $\Sigma$. If $\lambda \neq 1$, define $f(x)$ by $\lambda^\alpha$. Then

$$f(x+1) - \lambda f(x) = \lambda^{\alpha+1} - \lambda\cdot\lambda^\alpha = 0.$$

If $y = \alpha' 1 + \beta' b + \cdots + \gamma' c$ is the representation of $y$ then

$$f\left(\frac{x+y}{2}\right) = (\lambda^\alpha\cdot\lambda^{\alpha'})^{1/2} \leqq \frac{\lambda^\alpha + \lambda^{\alpha'}}{2} = \frac{f(x)+f(y)}{2}.$$

Hence $f(x)$ is a convex solution of (6). $f(x)$ is discontinuous in every interval, as for an element $b \neq 1$ of $\Sigma$

$$f(\beta b) = 1$$

for all rational $\beta$, whereas

$$f(\alpha) = \lambda^\alpha \neq 1$$

for all rational $\alpha \neq 0$. Similarly, if $\lambda = 1$, we define $f(x) = \beta$, where $\beta$ is the coefficient of the fixed element $b \neq 1$ of $\Sigma$. Then

$$f(x+1) - f(x) \equiv 0, \qquad f\left(\frac{x+y}{2}\right) \equiv \frac{f(x)+f(y)}{2},$$

$f(\alpha) = 0$ for all rational $\alpha$, but $f(\beta b) = \beta \neq 0$ for all rational $\beta \neq 0$.

Note added March 15: I am indebted to Professors O. Szász and G. Szegö for the information that the Lemma had been proved by E. Meissner, Mathematische Annalen, vol. 70 (1911), pp. 223–235. See also Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 2, p. 730, no. 190.

## BIBLIOGRAPHY

1. F. Bernstein, *Über das Gauss'sche Fehlergesetz*, Mathematische Annalen, vol. 64 (1907), pp. 417–447.

2. G. Hamel, *Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung* $f(x+y)=f(x)+f(y)$, Mathematische Annalen, vol. 60 (1905), pp. 459–462.

3. F. John, *Special solutions of certain difference equations*, Acta Mathematica, vol. 71 (1939), pp. 175–189.

4. A. E. Meyer, *Konvexe Lösung der Funktionalgleichung* $1/f(x+1)=xf(x)$, Acta Mathematica, vol. 70 (1939), pp. 57–62.

5. H. P. Thielman, *On the convex solution of a certain functional equation*, this Bulletin, vol. 47 (1941), pp. 118–120.

UNIVERSITY OF KENTUCKY

# MOMENT PROBLEM FOR A BOUNDED REGION[1]

L. B. HEDGE

1. **Introduction.** In this paper a solution of the moment problem given by Hausdorff[2] for a bounded interval is extended to any bounded region in euclidean $n$-space, under certain conditions on polynomial expansions over the region. The resulting solution is valid for the $n$-dimensional sphere, and includes the Hausdorff case as well as the known conditions on the "class" of Fourier and Fourier-Stieltjes series.[3]

2. **Definitions and notation.** Let $n$ be a positive integer, fixed but arbitrary. $R^n$ will denote the euclidean $n$-space, $(x)$ and $(y)$ will stand for $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$, points of $R^n$, and $E$ a bounded, closed subset of $R^n$. $\nu, \tau, i, j, k$, and $s$, will be used for non-negative integers, and $(k)$, $(s)$, and so on, will denote ordered $n$-tuples of non-negative integers $(k_1, k_2, \cdots, k_n)$, $(s_1, s_2, \cdots, s_n)$, and so on, $(k)=(s)$ will mean $k_i=s_i$, $i=1, 2, \cdots, n$. $(0)$ will mean $(0, 0, \cdots, 0)$, $\{\mu_{(m)}\}$ will be a sequence of real numbers, and $\{U_{(k)}(x)\}$ and $\{V_{(k)}(x)\}$ will be two sequences of polynomials such that

$$U_{(0)}(x) = V_{(0)}(x) = \text{const.},$$
$$\int_E U_{(k)}(x)V_{(s)}(x)dx = \begin{cases} 0, & (k) \neq (s), \\ 1, & (k) = (s), \end{cases} \tag{1}$$

and by $\int_{E(y)} f(x, y)d\Phi(E)$ will be meant the Lebesgue-Stieltjes integral over $E$ of $f$ considered as a function of a point $(y)$. $B$ will be used for any Borel set with $B \subseteq E$.

If $f$ is integrable over $E$ we define

$$\mathfrak{S}(f, x) \simeq \sum_{(k)} A_{(k)}V_{(k)}(x), \qquad A_{(k)} = \int_E f(x)U_{(k)}(x)dx,$$
$$S(x, y) \simeq \sum_{(k)} U_{(k)}(x)V_{(k)}(y).$$

Let $L_\nu$ for every $\nu$ be a partition of $R^n$ into two subsets, one closed and bounded. We write $(k) \in L_\nu$ to indicate that $(k)$ belongs to the bounded subset defined by $L_\nu$, and require that for every $(k)$ there exist a $\nu$ such that $(k)\in L_\nu$, and that $(k)\in L_\nu$ shall imply $(k)\in L_{\nu'}$ for all $\nu'\geqq\nu$. Now let

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[1] Presented to the Society, June 20, 1940.

[2] F. Hausdorff, *Momentprobleme für ein endliches Intervall*, Mathematische Zeitschrift, vol. 16 (1923), pp. 220–248.

[3] See, for example, A. Zygmund, *Trigonometrical Series*, Monografje Matematyczne, vol. 5, Warsaw, 1935, pp. 79–86.

$$S_\nu(x, y) = \sum_{(k)\,\epsilon\, L_\nu} U_{(k)}(x)V_{(k)}(y),$$

$$\mathfrak{S}_\nu(f, x) = \sum_{(k)\,\epsilon\, L_\nu} A_{(k)}V_{(k)}(x) = \int_E S_\nu(x, y)f(y)dy.$$

If $T$: $\|a_{ij}\|$ is any regular Toeplitz transformation,[4] we write

$$T\mathfrak{S}_\nu(f, x) = \int_E TS_\nu(x, y)f(y)dy = \int_E K_\nu(x, y)f(y)dy.$$

If $P$ is a polynomial in $(x)$ we denote by $\mu_{(x)}(P)$ the expression resulting from the substitution of $\mu_{m_1,m_2,\cdots,m_n}$ for $x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}$ in $P$.

3. **Moment problem.** A solution of the moment problem for the set $E$ is given in the following theorem:

THEOREM. *Given* $\{U_{(k)}(x)\}$, $\{V_{(k)}(x)\}$, $\{L_\nu\}$, *and* $T$ *satisfying the conditions above, and such that* $TS_\nu(x, y)=K_\nu(x, y)\geqq 0$ *for all* $(x)$, $(y)\in E$, *and all* $\nu$, *and such that for any* $f$ *integrable over* $E$ $T\mathfrak{S}_\nu(f, x)\to f(x)$ *for every* $(x)\in E$ *for which* $f$ *is continuous, and uniformly on* $E$ *if* $f$ *is continuous on* $E$, *then in order that a sequence* $\{\mu_{(m)}\}$ *be expressible in the form*

$$\mu_{m_1,m_2,\cdots,m_n} = \int_E x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}d\Phi(E),$$

*where* $\Phi$ *is completely additive, defined over at least all Borel sets of* $R^n$, *and with*

(1) $\int_E|d\Phi(E)|\leqq M$,
(2) $\Phi(B)\geqq 0$,
(3) $\Phi(B)=\int_B\phi(x)dx$ *and with*
(3a) $\phi\in L_E^p, p>1$,
(3b) $\phi\in L_E$,
(3c) $|\phi|\leqq M$,
(3d) $\phi\in C_E$,

*it is necessary and sufficient that*

(1) $\int_E|\mu_{(y)}\{K_\nu(x, y)\}|dx\leqq M$ *for all* $\nu$,
(2) $\mu_{(y)}\{K_\nu(x, y)\}\geqq 0$ *for all* $(x)\in E$ *and all* $\nu$,
(3a) $\int_E|\mu_{(y)}\{K_\nu(x, y)\}|^p dx\leqq M$ *for all* $\nu$,

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[4] Zygmund, loc. cit., pp. 40–43.

(3b) $\lim_{\nu,\tau\to\infty}\int_B|\mu_{(y)}\{K_\nu(x, y)\}-\mu_{(y)}\{K_\tau(x, y)\}|dx=0$,
(3c) $|\mu_{(y)}\{K_\nu(x, y)\}|\leqq M$ *for all* $(x)\in E$ *and all* $\nu$,
(3d) $\lim_{\nu,\tau\to\infty}|\mu_{(y)}\{K_\nu(x, y)\}-\mu_{(y)}\{K_\tau(x, y)\}|=0$ *uniformly in* $(x)\in E$.

The proof in each of the six cases closely parallels that of Hausdorff.[3] The proof is given for case (1) to indicate the modifications:

*Necessity.* We have

$$|\mu_{(y)}\{K_\nu(x, y)\}| = \left|\int_{E(y)} K_\nu(x, y)d\Phi(E)\right|$$

$$\leqq \int_{E(y)} K_\nu(x, y)\,|d\Phi(E)|,$$

$$\int_B |\mu_{(y)}\{K_\nu(x, y)\}|\,dx \leqq \int_B\left\{\int_B K_\nu(x, y)dx\right\}|d\Phi(E)|$$

$$\leqq C\int_B |d\Phi(E)| \leqq M$$

for

$$K_\nu(x, y) = \sum_{j=0}^{\infty} a_{\nu j}\sum_{(k)\in L_j} U_{(k)}(x)V_{(k)}(y)$$

$$\int_B K_\nu(x, y)dx = \sum_{j=0}^{\infty} a_{\nu j}\sum_{(k)\in L_j} V_{(k)}(y)\int_B U_{(k)}(x)dx$$

$$= \sum_{j=0}^{\infty} a_{\nu j} \leqq \sum_{j=0}^{\infty}|a_{\nu j}| \leqq C.$$

*Sufficiency.* Let

$$\Phi_\nu(B) = \int_B \mu_{(y)}\{K_\nu(x, y)\}dx,$$

$$\int_B |d\Phi_\nu(E)| = \int_B |\mu_{(y)}\{K_\nu(x, y)\}|\,dx \leqq M$$

and, by a well known theorem of Helly, there is a subsequence $\{\Phi_{\nu'}\}$ and a function $\Phi$ such that $\int_B|d\Phi(E)|\leqq M$ and $\Phi_{\nu'}(B)\to\Phi(B)$, and also $\int_B V_{(k)}(y)d\Phi_{\nu'}(E)\to\int_B V_{(k)}(y)d\Phi(E)$ whence $\mu_{(y)}\{V_{(k)}(y)\}=\int_B V_{(k)}(y)d\Phi(E)$, and $\Phi$ is a solution.

4. **Examples and conclusion.** If $E$ is the unit sphere in $R^n$, $\{U_{(k)}(x)\}$ and $\{V_{(k)}(x)\}$ may be taken as the normalized polynomials of Appell-

Didon,[5] $(k)\in L_\nu$ to mean $\sum_{i=1}^{n} k_i \leqq \nu$, and $T$ any $(C, r)$ with $r \geqq n+1$.[6] In particular, for $n=1$ this reduces to the Hausdorff solution for the unit interval. If $E$ is the circumference of the unit circle we may set $U_0(x)=V_0(x)=(2\pi)^{-1/2}$, and, for $k>0$,

$$U_{2k}(x) = V_{2k}(x) = (\pi)^{-1/2}\cos k\theta, \quad U_{2k-1}(x) = V_{2k-1}(x) = (\pi)^{-1/2}\sin k\theta$$

with $(s)\in L_\nu$ meaning $s \leqq 2\nu$, $T$ any $(C, r)$ with $r \geqq 1$.[7] Sequences $\{U_{(k)}(x)\}$ and $\{V_{(k)}(x)\}$ can be constructed by the Schmidt process for any bounded region in $R^n$. It would be interesting to know whether regular Toeplitz transformations of the type required for the present theorem exist in general.

BROWN UNIVERSITY

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[5] P. Appell and J. Kampé de Fériet, *Fonctions Hypergéométriques et Hypersphériques; Polynomes d'Hermite*, Paris, 1926.

[6] L. Koschmieder, *Über die C-Summierbarkeit gewisser Reihen von Didon und Appell*, Mathematische Annalen, vol. 104 (1931), pp. 387–402.

[7] L. Fejer's theorem. See, for instance, Zygmund, loc. cit., p. 45.

# NOTE ON AN INTEGRAL OF BIERENS DE HAAN

THOMAS JAMES HIGGINS

In connection with certain potential problems, the author has recently had occasion to evaluate integrals of the form

$$I_n = \int_0^\infty \frac{\epsilon^{-px}}{x^{n+1}} \sin q_0 x \cdot \sin q_1 x \cdot \sin q_2 x \cdots \sin q_n x \, dx.$$

In Table 371 of Bierens de Haan's *Nouvelles Tables d'Intégrales Définies* a rule is given for evaluation of this integral. An attempt to use it led to finding that it was incorrect.

This integral was originally considered by de Haan in a paper published in Dutch in the rather inaccessible Archief, Wiskundig Genootschap onder de Zinspreuk, part 1, 1856–1859, pp. 288–315. Reference to this article revealed that while de Haan did not give a specific rule for the above integral, he did discuss it in general. Evidently, in the compilation of these tables in French a garbled version of this discussion was inserted. As integrals of this form occur in several potential problems of current interest,[1] it is thought worthwhile to point out this error in these universally used tables and to give the following rule, ascertained partly from de Haan's discussion and partly from direct inspection of a number of specific cases given in the same article. The correct formula is:

$$I = \frac{1}{n!2^n} \sum_{j=1}^{j=2^n} (-1)^{r_j} \left[ T_j \arctan \frac{k_j}{p} - L_j \log (k_j^2 + p^2)^{1/2} \right],$$

where

(a) $k_j$ is any one of the $2^n$ possible sums obtainable from $(q_0 \pm q_1 \pm q_2 \pm \cdots \pm q_n)$ by selection of the $+$ and $-$ signs;

(b) $r_j$ is the number of negative signs in $k_j$;

(c) $T_j$ is the sum of the 1st, 3rd, 5th, $\cdots$ terms of

$$k_j^n + C_{n,n-1} p k_j^{n-1} - C_{n,n-2} p^2 k_j^{n-2} - C_{n,n-3} p^3 k_j^{n-3} + C_{n,n-4} p^4 k_j^{n-4} + C_{n,n-5} p^5 k_j^{n-5} - C_{n,n-6} p^6 k_j^{n-6} - \cdots p^n;$$

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[1] The author encountered these integrals in seeking to determine the vector potential of two parallel, infinitely long, tubular rectangular conductors carrying currents in opposite directions. In general, they arise in two dimensional problems, where it is necessary to obtain a potential satisfying Poisson's equation over certain rectangular areas and Laplace's equation over the remainder of space.

(d) $L_j$ is the sum of the remaining terms.

As an illustration we evaluate the integral

$$I_3 = \int_0^\infty \frac{\epsilon^{-px}}{x^3} \sin^2 ax \cdot \sin bx \, dx.$$

Now

$$\begin{aligned} k_1 &= a + a + b = 2a + b; \ (-1)^{r_1} = 1; \\ k_2 &= a + a - b = 2a - b; \ (-1)^{r_2} = -1; \\ k_3 &= a - a + b = b; \qquad (-1)^{r_3} = -1; \\ k_4 &= a - a - b = -b; \qquad (-1)^{r_4} = 1; \end{aligned}$$

and from

$$k_j^2 + C_{2,1} p k_j - p^2,$$

we have

$$T_j = k_j^2 - p^2, \qquad L_j = 2pk_j.$$

Hence

$$I_3 = \frac{1}{2^2 2!} \sum_{j=1}^{j=4} (-1)^{r_j} \left[ (k_j^2 - p^2) \operatorname{arc\,tan} \frac{k_j}{p} - 2pk_j \log (k_j^2 + p^2)^{1/2} \right].$$

This confirms the integral #2, Table 370.

PURDUE UNIVERSITY

# ELEMENTARY PROOF OF A THEOREM ON LORENTZ MATRICES

E. R. VAN KAMPEN

Let $x$ and $y$ be real $n$- and $m$-vectors and $x^2$, $y^2$ the scalar squares of $x$, $y$. The corresponding Lorentz matrices are matrices of $(n+m)$-dimensional real linear transformations which leave the quadratic form $x^2-y^2$ invariant. Let the transformation be written in the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax+By \\ Cx+Dy \end{pmatrix}. \tag{1}$$

Then *the signs of the determinants* $|A|$ *and* $|D|$ *form two* 1*-dimensional representations of the Lorentz group.* Two algebraic proofs at present available for this fact[1] depend on a recursive factorization of the Lorentz matrix into simple factors or on deeper facts from the theory of representations. On the other hand, a simple topological proof may be given in quite an obvious manner. In this note the topological proof is briefly sketched and then a simple algebraic proof is given which does not depend on recursive factorization or representation theory and is valid in any real field.

The set defined by $x^2-y^2\geqq 1$ in the real $(n+m)$-dimensional space possesses one basic $(n-1)$-dimensional (finite) cycle $\Gamma$ which can most easily be represented by the $(n-1)$-dimensional basic cycle of the $(n-1)$-dimensional sphere $x^2=1$, $y=0$. Now $\Gamma$ is transformed by (1) into a cycle homologous to $+\Gamma$ or to $-\Gamma$ according as $|A|$ is positive or negative. The formal proofs of these topological facts are obtained most easily from the remark that the whole space $x^2-y^2\geqq 1$ can be retracted into its subset $x^2=1$, $y=0$ by a deformation which does not change the value of $x/(x^2)^{1/2}$ for any point. That *sign* $|A|$ is a one-dimensional representation of the Lorentz group is of course evident from the fact that $\Gamma$ is transformed by (1) into a cycle homologous to *sign* $|A|\cdot\Gamma$. The statement concerning the signature of $|D|$ depends on a similar consideration for the set defined by $x^2-y^2\leqq -1$.

Now let the elements of the matrix in (1) belong to any real field. Let the unit matrices of dimensions $n$ and $m$ be denoted by $E_n$ and $E_m$. The fact that the matrix in (1) is a Lorentz matrix may be expressed by the relations:

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[1] Cf. W. Givens, *Factorization and signatures of Lorentz matrices*, this Bulletin, vol. 46 (1940), pp. 81–85, where other references are given. My thanks are due to Dr. Murnaghan who drew my attention to the above theorem.

$$(2)\qquad A'A - C'C = E_n, \qquad D'D - B'B = E_m, \qquad A'B = C'D,$$

which may be obtained by forming the expression $x^2-y^2$ for the vector on the right in (1).

If $P$ is a matrix of $m$ rows and $n$ columns, such that $E_n-P'P$ is positive definite, then the sign of the determinant $|A+BP|$ is independent of $P$; in particular $|A+BP|\neq 0$ and $|A|\neq 0$.

In fact, from (2) one easily obtains the identity

$$(A + BP)'(A + BP) = (C + DP)'(C + DP) + E_n - P'P.$$

Since $E_n-P'P$ is assumed to be positive definite, this implies that $(A+BP)'(A+BP)$ is positive definite. Thus $|A+BP|\neq 0$ and, by choosing $P=0$, also $|A|\neq 0$. On replacing $P$ by $tP$, one sees that the determinant $|A+tBP|$, which is a polynomial in the parameter $t$, is never 0 while $-1\leqq t\leqq 1$. For $E_n-t^2P'P=E_n-P'P+(1-t^2)P'P$ is positive definite if $-1\leqq t\leqq 1$. Thus the polynomial $|A+tBP|$ cannot change its sign as $t$ varies between 0 and 1. In the field of real numbers this is evident. If the underlying field is any real field, and if the polynomial $|A+tBP|$ took both possible signs for $-1\leqq t\leqq 1$, then one could adjoin to the field a root of $|A+tBP|=0$ which lies between $-1$ and 1. In the enlarged field one obtains of course a contradiction with the fact that $|A+tBP|\neq 0$ for $-1\leqq t\leqq 1$.

Let the product of two Lorentz matrices be written in the form

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{pmatrix}.$$

Then one has

$$A_1A_2 + B_1C_2 = (A_1 + B_1C_2A_2^{-1})A_2 = (A_1 + B_1P)A_2,$$

where $P=C_2A_2^{-1}$. But

$$\begin{aligned} E_n - P'P &= E_n - (A_2')^{-1}C_2'C_2A_2^{-1} = (A_2')^{-1}(A_2'A_2 - C_2'C_2)A_2^{-1} \\ &= (A_2')^{-1}A_2^{-1} \end{aligned}$$

is a positive definite matrix, so that sign $|A_1+B_1C_2A_2^{-1}|=$sign $|A_1+B_1P|=$sign $|A_1|$. Thus

$$\operatorname{sign}|A_1A_2 + B_1C_2| = \operatorname{sign}|A_1|\cdot\operatorname{sign}|A_2|.$$

This completes the algebraic proof of the above theorem.

The geometrical content of the proof becomes clearer, if one realizes that the $n$-dimensional linear spaces with the equations $y=Px$ (where $E_n-P'P$ is positive definite) are precisely those spaces through the origin which meet the quadratic $x^2-y^2=1$ in a completely elliptical quadratic (and the cone $x^2 - y^2 = 0$ in its vertex only). Thus this system of linear spaces is invariant under the Lorentz group. That the sign of $|A + BP|$ is independent of $P$ means that the orientation of all spaces $y = Px$ is left invariant by a Lorentz matrix with $|A| > 0$ and is changed into its opposite by a Lorentz matrix with $|A| < 0$. Complications in preceding proofs of the theorem apparently originate either from the inclusion of the proof that every matrix $P$ with positive definite $E - P'P$ is the matrix $CA^{-1}$ of a Lorentz transformation (1) and/or of the proof that the subgroup of the Lorentz group defined by $|A| > 0$, $|D| > 0$ is connected.

*The* 1-*dimensional representation of the Lorentz group given by the determinants of the Lorentz matrix* (1) *is the product of the two representations given by the signs of* $|A|$ *and* $|D|$. In fact, $D$ as well as $A$ is nonsingular. Thus[2]

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D| \cdot \begin{vmatrix} A & B \\ D^{-1}C & E_m \end{vmatrix} = |D| \cdot \begin{vmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & E_m \end{vmatrix}$$
$$= |D| \cdot |A - BD^{-1}C|,$$

so that, since $BD^{-1} = A'^{-1}C'$ and $|A'| = |A|$,

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D| \cdot |A - A'^{-1}C'C| = \frac{|D|}{|A|}|A'A - C'C| = \frac{|D|}{|A|}.$$

Thus the sign of

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix}$$

is the product of the signs of $|D|$ and $|A|$. Since

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \frac{|A|}{|D|}$$

may be similarly proved, one has

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} = \operatorname{sign} |A| \cdot \operatorname{sign} |D|.$$

THE JOHNS HOPKINS UNIVERSITY

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[2] Cf. J. Williamson, *The expansion of determinants of composite order*, American Mathematical Monthly, vol. 40 (1933), pp. 65–69.

# ERRORS IN HAYASHI'S TABLE OF BESSEL FUNCTIONS FOR COMPLEX ARGUMENTS[1]

ARNOLD N. LOWAN AND GERTRUDE BLANCH[2]

The Project for the Computation of Mathematical Tables is engaged in the computation of an extensive table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for complex arguments.[3] Certain formulas developed for the purpose of checking the tables could also be applied to checking systematically the above-mentioned table of Hayashi (credited to A. Dinnik) and it was deemed worth while to do so.

All values on the rays $\phi=\frac{1}{2}\pi$ and $\phi=\frac{1}{4}\pi$ were checked against the 15-decimal place values computed by us. Those on the ray $\phi=\frac{1}{2}\pi-0.001$ were differenced as a function of $r$; one error was discovered and corrected.

The entries on all the other rays were differenced as a function of $r$. In addition, "summation" tests were applied, based on the following theorem due to G. Blanch:

*For a fixed $r$, let $U_0(r, \phi_p)$ represent the real component of $J_0(r, \phi_p)$, where $\phi_p=p\pi/16$, and $p$ ranges from 0 to 7. Similarly, let $V_0(r, \phi_p)$, $U_1(r, \phi_p)$ represent the corresponding values of these functions for a fixed $r$, and $\phi$ ranging as before. Then the following formulas hold:*

$$\sum_{p=0}^{7} U_0(r, \phi_p) = 8 + \tfrac{1}{2}[J_0(r) - I_0(r)] + R_1, \tag{1}$$

$$\sum_{p=1}^{7} V_0(r, \phi_p) \sin(2\phi_p) = -r^2 + R_2, \tag{2}$$

$$\sum_{p=1}^{7} U_1(r, \phi_p) \cos\phi_p = 2r - \tfrac{1}{2}J_1(r) + R_3, \tag{3}$$

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[1] Published in *Fünfstellige Funktionentafeln*, by K. Hayashi, Berlin, Springer, 1930, pp. 105–109 (from Prof. A. Dinnik, Yekaterinoslaw, 1922). Tabulated in the form $J_0(z)=U_0+iV_0$; $J_1(z)=U_1+iV_1$ for $z=r\exp(i\phi)$. Range of $r$: 0(0.2)8. Range of $\phi$: $0(\pi/16)\pi/2$ for $U_0$ and $V_1$; $\pi/16(\pi/16)7\pi/16$ and $\pi/2-0.001$ for $V_0$ and $U_1$. To 4 decimal places.

[2] The results reported in this paper were obtained in the course of work done by the Project for the Computation of Mathematical Tables, O.P. No. 65-2-97-33, Work Projects Administration, New York, N. Y., operated under the sponsorship of Dr. Lyman J. Briggs, Director of the National Bureau of Standards. The authors wish to express their appreciation to the W.P.A. and to the Sponsor of this Project for permission to publish these results.

[3] This work has since been completed.

$$(4) \qquad \sum_{p=1}^{7} V_1(r, \phi_p) \sin \phi_p = 2r - \tfrac{1}{2} I_1(r) + R_4,$$

*where* $|R_i| < 7\times 10^{-7}$ *for* $r \leqq 8$, $i = 1, 2, 3, 4$.

The summations indicated above were actually performed, and wherever the discrepancy between the theoretical and actual sums exceeded one unit in the fourth decimal place, the source of a possible error was sought, along rays where the differences were irregular. Since each sum contained a group of 7 rounded entries, an occasional discrepancy of two units was to be expected. However, in such cases, the entries giving rise to the discrepancy were checked with special precaution, especially in formulas (2), (3), and (4), where multiplication of the entries by sin $\phi_p$ or cos $\phi_p$ might possibly mask an error larger than that shown by the discrepancy.

Because of the great number of errors in the table, it would have been very difficult to localize them by differencing methods alone; but with the aid of the summation tests, the erroneous entries were picked out systematically. The values in error were then re-computed. After all errors were corrected, the differences along the rays were sufficiently regular, and the discrepancies in the summation tests were no larger than 2 units in the fourth decimal place anywhere.

It was obviously impossible to check last-place errors of about a unit, except along the rays which we ourselves have computed. A number of additional errors of one unit were discovered during the recomputation of entries on complementary rays. [Thus, for example, $U_0(r, \phi) = A + B$, where $A = \sum_{k=0}^{\infty} - r^{4k+2} \cos (4k+2)\phi/(k!)^2$, and $B = \sum_{k=0}^{\infty} r^{4k} \cos 4k\phi/(k!)^2$; and $-A + B = U_0(r, \frac{1}{2}\pi - \phi)$. Hence one set of values of $A$ and $B$ yields the entries on two (complementary) rays.] However, it is not deemed worth while to report these errors of one unit. Readers interested in obtaining a list of those which came to light might communicate with the authors. Errors of more than one unit are given below:

Errors in Hayashi's table of Bessel functions
for complex arguments

| | | $U_0$ | | | | $V_0$ | |
|---|---|---|---|---|---|---|---|
| $r$ | $\phi$ | *Hayashi's Value* | *Correct Value* | $r$ | $\phi$ | *Hayashi's Value* | *Correct Value* |
| 6.8 | $\pi/16$ | 0.5338 | 0.5837 | 1.0 | $\pi/16$ | −0.0880 | −0.0850 |
| 5.2 | $2\pi/16$ | −0.6625 | −0.6601 | | | | |
| 7.4 | $2\pi/16$ | 0.5079 | 2.5079 | | | | |
| 1.4 | $3\pi/16$ | 0.7781 | 0.7731 | 4.6 | $3\pi/16$ | 0.4125 | 0.4095 |
| 5.6 | $3\pi/16$ | −2.0436 | −2.0636 | 6.0 | $3\pi/16$ | 4.4654 | 4.4854 |

$U_0$

| $r$ | $\phi$ | *Hayashi's Value* | *Correct Value* |
|---|---|---|---|
| 5.8 | $3\pi/16$ | −1.6127 | −1.6428 |
| 6.8 | $3\pi/16$ | 2.4803 | 2.8403 |
| 5.4 | $4\pi/16$ | −7.6676 | −7.6674 |
| 3.8 | $5\pi/16$ | −0.9092 | −1.0874 |
| 5.0 | $5\pi/16$ | −8.9930 | −9.0929 |
| 6.2 | $5\pi/16$ | −26.2519 | −28.2518 |
| 5.2 | $8\pi/16$ | 32.5636 | 32.5836 |
| 6.8 | $8\pi/16$ | 140.1364 | 140.1362 |
| 8.0 | $8\pi/16$ | 427.5529 | 427.5641 |

$V_0$

| $r$ | $\phi$ | *Hayashi's Value* | *Correct Value* |
|---|---|---|---|
| 6.8 | $4\pi/16$ | 18.0734 | 18.0736 |
| 7.4 | $4\pi/16$ | 28.6088 | 27.6088 |
| 8.0 | $4\pi/16$ | 35.0280 | 35.0167 |
| 0.8 | $5\pi/16$ | −0.1528 | −0.1523 |
| 6.0 | $5\pi/16$ | −2.8406 | −2.8206 |
| 6.8 | $6\pi/16$ | −56.4404 | −56.4402 |
| 5.8 | $7\pi/16$ | −40.0003 | −42.8973 |
| 5.6 | $7\pi/16$ | −35.0061 | −35.0360 |
| 8.0 | $\pi/2-0.001$ | −3.3587 | −3.1989 |

$U_1$

| $r$ | $\phi$ | *Hayashi's Value* | *Correct Value* |
|---|---|---|---|
| 0.2 | $\pi/16$ | 0.0966 | 0.0977 |
| 5.2 | $\pi/16$ | −0.5248 | −0.5255 |
| 7.6 | $3\pi/16$ | −3.8432 | −3.8428 |
| 7.8 | $3\pi/16$ | −2.5846 | −2.5841 |
| 8.0 | $3\pi/16$ | −0.9458 | −0.9448 |
| 5.6 | $5\pi/16$ | 4.6486 | 4.6489 |
| 7.4 | $5\pi/16$ | −44.1809 | −42.8744 |
| 2.6 | $6\pi/16$ | 1.8123 | 1.8133 |
| 6.2 | $6\pi/16$ | 37.4187 | 37.4126 |
| 8.0 | $6\pi/16$ | 55.4879 | 55.3546 |
| 1.4 | $7\pi/16$ | 0.2241 | 0.2441 |

$V_1$

| $r$ | $\phi$ | *Hayashi's Value* | *Correct Value* |
|---|---|---|---|
| 5.6 | $\pi/16$ | 0.0118 | 0.0818 |
| 2.2 | $2\pi/16$ | −0.0745 | −0.0754 |
| 8.0 | $2\pi/16$ | 2.6334 | 2.5000 |
| 5.0 | $3\pi/16$ | −2.2772 | −2.2792 |
| 5.6 | $3\pi/16$ | −1.7016 | −1.7014 |
| 7.4 | $3\pi/16$ | 8.6155 | 7.3091 |
| 7.8 | $3\pi/16$ | 10.2946 | 10.2950 |
| 1.0 | $4\pi/16$ | 0.3005 | 0.3076 |
| 2.6 | $5\pi/16$ | 0.5418 | 0.5412 |
| 6.6 | $5\pi/16$ | −34.4889 | −34.4849 |
| 7.6 | $5\pi/16$ | −52.7084 | −52.7088 |
| 7.8 | $5\pi/16$ | −53.9269 | −53.9275 |
| 8.0 | $5\pi/16$ | −53.3290 | −53.3300 |
| 0.2 | $7\pi/16$ | 0.0945 | 0.0985 |
| 2.8 | $7\pi/16$ | 1.7755 | 2.7755 |
| 5.2 | $7\pi/16$ | 15.7771 | 15.7764 |

NEW YORK CITY

# RECURRENCE OF SYMBOLIC ELEMENTS IN DYNAMICS[1]

RUFUS OLDENBURGER

1. **Introduction.** Morse and Hedlund [1] have given a symbolic treatment of modern theoretical dynamics as developed by Birkhoff [2] and others. In the Morse-Hedlund viewpoint the concept of recurrence plays an important role. To establish various theorems on symbolic trajectories Morse and Hedlund introduced "symbolic elements," the analogues of line elements on ordinary trajectories, a symbolic element being the notion of a trajectory $T$ and a particular symbol in $T$. In the present paper we shall be concerned primarily with the question: *"How are recurrence of a trajectory $T$ and elements based on $T$ related?"*

2. **Definitions.** For terms defined elsewhere and used here the reader is referred to other papers [1, 3]. Let rays $R_i$ $(i=1, 2, 3, 4)$ be given by

$$\alpha_{i1}\, \alpha_{i2}\, \alpha_{i3} \cdots .$$

The distance $E_1E_2$ between the elements $E_1=(R_1, R_2)$ and $E_2=(R_3, R_4)$ is defined to be $1/n$ where $n$ is such that $a_{1j}=a_{3j}$, $a_{2j}=a_{4j}$ for each value of $j$ in the range $1, 2, \cdots, n$, while

$$(a_{1,n+1}, a_{2,n+1}) \neq (a_{3,n+1}, a_{4,n+1}).$$

The element $E_1$ is the ray

$$A_1A_2A_3 \cdots , \tag{1}$$

where $A_j$ denotes the pair of symbols $(a_{1j}, a_{2j})$. In what follows the term "recurrence of $E_1$" designates the recurrence of the ray (1). We recall that a ray (1) is *recurrent* if for each $n$ there is an $m$ such that each $n$-block in (1) is contained in each $m$-block of (1). For each $n$ the value $R(n)$ of the *recurrency* function of (1) is the smallest $m$ with the property just mentioned.

3. **Recurrence.** It is evident that the recurrence of a single element based on a trajectory $T$ does not imply the recurrence of $T$.

THEOREM 1. *If each element based on a trajectory $T$ is recurrent, $T$ is recurrent.*

The recurrence of each ray based on $T$ obviously implies the recurrence of $T$. The recurrence of each element based on $T$ implies the recurrence of each ray based on $T$, whence $T$ is recurrent.

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[1] Presented to the Society, December 28, 1939, under the title *Symbolic elements in dynamics.*

THEOREM 2. *There exists a recurrent trajectory with a non-recurrent element.*

If $\alpha$ is irrational, the trajectory $T(0, \alpha)$ of Morse and Hedlund [4] is recurrent and is of the form $R^{-1}abR$. The element $(bR, aR)$ is not recurrent.

The proof of the theorem to follow is simple and will be omitted.

THEOREM 3. *A trajectory $T$ is identical with its inverse if and only if $T=R^{-1}R$ or $T=R^{-1}aR$ for some ray $R$ and symbol $a$.*

THEOREM 4. *If a trajectory $T$ is recurrent and is identical with its inverse, each element based on $T$ is recurrent.*

We assume first that $T=R^{-1}R$ for some ray $R$. Consider an element $E=(B^{-1}R, R_1)$, where $B$ is such that $R=BR_1$. We permit $B$ to be vacuous, in which case $E=(R, R_1)$. Let $G$ denote the leading $s$-block of $R_1$ for a given value of $s$, and let $r$ denote the length of $B$. Let $J$ denote the leading $s$-block of $B^{-1}R$. Then $J$ in $B^{-1}R$ matches $G$ in $R_1$, and the pair of blocks $J$, $G$ corresponds to the block $A_1 \cdots A_s$ in the representation (1) of $E$. The block $G^{-1}B^{-1}BG$ is a subblock of $T$. Let $R(n)$ denote the recurrency function of $T$, whence $G^{-1}B^{-1}BG$ occurs in each $R(2r+2s)$-block of $T$. Let $n$ be the number of symbols preceding one such block $G^{-1}B^{-1}BG$ in $R$, so that

$$R = HG^{-1}B^{-1}BGR_2,$$

where $H$ is a block of length $n$ and $R_2$ is a subray of $R$. For brevity we introduce the notation $K=G^{-1}B^{-1}BG$ so that $R=HKR_2$. It is no restriction on the generality of the argument to follow to suppose that $n \geqq r$ so that the block $K$ is a subblock of $R_1$. The final block $G$ in $K$ is preceded in $R_1$ by a subblock of length $n+r+s$. In the ray $B^{-1}R$ the final subblock $B^{-1}BG$ of $K$ is preceded by a block of length $n+r+s$. Thus the initial $s$-block $J$ of the subblock $B^{-1}BG$ of $K$ in $B^{-1}R$ matches the final block $G$ of $K$ in $R_1$. We have proved that in each $R(2r+2s)$-block of $R_1$ there is a subblock $G$ matching a subblock $J$ of $B^{-1}R$ whence each $R(2r+2s)$-block in the representation (1) contains a block identical with $A_1 \cdots A_s$. Since each block in (1) is contained in the block $A_1 \cdots A_s$ for large enough $s$, it follows that $E$ is recurrent.

We suppose, finally, that $T=R^{-1}aR$ for some ray $R$ and symbol $a$. Let $r$ and $s$ be arbitrary non-negative integers. We consider an element $E=(D^{-1}aR, R_1)$ where $D$ is the leading $r$-block of $R$ so that $R=DR_1$, $R_1$ being a subray of $R$. We let $G$ denote the leading $s$-block of $R_1$ so that $R_1=GR_2$ where $R_2$ is a subray of $R_1$. Evidently, $T$ contains the subblock $G^{-1}D^{-1}aDG$ and the initial block $G$ of $R_1$ matches the leading $s$-block $J$ of $D^{-1}aR$. The last block $G$ in any block $G^{-1}D^{-1}aDG$ of $R_1$ matches the leading block $J$ of a subblock $D^{-1}aDG$ in $D^{-1}aR$, whence $E$ is recurrent.

By Theorem 3 the above argument proves that each element $E$ based on $T$ is recurrent.

THEOREM 5. *If for some block $C$ and ray $R$ a trajectory $T$ is of the form $R^{-1}CR$, and if each element based on $T$ is recurrent, the trajectory $T$ is identical with its inverse.*

We suppose first that $C$ is of even length $2q$, and we accordingly write $C=C_1C_2$ where $C_1$ and $C_2$ are of length $q$. We consider the element $E=(C_1^{-1}R, C_2R)$ based on $T$. Since $E$ is recurrent, $C_2$ occurs in the subray $R$ matched in $C_2R$ by a block $C_1^{-1}$ in $C_1^{-1}R$. It follows that $C_1^{-1}$ matches $C_2$ in $R$, whence $C_1=C_2^{-1}$ and $T=T^{-1}$.

If now $C$ is of odd length $2q+1$, we write $C=C_1aC_3$ where $C_1$ and $C_3$ are of length $q$. Let the initial symbol of $R$ be denoted by $b$. Since the element $E=(C_1^{-1}R, aC_3R)$ is recurrent, there is a subblock $aC_3$ of $R$ matched in $aC_3R$ by a block $C_1^{-1}b$ in the ray $C_1^{-1}R$. Hence there is a block $C_3$ in $R$ matching a block $C_1^{-1}$ in $R$. Thus $C_1=C_3^{-1}$ and $T=T^{-1}$.

The Morse recurrent trajectory $T$ extensively studied in the literature is a trajectory identical with its inverse, whence each element based on $T$ is recurrent. Morse and Hedlund gave methods termed projection, reduction, association, and substitution by which one can obtain a recurrent trajectory from a given recurrent trajectory. One can prove readily that the elements based on the trajectories obtained by derivation from the Morse recurrent trajectory are recurrent.

Elsewhere the author proved (3) that the exponent trajectory $T_e$ of a recurrent trajectory $T$ is generated by a finite number of symbols, and is recurrent. Let $T_1, T_2, \cdots$ be a sequence of trajectories such that $T_i$ is the exponent trajectory of $T_{i-1}$ for each $i$. We term $T_2, T_3, \cdots$ the *successive exponent trajectories* of $T$. One can prove readily that the successive exponent trajectories of the Morse recurrent trajectory are each of the form $R^{-1}aR$ where $a$ is a symbol and $R$ is a ray, whence the elements based on these successive trajectories are recurrent.

We let $R_i(n)$ denote the recurrency function of the element $E_i$ based on a trajectory $T$. If for each $i$ the set $R_i(1), R_i(2), \cdots$ is bounded, $T$ is said to be *E-recurrent*. For the Morse recurrent trajectory one can prove that the set $R_i(1)$ is not bounded whence this trajectory is not $E$-recurrent. The existence of a nonperiodic $E$-recurrent trajectory is still an open question.

## REFERENCES

1. M. Morse and G. A. Hedlund, *Symbolic dynamics*, American Journal of Mathematics, vol. 60 (1938), pp. 815–866.

2. G. D. Birkhoff, *Dynamical Systems*, American Mathematical Society Colloquium Publications, vol. 9, New York, 1927.

3. R. Oldenburger, *Exponent trajectories in symbolic dynamics*, Transactions of this Society, vol. 46 (1939), pp. 453–466.

4. Morse and Hedlund, *Symbolic dynamics*, II. *Sturmian trajectories*, American Journal of Mathematics, vol. 52 (1940), pp. 1–42.

ILLINOIS INSTITUTE OF TECHNOLOGY

# COMMENTS ON CANONICAL LINES

R. B. RASMUSEN AND B. L. HAGEN

1. **Introduction.** In this paper we propose to find the equations of the canonical edges of Green using a conjugate net as the parametric net of an analytic surface; to give a new interpretation to the line called by Davis the associate line of collineation; and, finally, to make a generalization of the canonical quadric of Davis.

2. **Analytic basis.** Let the projective homogeneous coordinates $x^1, \cdots, x^4$ of a point $P_x$ on a surface $S$ in ordinary space be analytic functions of two independent variables $u$, $v$. In the notation[1] of Lane if the parametric curves on $S$ form a conjugate net, the coordinates $x$ of the point $P_x$ and the coordinates $y$ of a point which is the harmonic conjugate of the point $P_x$ with respect to the foci of the axis of the point $P_x$, satisfy a system of equations of the form

$$
\begin{aligned}
x_{uu} &= px + \alpha x_u + Ly, \\
x_{uv} &= cx + ax_u + bx_v, \qquad\qquad (1)\\
x_{vv} &= qx + \delta x_v + Ny, \qquad\qquad LN \neq 0.
\end{aligned}
$$

The ray-points of the net at the point $P_x$ are given by the formulas

$$x_{-1} = x_u - bx, \qquad x_1 = x_v - ax.$$

Some of the invariants of the net are

$$
\begin{aligned}
&H = c + ab - a_u, && K = c + ab - b_v, \\
&\mathfrak{H} = c + ab + b_v - \delta_u, && \mathfrak{K} = c + ab + a_u - \alpha_v, \qquad (2)\\
&8\mathfrak{B}' = 4a - 2\delta + (\log r)_v, && r = N/L, \\
&8\mathfrak{C}' = 4b - 2\alpha - (\log r)_u.
\end{aligned}
$$

If the covariant tetrahedron, $x$, $x_1$, $x_{-1}$, $y$ is used as the local tetrahedron of reference, a power series expansion[2] for one nonhomogeneous coordinate $z$ of a point on the surface in terms of the other two coordinates $x$, $y$ is

$$
\begin{aligned}
z = \tfrac{1}{2}(Lx^2 + Ny^2) &+ \tfrac{1}{3}(L\mathfrak{C}'x^3 + N\mathfrak{B}y^3) + c_0x^4 \qquad (3)\\
&+ 4c_1x^3y + 4c_3xy^3 + c_4y^4 + \cdots,
\end{aligned}
$$

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[1] Lane, *Conjugate nets and the lines of curvature*, American Journal of Mathematics, vol. 53 (1931), p. 574.

[2] Lane, *A canonical power series expansion for a surface*, Transactions of this Society, vol. 37 (1935), p. 481.

where the coefficients $c_0$, $c_1$, $c_3$, $c_4$ are given by

$$c_0 = \tfrac{1}{3}L\mathfrak{C}'[12\mathfrak{C}' + (\log \mathfrak{C}'r^{1/2})_u], \qquad 4c_1 = \tfrac{1}{6}L(H - \mathfrak{H}),$$
$$c_4 = \tfrac{1}{3}N\mathfrak{B}'[12\mathfrak{B}' + (\log \mathfrak{B}'r^{-1/2})_v], \quad 4c_3 = \tfrac{1}{6}N(K - \mathfrak{K}),$$

and the coefficient of $x^2y^2$ is zero.

We will assume that the surface $S$ is non-ruled,[3] that is, that $\mathfrak{B}'^2 + r\mathfrak{C}'^2 \neq 0$.

3. **The second canonical edge of Green.** Segre has shown[4] that the second[5] edge of Green is the line of inflexion of the nodal cubic in which the tangent plane at a point $P_x$ of an analytic surface $S$ intersects the four-parameter family of nondegenerate algebraic cubic surfaces having fourth order contact with $S$ at $P_x$.

The equation in nonhomogeneous coordinates of a general algebraic cubic surface through the point $P_x$ is

$$\begin{aligned} a_1x + a_2y + a_3z + b_1x^2 + b_2y^2 + b_3z^2 + b_4xy + b_5xz \\ + b_6yz + d_1x^3 + d_2y^3 + d_3z^3 + d_4x^2y + d_5x^2z + d_6xy^2 \\ + d_7y^2z + d_8xz^2 + d_9yz^2 + d_{10}xyz = 0. \end{aligned} \tag{4}$$

If this surface has fourth order contact with the surface $S$ at the point $P_x$, its coefficients satisfy the following conditions:

$$\begin{aligned} & a_1 = a_2 = b_4 = 0, \qquad a_3L + 2b_1 = 0, \qquad a_3N + 2b_2 = 0, \\ & 8L\mathfrak{C}'a_3 + 3Lb_5 + d_1 = 0, \qquad Lb_6 + 2d_4 = 0, \\ & 8N\mathfrak{B}'a_3 + 3Nb_6 + d_2 = 0, \qquad Nb_5 + 2d_6 = 0, \\ & 12c_0a_3 + 3L^2b_3 + 16L\mathfrak{C}'b_5 + 6Ld_5 = 0, \\ & 12c_4a_3 + 3N^2b_3 + 16N\mathfrak{B}'b_6 + 6Nd_7 = 0, \\ & 12c_1a_3 + 8L\mathfrak{C}'b_6 + 3Ld_{10} = 0, \\ & 12c_3a_3 + 8N\mathfrak{B}'b_5 + 3Nd_{10} = 0, \\ & LNb_3 + Nd_5 + Ld_7 = 0. \end{aligned} \tag{5}$$

The tangent plane, $z=0$, intersects this cubic surface in the cubic curve

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[3] Green, *Projective differential geometry of one-parameter families of curves and conjugate nets on a curved surface*, First Memoir, American Journal of Mathematics, vol. 37 (1915), p. 239.

[4] B. Segre, *La cubique indicatrice de l'élément linéaire projectif d'une surface*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 184, p. 729.

[5] This classification of the edges of Green is the reverse of that introduced by Green.

$$z = b_1x^2 + b_2y^2 + d_1x^3 + d_2y^3 + d_4x^2y + d_6xy^2 = 0, \tag{6}$$

where the coefficients $b_1$, $b_2$, $d_1$, $d_2$, $d_4$, $d_6$ can be expressed in terms of the invariants given by equations (2) and $a_3$.

Following the usual method of solving a cubic with its Hessian and discarding the solution giving the double point tangents, we obtain the following equation of the line of inflexions of the cubic:

$$z = a_3 + (b_5 + 2\mathfrak{C}'a_3)x + (b_6 + 2\mathfrak{B}'a_3)y = 0, \tag{7}$$

where

$$\begin{aligned} b_5 = \frac{-a_3}{8(N\mathfrak{C}'^2 + L\mathfrak{B}'^2)} \{ &- L\mathfrak{B}'[(H - \mathfrak{H}) - (K - \mathfrak{K})] \\ &+ 2\mathfrak{C}'[12(N\mathfrak{C}'^2 + L\mathfrak{B}'^2) + N\mathfrak{C}'(\log \mathfrak{C}'r^{1/2})_u \\ &+ L\mathfrak{B}'(\log \mathfrak{B}'r^{-1/2})_v]\}, \\ b_6 = \frac{-a_3}{8(N\mathfrak{C}'^2 + L\mathfrak{B}'^2)} \{ &N\mathfrak{C}'[(H - \mathfrak{H}) - (K - \mathfrak{K})] \\ &+ 2\mathfrak{B}'[12(N\mathfrak{C}'^2 + L\mathfrak{B}'^2) + N\mathfrak{C}'(\log \mathfrak{C}'r^{1/2})_u \\ &+ L\mathfrak{B}'(\log \mathfrak{B}'r^{-1/2})_v]\}. \end{aligned} \tag{8}$$

Any line of the second canonical pencil[6] of Davis may have its equation in the form

$$z = 1 + k(\mathfrak{C}'x + \mathfrak{B}'y) = 0,$$

where $k$ is a constant. The flex-ray (7) is a line of this pencil in case

$$2\mathfrak{C}' + b_5/a_3 = k\mathfrak{C}', \qquad 2\mathfrak{B}' + b_6/a_3 = k\mathfrak{B}'.$$

Eliminating $k$ from these equations, we obtain

$$(\mathfrak{B}'^2 + r\mathfrak{C}'^2)[(H - \mathfrak{H}) - (K - \mathfrak{K})] = 0.$$

Therefore, *necessary and sufficient conditions for the second edge of Green to be a line of the second canonical pencil of Davis are*

$$\begin{gathered} H - \mathfrak{H} = K - \mathfrak{K}, \\ [r\mathfrak{C}'(\log \mathfrak{C}'r^{1/2})_u + \mathfrak{B}'(\log \mathfrak{B}'r^{-1/2})_v]/(\mathfrak{B}'^2 + r\mathfrak{C}'^2) = \text{const.} \end{gathered} \tag{9}$$

A line of the first canonical pencil[7] of Davis joins the points $(1, 0, 0, 0)$ and $(0, -k\mathfrak{C}'/L, -k\mathfrak{B}'/N, 1)$. It can be verified that the

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[6] W. M. Davis, *Contributions to the Theory of Conjugate Nets*, Chicago Doctor's Thesis, 1933, p. 18.

[7] Davis, ibid., p. 20.

*first edge of Green* is the line joining the point (1, 0, 0, 0) to the point $[0,\ -(b_5/a_3+4\mathfrak{C}')/L,\ -(b_6/a_3+4\mathfrak{B}')/N,\ 1]$. The first edge of Green is a line of the first canonical pencil of Davis if conditions (9) hold. This is a special case of the following theorem.[8]

*If one line of an arbitrarily selected pair of lines which are reciprocal with respect to a quadric of Darboux is a line of the second canonical pencil of Davis, its reciprocal is a line of the first canonical pencil of Davis, and conversely.*

4. **The associate line of collineation.** The cubic surface

$$z = \tfrac{1}{2}(Lx^2 + Ny^2) + \tfrac{4}{3}(L\mathfrak{C}'x^3 + N\mathfrak{B}'y^3)$$

obtained by truncating the power series (3) can be interpreted[9] geometrically as follows: first, it has third order contact with the analytic surface $S$ given by equations (3) at the point (1, 0, 0, 0); second, it has a unode of second order at the point (0, 0, 0, 1). This cubic surface intersects the tangent plane $z=0$ in the nodal cubic

$$z = 3(x^2 + ry^2) + 8(\mathfrak{C}'x^3 + r\mathfrak{B}'y^3) = 0$$

which has for its line of inflexion $z= 1+2(\mathfrak{C}'x+\mathfrak{B}'y)=0$. This is the *associate line of collineation*,[10] and is a line of the second canonical pencil of Davis.

5. **A generalization of the canonical quadric of Davis.** Davis has shown[11] that at each point $P_x$ of a conjugate net on an analytic surface $S$ there exists a unique quadric surface which has the following properties: the quadric has second order contact with the surface sustaining the net; the axis and ray of the net at $P_x$ are reciprocal polars with respect to the quadric; and the quadric passes through the point $P_y$ which is the harmonic conjugate of $P_x$ with respect to the focal points of the axis. We shall generalize this definition by showing that at each point $P_x$ of a conjugate net there exists a two-parameter family of quadric surfaces having the following properties: *the quadrics have second order contact with the surface sustaining the net and any line of the second canonical pencil of Davis has as its polar reciprocal with respect to any quadric of this family the line of the first canonical pencil defined by the same value of k.*

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[8] We are indebted to P. O. Bell for this theorem in its general form.

[9] Lane, *Projective Differential Geometry of Curves and Surfaces*, The University of Chicago Press, 1932, p. 296.

[10] Davis, ibid., p. 25.

[11] Davis, ibid., p. 11.

The equation[12] of all quadric surfaces having second order contact with the analytic surface $S$, given by equation (3), is

$$Lx^2 + Ny^2 + z(-2 + k_2x + k_3y + k_4z) = 0, \tag{10}$$

where $k_2$, $k_3$, $k_4$ are arbitrary constants when $P_x$ is fixed, and functions of $u$, $v$ when $P_x$ varies over $S$.

If we demand that the polar reciprocals of any line of the first canonical pencil of Davis with respect to a quadric given by equation (10) be the line of the second canonical pencil of David defined by the same value of $k$, we obtain $k_2 = -4k\mathfrak{C}'$, $k_3 = -4k\mathfrak{B}'$. Substituting these values in (10), we have

$$Lx^2 + Ny^2 + z(-2 - 4k\mathfrak{C}'x - 4k\mathfrak{B}'y + k_4z) = 0. \tag{11}$$

If $k=0$, and the quadric passes through the point $P_y$ we have the canonical quadric of Davis; also if $k=-1$, we have the quadrics of Darboux.

It is interesting to note that for $k=-1$ the line in the second canonical pencil of Davis is the line of inflexions of the ray-point cubic.[13]

The second canonical tangent[14] of Davis is the line joining the canonical point $(0, \mathfrak{B}', -\mathfrak{C}', 0)$ to the point $P(1, 0, 0, 0)$. The coordinates of any point on this line are

$$(\lambda, \mu\mathfrak{B}', -\mu\mathfrak{C}', 0), \tag{12}$$

where $\lambda$ and $\mu$ are arbitrary constants. The polar plane of this point with respect to the quadrics whose equation is given by (11) is

$$\mu(L\mathfrak{B}'x - N\mathfrak{C}'y) - \lambda z = 0, \tag{13}$$

which is independent of $k$.

For $\lambda=0$, the point given by (12) is the canonical point of Davis and the plane (13) is the canonical plane of Davis.

Consequently, we have the theorem: *the polar plane of a point on the second canonical tangent of Davis with respect to a quadric of the family defined by equation* (11) *is a member of a pencil whose axis is the first canonical tangent of Davis. This plane is dependent on the selection of the point but is independent of the selection of the quadric of the family* (11).

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PENNSYLVANIA STATE COLLEGE

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[12] Davis, ibid., p. 11.

[13] Lane, *Conjugate nets and the lines of curvature*, American Journal of Mathematics, vol. 53 (1931), p. 577.

[14] Davis, ibid., pp. 17–18.

# A NOTE ON A THEOREM BY WITT[1]

ROBERT M. THRALL

1. **Introduction.** Let $F$ denote the free group with $n$ generators and let $F^c$ be the $c$th member of the lower central series[2] of $F$. Witt[3] has shown that $Q^c = F^c/F^{c+1}$ is a free abelian group with $\psi_c(n) = (1/c)\sum\mu(c/d)n^d$ generators (the summation is over all divisors $d$ of $c$ and $\mu$ is the Möbius $\mu$-function).

The set of $k$th powers in $F$ generates a normal subgroup $H_k$. Let $F_k = F/H_k$ and $G_{k,c} = F_k/F_k^{c+1}$. We shall call $F_k$ the *free k-group* and $G_{k,c}$ the *free k-group of class c*. It is a consequence of Witt's result that $F_k^c/F_k^{c+1}$, the central of $G_{k,c}$, is abelian and has at most $\psi_c(n)$ generators. In this note we show that if $p$ is a prime greater than $c$, and $q = p^\alpha$, then the central of $G_{q,c}$ is of order $q^N$ where $N = \psi_c(n)$. If the prime divisors of $k$ are all greater than $c$, an analogous result holds for the central of $G_{k,c}$ as a consequence of Burnside's theorem that a nilpotent group is the direct product of its Sylow subgroups.

Let $M_c$ denote the space of tensors of rank $c$ over the $GF[p]$. A homomorphic mapping of $M_c$ upon the central of $G_{p,c}$ is set up and enables one to apply the theory of decompositions of tensor space under the full linear group mod $p$, to determine all characteristic subgroups of $G_{p,c}$ which lie in its central. This theory is applied to determine all the characteristic subgroups of $G_{p,c}$ for $c < 5$ and a multiplication table is constructed for $G_{p,3}$.

2. **Commutator calculus.**[4] Let $s_1, s_2, \cdots$ be operators in any group $G$ and set $s_{12} = (s_1, s_2) = s_1^{-1}s_2^{-1}s_1s_2$ and $s_{12\cdots k} = (s_{12\cdots k-1}, s_k)$. $s_{12\cdots k}$ is called a *simple commutator* of *weight k* in the components $s_1, \cdots, s_k$. The group $G^k$ generated by the simple commutators of weight $k$ for all choices of $s_1, \cdots, s_k$ in $G$ is called the $k$th member of the *lower central series* of $G$. If $s \in G^k$ but $s \notin G^{k+1}$, then $s$ is said to have *weight k* in $G$.

For all $s_1, s_2, s_3$ in $G$ we have

$$(1) \qquad (s_1s_2, s_3) = s_{13}s_{132}s_{23}, \qquad (s_1, s_2s_3) = s_{13}s_{12}s_{123}.$$

Let the weight of $s_i$ be $\alpha_i$ and set $\alpha = \alpha_1 + \cdots + \alpha_k + 1$. The following relations are then true:

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[1] Presented to the Society, April 13, 1940.

[2] For definition see §2 below or [4, p. 49].

[3] [7, p. 153].

[4] The relations in this section are either taken directly from Hall, Magnus, or Witt or are immediate consequences of their theorems. See [4, 6 and 7].

$$s_{123\cdots k}s_{213\cdots k} \equiv I \bmod G^{\alpha}, \tag{2}$$

$$s_{123\cdots k}s_{231\cdots k}s_{312\cdots k} \equiv I \bmod G^{\alpha}, \tag{3}$$

$$(s_1^{a_1}, s_2^{a_2}, \cdots, s_k^{a_k}) \equiv (s_{12\cdots k})^{a_1a_2\cdots a_k} \bmod G^{\alpha}. \tag{4}$$

If now $\alpha-1=km$, $m=$ minimum $(\alpha_1, \cdots, \alpha_k)$ and $\rho_\beta=\prod_{\delta=1}^{\delta=n} s_\delta^{a_{\beta\delta}}$, $\beta=1, \cdots, k$, it follows that

$$\rho_{12\cdots k} \equiv \prod_{\beta_\gamma=1}^{n} (s_{\beta_1\cdots\beta_k})^{a_{1\beta_1}\cdots a_{k\beta_k}} \bmod G^{\alpha}. \tag{5}$$

3. **The groups** $F_q$. Let $F$ be the free group generated by $s_1, \cdots, s_n$, and denote by $\overline{H}_k$ the smallest normal subgroup containing the $k$th powers of all simple commutators of $s_1, \cdots, s_n$.

LEMMA I. *Let* $q=p^\alpha$, $p$ *any prime. Then* $s^q\in\overline{H}_q\cup F^p$ *for any element* $s\in F$.

PROOF BY INDUCTION. The lemma is trivial for $s$ of weight greater than $p-1$. Suppose the lemma true for all weight greater than $c$ and let $s$ be of weight $c$. By the definition of weight, $s$ can be written in the form $s=t_1\cdots t_mv_0$ where $v_0$ has weight greater than $c$ and the $t_i$ are of weight $c$ and are all simple commutators in $s_1, \cdots, s_n$. Then by the fundamental expansion formula[5] for $(PQ\cdots)^n$ we have

$$s^q = t_1^q \cdots t_m^q v_0^q v_1^q \cdots v_j^q w$$

where $w\in F^p$ and the $v_\beta$ are all of weight greater than $c$. By definition $t_\beta^q\in\overline{H}_q$ and by our induction hypothesis $v_\beta^q\in\overline{H}_q\cup F^p$ and so $s^q\in\overline{H}_q\cup F^p$.

COROLLARY I. *Let* $s$ *have weight* $c$, *for* $c<p$. *Then* $s^q\in\overline{H}_q\cup F^{c+1}$.

Set $H_{q,c}=H_q\cap F^c$ and $\overline{H}_{q,c}=\overline{H}_q\cap F^c$. Then we have

COROLLARY II. *For* $c<p$, $H_{q,c}\cup F^{c+1}=\overline{H}_{q,c}\cup F^{c+1}$.

LEMMA II. $F_q^c/F_q^{c+1}\simeq F^c/(F^{c+1}\cup H_{q,c})$.

We note first that applying the second homomorphism theorem[6] to Hall's formula[7] $F_q^c=(F^c\cup H_q)/H_q$ we obtain the result $F_q^c=F^c/H_{q,c}$ (for all $c$). Now

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[5] See [4, formula 3.51] or [6, p. 111].

[6] See [2, p. 32].

[7] See [4, formula 2.491] or [2, p. 119].

$$F^c/(F^{c+1} \cup H_{q,c}) \simeq (F^c/H_{q,c})/([F^{c+1} \cup H_{q,c}]/H_{q,c})$$
$$\simeq F_q^c/(F^{c+1}/[H_{q,c} \cap F^{c+1}]) = F_q^c/F_q^{c+1},$$

since $H_{q,c} \cap F^{c+1} = H_{q,c+1}$.

Set $Q_q^c = F_q^c/F_q^{c+1}$.

THEOREM I. *For* $c<p$, $Q_q^c$ *is abelian of order* $q^N$, $N=\psi_c(n)$.

DEFINITION. $t_1, \cdots, t_k$ *is said to be a basis for* $F^c$ mod $F^{c+1}$, *if any operator* $t$ *of weight* $c$ *can be written uniquely in the form* $t=\prod t_i^{d_i}\theta$ *where* $\theta \in F^{c+1}$.

Evidently such a basis exists, and by Witt's theorem[8] $k=N$; and we may choose the $t_i$ as simple commutators in the generators $s_1, \cdots, s_n$. Let $\rho_i$ be the image in $Q_q^c$ of $t_i$. Then since the $t_i$ are a basis for $F^c$ mod $F^{c+1}$, any operator $\rho \in Q$ can be written in the form $\rho = \prod \rho_i^{d_i}$ where $0 \leqq d_i < q$. Hence the order of $Q_q^c$ is at most $q^N$ for any $c$. If the order of $Q_q^c$ is less than $q^N$ there exists a relation $\prod \rho_i^{d_i} = I$ where, say, $d_j \neq 0$.

If now $p>c$, this relation together with Corollary II and Lemma II imply that $\prod t_i^{d_i} \in \overline{H}_{q,c} \cup F^{c+1}$, or $\prod t_i^{d_i} \equiv \prod t_i^{qe_i}$ mod $F^{c+1}$. Since the $t_i$ are a basis for $F^c$ mod $F^{c+1}$ this requires $d_i - qe_i = 0$, $i=1, \cdots, N$, which contradicts the assumption that $d_j$ and, therefore, $d_j - qe_j$ is not divisible by $q$. Hence there can be no relation between the $\rho_i$ and the theorem is proved.

COROLLARY III. *For* $p>c$, $G_{q,c}^j$ *is of order* $q^m$,

$$m = \psi_j(n) + \cdots + \psi_c(n), \qquad j = 1, \cdots, c.$$

4. **Characteristic subgroups of** $G=G_{p,c}$. A large variety of characteristic subgroups of $G$ can be obtained from the lower central series by sequences of joins, intersections, and commutations. In $G$ the upper and lower central series are identical; in particular, the central $C$ $(=C_{p,c})$ of $G$ is $G^p$. The central quotient group of $G$ is $G_{p,c-1}$, and any characteristic subgroup $H$ of $G$ is mapped into a characteristic subgroup $H' = H \cup C/C$ in $G_{p,c-1}$.

We say that $K$ is a *minimal characteristic subgroup* (m.c.s.) of $G$ if no proper subgroup of $K$ is characteristic in $G$. *For* $G=G_{p,c}$, *every m.c.s. lies in the central.* Indeed any normal subgroup of $G$ must contain commutators of weight $c$ and therefore must have an intersection not equal to $I$ with $C$. We turn now to the determination of all characteristic subgroups of $G$ which lie in $C$.

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[8] See [7, Theorems 3 and 4, pp. 152–153].

Let $\bar{A}$ be any automorphism of $G$, and $H$ any characteristic subgroup of $G$. $\bar{A}$ induces an automorphism $\bar{A}(H)$ on $G/H$ and an automorphism $\bar{A}[H]$ on $H$. If in particular $H$ is $G^2$, the commutator subgroup of $G$, then $G/H$ is the abelian group of order $p^n$ and type $1, 1, 1, \cdots$. Let the generators of $G$ be $s_i, \cdots, s_n$, and let $t_i$ be the image in $G/G^2$ of $s_i$. Then $\bar{A}(H)$ takes the form $t_i \rightarrow t_i'$ where

$$t_i' = \prod t_j^{a_{ij}}, \quad a_{ij} \in GF[p], \qquad |a_{ij}| \neq 0.$$

Hence $\bar{A}$ itself must be of the form $s_i \rightarrow s_i'$ where

$$s_i' = \prod s_j^{a_{ij}} r_i, \qquad r_i \in G^2.$$

To calculate $\bar{A}[C]$ we apply (5) with $k=c$. Since $G^{c+1}=I$, (5) is now an equality and shows that $\bar{A}[C]$ is independent of the $r_i$. Indeed if we set $A=(a_{ij})$ we see that the formal commutators $s_{i_1\cdots i_c}$ transform like tensors of rank $c$, that is, according to $A \times A \times \cdots \times A$ (Kronecker direct product with $c$ factors).

Denote by $M_c$ the whole space of tensors of rank $c$. It has dimension $n^c$. The group $A_c = \{A \times A \times \cdots \times A\}$ ($c$ factors) is homomorphic to the group $\{A\}$ of linear transformations, and hence $M_c$ is a representation space for $\{A\}$. Brauer[9] has proved the following theorem concerning the decompositions of this representation:

THEOREM II. *If $K$ is a field of characteristic $p \neq 0$, the representation $A_c$ is completely reducible for $c<p$, and it splits into irreducible parts in exactly the same way as in the case of characteristic zero.*

The mapping $x_{i_1\cdots i_c} \rightarrow s_{i_1\cdots i_c}$ (where of course products in $C$ are replaced by sums in $M_c$) establishes a homomorphic mapping of $M_c$ upon $C$ and this mapping is preserved under the group $A_c$, that is, $C$ is also a representation space for the group $A_c$. Let $\bar{C}$ denote $C$ written additively. Then $\bar{C}=M_c-W_c$, where $W_c$ contains all tensors whose image in $C$ is identity. We call $W_c$ the *space of commutator relations*, $W_c$ is evidently an invariant subspace of $M_c$ under the tensor group and by Theorem I it has dimension $n^c-\psi_c(n)$ if $p>c$. Because of the complete reducibility of the representation $A_c$ we can write $M_c=W_c+P_c$ where $P_c$ is likewise an invariant subspace of $M_c$, and furthermore the decomposition into irreducibly invariant subspaces of $P_c$ under $A_c$ will be the same as that of $C$ under the group of automorphisms of $G$. ($P_c$ is not uniquely determined by $W_c$ but its decompositions are.) Let $R_1, \cdots, R_t$ be irreducibly invariant subspaces of $M_c$ whose direct sum is $P_c$, and let $T_1, \cdots, T_t$ be the corresponding subgroups of $C$. Then the following theorem expresses the above arguments in group theoretic terms:

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[9] See [3, p. 867].

THEOREM III. *Any minimal characteristic subgroup is isomorphic to one of* $T_1, \cdots, T_t$ *and any characteristic subgroup* $K$ *of* $G$ *which lies in the central is the direct product of the minimal characteristic subgroups which it contains.* ($p>c$ *is assumed throughout.*)

The number of characteristic subgroups in $G$ is clearly independent of the number $n$ of generators provided that $n \geqq c$. Hence to obtain all characteristic subgroups of the set of groups $G_{p,c}$ with $p>c$ we need only consider those with $n=c$.

5. **The groups $G_{p,3}$ and $G_{p,4}$.** In this section we shall make use of the decomposition into irreducibly invariant subspaces of the tensor spaces $M_3$ and $M_4$. These can be readily obtained by a direct computation based upon the decomposition theorems of $M_c$ in general.[10] We suppose $n=3$ in $M_3$ and $n=4$ in $M_4$.

$M_3=\sum_1+\sum_{2,1}+\sum_{2,2}+\sum_3$ in which the summands have dimensions 10, 8, 8 and 1 respectively. $W_3=\sum_1+\sum_{2,1}+\sum_3$ and hence $G_{p,3}$ has just one m.c.s., its central.

$$M_4 = \sum\nolimits_1 + \sum\nolimits_{2,1} + \sum\nolimits_{2,2} + \sum\nolimits_{2,3} + \sum\nolimits_{3,1} + \sum\nolimits_{3,2}$$
$$+ \sum\nolimits_{4,1} + \sum\nolimits_{4,2} + \sum\nolimits_{4,3} + \sum\nolimits_5$$

in which the summands have dimensions 35, 45, 45, 45, 20, 20, 15, 15, 15, and 1 respectively. $W_4=\sum_1+\sum_{2,1}+\sum_{2,2}+\sum_{3,1}+\sum_{3,2}+\sum_{4,1}+\sum_{4,2}+\sum_5$ and hence $G_{p,4}$ has two m.c.s., one of which is its second derived group. Let us denote these by $D$ and $E$.

$G_{p,1}$ has no proper characteristic subgroups and the only proper characteristic subgroup of $G_{p,2}$ is its central $G^2_{p,2}$.

THEOREM IV. *The only characteristic subgroups of* $G_{p,3}$ *are the members of its lower central series.*

Let $H$ be characteristic in $G_{p,3}$. Then if $H \neq I$ or $C$, by Theorem III $H \supset C$. $H'=H/C$ must then be $G_{p,2}$ or its central. In the first case $H=G_{p,3}$ and in the second case $H=G^2_{p,3}$.

THEOREM V. *The only characteristic subgroups of* $G_{p,4}$ *are* $D$, $E$ *and the members of the lower central series.*

It is easy to see that if a characteristic subgroup $H \supset C$ then $H$ is in the lower central series. To complete the proof we show then that if $H \not\supset C$, $H = D$ or $E$. Since $H \not\supset C$, either $H' = I$; in which case $H \subset C$ and therefore $H = D$ or $E$; or $H' \supset G^3_{p,3}$ (by Theorem IV). It remains now only to show that $H' \supset G^3_{p,3}$ implies $H \supset C$. If now $H' \supset G^3_{p,3}$, then $H \cup C \supset G^3_{p,4}$ and hence for the commutator $s_{123}$ of weight 3 we have a factorization $s_{123} = hd$ where $h \in H$ and $d \in C$ (and so $d$ has weight not less than 4). Since $H$ is normal $(h, s_4) = (s_{123} \cdot d^{-1}, s_4) = s_{1234} \in H$. But the conjugates of $s_{1234}$ generate $C$ so that $H \supset C$ contrary to hypothesis, and the theorem is proved.

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[10] See for instance [1, Theorem 4.4D, p. 129].

For the sake of completeness we give a multiplication table for $G_{p,3}$. Applying the formulas of §2 and Theorem I we have for any operator $s$ of $G_{p,3}$ a unique expression in the form

$$s = s^A = \prod s_i^{a_i} \prod_{i<j} s_{ij}^{a_{ij}} \prod_{i \neq j} s_{ijj}^{a_{ijj}} \prod_{i<j<k} s_{ijk}^{a_{ijk}} s_{jki}^{a_{jki}}.$$

If now $s^C = s^A s^B$, then applying the readily verified formula $(s_1^\alpha, s_2^\beta) = s_{12}^{\alpha\beta} s_{121}^{\beta C_{\alpha,2}} s_{122}^{\alpha C_{\beta,2}}$ we obtain[11] $(i<j<k)$

$$\begin{aligned} c_i &= a_i + b_i, \qquad c_{ij} = a_{ij} + b_{ij} - b_i a_j, \\ c_{ijj} &= a_{ijj} + b_{ijj} - b_i C_{a_j,2} + b_j a_{ij} - b_i b_j a_j, \\ c_{jii} &= a_{jii} + b_{jii} + a_j C_{b_j,2} - b_i a_{ij}, \qquad (6) \\ c_{ijk} &= a_{ijk} + b_{ijk} + b_j a_{ik} + b_k a_{ij} - b_i a_j a_k - b_i b_j a_k - b_i b_k a_j, \\ c_{jki} &= a_{jki} + b_{jki} + b_i a_{jk} + b_j a_{ik} - b_i b_j a_k. \end{aligned}$$

## BIBLIOGRAPHY

1. H. Weyl, *The Classical Groups*, Princeton University Press, 1939.

2. H. Zassenhaus, *Lehrbuch der Gruppentheorie*, Hamburger Mathematische Einzelschriften, vol. 21, 1937.

3. R. Brauer, *Algebras connected with semi-simple groups*, Annals of Mathematics, vol. 38 (1937), pp. 857–872.

4. P. Hall, *Groups of prime power order*, Proceedings of the London Mathematical Society, vol. 36 (1934), pp. 29–95.

5. Levi and van der Waerden, *Über eine besondere Klasse von Gruppen*, Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität, vol. 9 (1933), pp. 154–158.

6. W. Magnus, *Über Beziehungen zwischen höheren Kommutatoren*, Journal für die reine und angewandte Mathematik, vol. 177 (1937), pp. 105–115.

7. E. Witt, *Treue Darstellung Liescher Ringe*, Journal für die reine und angewandte Mathematik, vol. 177 (1937), pp. 152–160.

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[11] For $p=3$, $s_{ijj} = s_{jii} = I$ and $s_{ijk} = s_{jki}$ so that (6) reduces to formula 9 of Levi and van der Waerden [5, p. 156].

# NON-INVOLUTORIAL SPACE TRANSFORMATIONS ASSOCIATED WITH A $Q_{1,2}$ CONGRUENCE

A. B. CUNNINGHAM

De Paolis[1] discussed the involutorial transformations associated with the congruence of lines meeting a curve of order $m$ and an $(m-1)$-fold secant, while Vogt[2] studied the transformation $T$ for a linear congruence and bundle of lines. In the present paper the transformations associated with the congruence of lines on a conic and a secant of it are discussed.

Given a conic $r$, a line $s$ meeting $r$ once, and two projective pencils of surfaces

$$|F_{n+m+1}|: \quad r^n s^m g; \qquad |F'_{n'+m'+1}|: \quad r^{n'} s^{m'} g',$$

where $n \leqq m+1$, $n' \leqq m'+1$, $[r, s]=A$, and $g$, $g'$ the residual base curves.

Through a generic point $P$, there passes a single surface $F$ of $|F|$. The unique line $t$ through $P$, $r$, $s$ meets the associated $F'$ in one residual point $P'$, image $(T)$ of $P$. The transformations to be considered are of three types:

*Case* I. $n=m+1, n'=m'+1$.
*Case* II. $n<m+1, n'<m'+1$.
*Case* III. $n=m+1, n'<m'+1$.

## CASE I

Given

$$|F_{2n}|: \quad r^n s^{n-1} g; \qquad |F'_{2n'}|: \quad r^{n'} s^{n'-1} g';$$

where $g$, $g'$ are of order $n^2+2n-1$, $n'^2+2n'-1$. The curve $g$ meets $r$, $s$ in $n^2+2n-1$, $n^2-1$ points respectively.

The conic $r$ is a fundamental curve whose image $(T^{-1})$ is $R: r^{n+n'}$, since there are $(n+n')$ invariant directions through each point on $r$. $R$ is generated by a monoidal plane curve of order $n+n'+1$, one curve on each plane of the pencil $(O_r s)=w$, as $O_r$ describes $r$. The fundamental line $s$ has for image $(T^{-1})$ a surface $S$: $s^{n+n'-1}$, of which $n+n'-2$ branches are invariant. $A$ is a fundamental point of the first kind, whose image $(T^{-1})$ is the plane $u: r$. In the plane $v: s$ and tangent to $r$ there is a curve $C_{n+n'}$, image $(T^{-1})$ of the intersection of $r$, $s$ at $A$, which lies on $R$, $S$. The tangent line $[u, v]$ to $r$ at $A$ lies on the surface $R$.

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[1] De Paolis, *Alcuni particolari trasformazioni involutori dello spazio*, Rendiconti dell' Accademia dei Lincei, Rome, (4), vol. 1 (1885), pp. 735–742, 754–758.

[2] Vogt, *Zentrale und windschiefe Raum-Verwandtschaften*, Jahresbericht der Schlesischen Gesellschaft für Vaterländische Kultur, class 84, 1906, pp. 8–16.

From any point $Q'$ on $g'$, there is a unique transversal $t$ meeting $r, s$. Any point $P$ on $t$ determines an $F$ and $t$ meets the associated $F'$ in a residual point $Q'$, thus $Q' \sim (T^{-1})t$. Every point $P'$ on $t$ determines the same $F'$ and $t$ meets the associated $F$ in one point $\bar{P}$; thus $\bar{P} \sim (T)t$. Considering all points on $g'$

$$g' \sim (T^{-1})G; \qquad \bar{g}_x \sim (T)G,$$

where $\bar{g}_x$ is the locus of points $\bar{P}$. Similarly

$$g \sim (T)G'; \qquad \bar{g}_y' \sim (T^{-1})G'.$$

The eliminant of the parameter from $|F|$, $|F'|$ is a point-wise invariant surface $K_{2n+2n'}$. A generic plane meets every line of the pencil $(Au)$, hence the homaloidal surfaces have an additional fixed direction $d$ through $A$.

The table of characteristics for $T^{-1}$ is

$$\begin{array}{rrllllllll}
\pi' \sim & \phi_{2n+2n'+2}: & A^{n+n'+1+d} & r^{n+n'+1} & s^{n+n'} & g & \bar{g}_x, & & \\
K \sim & K_{2n+2n'}: & A^{n+n'} & r^{n+n'} & s^{n+n'-2} & g & \bar{g}_x & g' & \bar{g}_y', \\
r \sim & R_{2n+2n'+1}: & A^{n+n'+d} & r^{n+n'} & s^{n+n'} & g & \bar{g}_x & C_{n+n'} & [u, v], \\
s \sim & S_{2n+2n'}: & A^{n+n'} & r^{n+n'} & s^{n+n'-1} & g & \bar{g}_x & C, & \\
g' \sim & G_{4n'}: & A^{2n'} & r^{2n'} & s^{2n'} & g' & \bar{g}_x, & & \\
\bar{g}_y' \sim & G_{4n}: & A^{2n} & r^{2n} & s^{2n} & g & \bar{g}_y', & & \\
A \sim & u: & A & r, & & & & &
\end{array}$$

$$J \equiv u^3RSGG'.$$

The intersection of two $\phi'$-surfaces gives the order of $\bar{g}_y'$, $y = n^2+2nn'+2n+1$. The curve $\bar{g}_y'$ meets $r$, $s$ in $y$, $y-2n$ points respectively.

The equations of $T^{-1}$ are $\tau x_i = Ry_i - Kz_i = Suy_i + Kw_i$, where $z_i$, $w_i$ are the points $[t, r]$, $[t, s]$.

## CASE II

Given

$$|F_{n+m+1}| : \ r^n s^m g; \qquad |F'_{n'+m'+1}| : \ r^{n'} s^{m'} g',$$

where $g$, $g'$ are of order $2mn+2m+2n-n^2+1$, $2m'n'+2m'+2n'-n'^2+1$. The curve $g$ meets $r$, $s$ in $2mn+4n-n^2$, $2mn+2m-n^2$ points respectively.

$A$ is a fundamental point of the second kind with image $(T^{-1})\,C_{n+n'+1}$: $A^{n+n'}$ in the plane $v$.

The image $(T^{-1})$ of a point on $s$ is a curve $s_{m+m'+2}$ on the quadric cone on $r$, with a $(m+m')$-fold point at the vertex and one point on each generator. This curve generates the surface $S$. The equations of $T$ are

$$\tau x = Ry_i - Kz_i = Sy_i + Kw_i.$$

The table of characteristics for $T^{-1}$ is

$$\begin{array}{rlllllll}
\pi' \sim & \phi_{n+n'+m+m'+4}: & r^{n+n'+1} & s^{m+m'+2} & g & \bar{g}_x, & & \\
K \sim & K_{n+n'+m+m'+2}: & r^{n+n'} & s^{m+m'} & g & \bar{g}_x & g' & \bar{g}'_y, \\
r \sim & R_{n+n'+m+m'+3}: & r^{n+n'} & s^{m+m'+2} & g & \bar{g}_x & C_{n+n'+1}, & \\
s \sim & S_{n+n'+m+m'+3}: & r^{n+n'+1} & s^{m+m'+1} & g & \bar{g}_x & C_{n+n'+1}, & \\
g' \sim & G_{2n'+2m'+3}: & r^{2n'+1} & s^{2m'+2} & g' & \bar{g}_x, & & \\
\bar{g}'_y \sim & G'_{2n+2m+8}: & r^{2n+1} & s^{2m+2} & g & \bar{g}'_y, & & \\
J \equiv & RSGG', & & & & & &
\end{array}$$

where $y = 2mn + 2m'n + 2mn' + 3m + 3n + m' + n' - n + 5 - 2nn'$. The curve $\bar{g}'_y$ meets $r$, $s$ in $[y - (2m - 2n + 1)]$, $[y - (2n + 1)]$ points respectively.

## CASE III

Given

$$|F_{2n}|: \quad r^n s^{n-1} g; \qquad |F'_{n'+m'+1}|: \quad r^{n'} s^{m'} g',$$

where $g$, $g'$ are of order $n^2 + 2n - 1$, $2m'n' + 2m' + 2n' - n'^2 + 1$. The curve $g$ meets $r$, $s$ in $n^2 + 2n - 1$, $n^2 - 1$ points, and $g'$ meets $r$, $s$ in $2m'n' + 4n' - n'^2$, $2m'n' + 2m' - n'^2$ points respectively.

In $T^{-1}$ $(T)$ $A$ is a fundamental point of the second (first) kind with image $C'_{n+n'}$ $(u)$. For some point $D$ on a line $\overline{P'A}$ of the pencil $(Au)$, the associated $F$ is the one determined by the direction $\overline{P'A}$; thus $D \sim (T^{-1})\overline{P'A}$. The locus of $D$ is a curve $\delta_{m'-n'+1}$: $A^{m'-n'}$ such that $\delta \sim (T^{-1})u$.

Since $[r, \delta] = (m' - n' + 2)$ points aside from $A$, $R$: $(m' - n' + 2)$ lines of the pencil $(Au)$, hence $R$: $A^{n+m'+2}$. The image $(T^{-1})$ of $A$ as a point on $s$ is $C_{n+n'+1}$ and the $(m' - n')$ tangents to $\delta$ at $A$, hence $S$: $A^{n+m'+1}$.

For the $(2m' - 2n' + 1)$ points, aside from those on $r$, in which $g'$ meets $u$, $t$ becomes a line of the pencil $(Au)$. Therefore $\bar{g}_x$: $A^{2m'-2n'+1}$ and $[g', \delta] = (2m' - 2n' + 1)$ points.

The table of characteristics for $T^{-1}$ is

$$\begin{array}{rrllllll}
\pi' \sim & \phi_{2n+n'+m'+3}: & A^{n+m'+1} & r^{n+n'+1} & s^{n+m'+1} & g & \bar{g}_x, & \\
K \sim & K_{2n+n'+m'+1}: & A^{n+m'} & r^{n+n'} & s^{n+m'-1} & g & \bar{g}_x & g' \quad \bar{g}_y'\delta, \\
r \sim & R_{2n+n'+m'+2}: & A^{n+m'+2} & r^{n+n'} & s^{n+m'+1} & g & \bar{g}_x & C_{n+n'+1}, \\
s \sim & S_{2n+n'+m'+2}: & A^{n+m'+1} & r^{n+n'+1} & s^{n+m'} & g & \bar{g}_x & C_{n+n'+1}, \\
g' \sim & G_{2n'+2m'+3}: & A^{2m'+2} & r^{2n'+1} & s^{2m'+2} & g' & \bar{g}_x, & \\
\bar{g}_y \sim & G'_{4n}: & A^{2n} & r^{2n} & s^{2n} & g & \bar{g}'_y, & \\
\delta \sim & u: & A & r & \delta, & & & \\
\end{array}$$

$$J \equiv uRSGG',$$

where $y=n^2+2m'n+4n+1$. The curve $\bar{g}_y'$ meets $r$, $s$ in $y$, $y-2n$ points respectively. The equations of $T^{-1}$ are $\tau x=Ry_i-Kz_i=Sy_i+Kw_i$.

The table of characteristics for $T$ is

$$\begin{array}{rrlllllll}
\pi \sim & \phi'_{2n+n'+m'+3}: & r^{n+n'+1} & s^{n+m'+1} & g' & \bar{g}'_y & \delta, & \\
r \sim & R_{2n+n'+m'+2}: & r^{n+n'} & s^{n+m'+1} & g' & \bar{g}'_y & C'_{n+n'} & [u, v]\delta, \\
s \sim & S'_{2n+n'+m'+1}: & r^{n+n'} & s^{n+m'} & g' & \bar{g}'_y & C'_{n+n'}, & \\
\end{array}$$

$$g \sim G'_{4n}, \qquad \bar{g}_x \sim G_{2n'+2m'+3},$$

$$A \sim u: \quad Ar\delta, \qquad J' \equiv u^2R'S'G'G,$$

where $x=2m'n'+2m'n-n'^2+3m'+n'+2n+4$. The curve $\bar{g}_x$ meets $r$, $s$ in $x-(2m'-2n'+1)$, $x-(2m+2)$ points respectively. The equations of $T$ are $\tau'y=R'x_i+Kz_i'=S'ux_i-Kw_i'$.

In each of the three cases there exists a monoidal transformation in the plane $w$. The space transformations are generated by allowing the vertex to describe the conic $r$, and the plane to generate the pencil on $s$.

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# REFLEXIVE BANACH SPACES NOT ISOMORPHIC TO UNIFORMLY CONVEX SPACES[1]

MAHLON M. DAY[2]

Clarkson[3] introduced the notion of uniform convexity of a Banach space: $B$ is uniformly convex if for each $\epsilon$ with $0<\epsilon\leqq 2$ there is a $\delta(\epsilon)>0$ such that whenever $\|b\|=\|b'\|=1$ and $\|b-b'\|\geqq\epsilon$, then $\|b+b'\|\leqq 2(1-\delta(\epsilon))$. Milman[4] and Pettis[5] have demonstrated that any uniformly convex space is reflexive;[6] that is, that for each $\beta\in B^{**}$ there is a $b\in B$ with $\beta(f)=f(b)$ for every $f\in B^*$. The same result clearly holds if $B$ is not uniformly convex but can be given a new norm defining the same topology under which the space is uniformly convex. It has been conjectured that every reflexive space can be given such a topologically equivalent uniformly convex norm; that is, that, in Banach's terminology,[7] every reflexive space is isomorphic to a uniformly convex space. We shall show by a large class of examples that this is not the case; in fact the following result holds:

THEOREM 1. *There exist Banach spaces which are separable, reflexive, and strictly convex,*[8] *but are not isomorphic to any uniformly convex space.*

We shall start with a class of Banach spaces and pick out a simple example having all but the strict convexity property; with this as a sample of what can happen we easily find a large number of spaces satisfying all the conditions of the theorem. As an application of our results we show that certain ergodic theorems of Alaoglu and Birkhoff[9] can be extended to some non-uniformly convex spaces.

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[1] Presented to the Society, November 23, 1940, under the title *Some more uniformly convex spaces.*

[2] Corinna Borden Keen Research Fellow of Brown University.

[3] J. A. Clarkson, *Uniformly convex spaces*, Transactions of this Society, vol. 40 (1936), pp. 396–414.

[4] D. Milman, *On some criteria for the regularity of spaces of the type* ($B$), Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS, new series, vol. 20 (1938), pp. 243–246.

[5] B. J. Pettis, *A proof that every uniformly convex space is reflexive*, Duke Mathematical Journal, vol. 5 (1939), pp. 249–253.

[6] That is, "regular" in the terminology of Hahn. We use $B^*$ to denote the conjugate space of the Banach space $B$; $B^{**}=(B^*)^*$.

[7] S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, chap. 11.

[8] $B$ is strictly convex if the set of points with $\|b\|=1$ contains no line segments; that is, if $\|b\|=\|b'\|=1$ and $b''=tb+(1-t)b'$ with $0<t<1$, then $\|b''\|<1$.

[9] L. Alaoglu and G. Birkhoff, *General ergodic theorems*, Annals of Mathematics, (2), vol. 41 (1940), pp. 293–309.

Let $\{B_i\}$, $i=1, 2, \cdots$, be a sequence of Banach spaces and let $p$ be a real number $>1$. Let $B=\mathfrak{P}^p\{B_i\}$ be the class of all those sequences $\mathbf{b}=\{b_i\}$ with $b_i \in B_i$ and $\|\mathbf{b}\|=(\sum_{i<\infty}\|b_i\|^p)^{1/p}<\infty$. It is easily verified that, with this norm and the obvious definitions of vector operations, $B$ is a Banach space.

LEMMA. *$\mathfrak{P}^p\{B_i\}$ is reflexive (strictly convex) if and only if all $B_i$ are reflexive (strictly convex); in fact, $(\mathfrak{P}^p\{B_i\})^*=\mathfrak{P}^{p'}\{B_i^*\}$, where $1/p+1/p'=1$.*

For reflexivity we need only prove the last statement. If $\mathbf{f}$ is linear on $B=\mathfrak{P}^p\{B_i\}$, let $\mathbf{b}_i$ be any element of $B$ of the form $(0, 0, \cdots; b_i, 0, 0, \cdots)$ and define $f_i \in B_i^*$ by $f_i(b_i)=\mathbf{f}(\mathbf{b}_i)$; then there exist $b_i \in B_i$ with $\|b_i\|=1$ and $\|f_i\|/2 \leqq f_i(b_i) \leqq \|f_i\|$ for all $i$. Since $\|b_i\|=1$ the sequence $\{\alpha_i b_i\}$ is in $\mathfrak{P}^p\{B_i\}$ if and only if the sequence of real numbers $\{\alpha_i\}$ is in $l^p$. If $\mathbf{b}^j=(\alpha_1 b_1, \alpha_2 b_2, \cdots, \alpha_j b_j, 0, 0, \cdots)$ and $\mathbf{b}=\{\alpha_i b_i\}\in\mathfrak{P}^p\{B_i\}$, then $\|\mathbf{b}^j-\mathbf{b}\|\rightarrow 0$ so $\mathbf{f}(\mathbf{b}^j)\rightarrow\mathbf{f}(\mathbf{b})$; clearly $\mathbf{f}(\mathbf{b}^j)=\sum_{i\leqq j}\alpha_i f_i(b_i)$, so the series $\sum_{i<\infty}\alpha_i f_i(b_i)$ converges if $\sum|\alpha_i|^p<\infty$. Hence the sequence $\{f_i(b_i)\}\in l^{p'}$ and therefore the sequence $\{\|f_i\|\}\in l^{p'}$; so to each $\mathbf{f}\in[\mathfrak{P}^p\{B_i\}]^*$ we have associated a point $\{f_i\}$ of $\mathfrak{P}^{p'}\{B_i^*\}$ such that for any $\mathbf{b}=\{b_i\}$, $\mathbf{f}(\mathbf{b})=\sum_i f_i(b_i)$. Conversely if $\{f_i\}\in\mathfrak{P}^{p'}\{B_i^*\}$,

$$\sum_{i<\infty}|f_i(b_i)| \leqq \sum_{i<\infty}\|f_i\|\cdot\|b_i\| \leqq \left(\sum_i\|f_i\|^{p'}\right)^{1/p'}\left(\sum_i\|b_i\|^p\right)^{1/p}$$

by Hölder's inequality; so this sequence $\{f_i\}$ defines an $\mathbf{f}\in(\mathfrak{P}^p\{B_i\})^*$, with $\mathbf{f}(\mathbf{b})=\sum_i f_i(b_i)$; clearly $\|\mathbf{f}\|=(\sum_i\|f_i\|^{p'})^{1/p'}$, so we have completed our proof of reflexivity.

The proof of strict convexity is straightforward if we notice that strict convexity is equivalent to what Krein and Šmulian[10] call "strict normalization"; $B$ is strictly normalized if $\|b+b'\|=\|b\|+\|b'\|$ and $\|b'\|\neq 0$ imply $b=tb'$, $t\geqq 0$.

We now make a simple choice of the spaces $B_i$ to get an example of a reflexive space not isomorphic to a uniformly convex space. Let $B_i$ be the $i$-dimensional Minkowski space of points $b_i=(b_{i1}, \cdots, b_{ii})$ in which $\|b_i\|=\sup_{j\leqq i}|b_{ij}|$; then each $B_i$ is reflexive since it is finite dimensional, so $B=\mathfrak{P}^p\{B_i\}$ is reflexive, by our lemma, for any choice of $1<p<\infty$. This $B$ can not be given a uniformly convex norm defining the same topology.

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[10] V. Šmulian, *On some geometrical properties of the unit sphere in the space of the type* (*B*), Recueil Mathematique (i.e., Matematicheskii Sbornik), new series, vol. 6 (48) (1940), pp. 90–94.

Suppose that there is such a norm—we shall represent it by $|b|$—defined in $B$. For $0<\epsilon\leqq 2$ we define $\delta(\epsilon)=\frac{1}{2}\inf\,(2-|b+b'|)$, where the inf is taken over $b$, $b'\in B$ with $|b|\leqq 1\geqq |b'|$ and $|b-b'|\geqq\epsilon$; it can easily be shown that $\delta(\epsilon)>0$ for all such $\epsilon$ if and only if the norm $|b|$ is uniformly convex. For the topologies to be the same it is clear that there must exist $m$, $M$ with $0<m\leqq M<\infty$ such that $m\|b\|\leqq |b| \leqq M\|b\|$; without loss of generality we can make $m=1$ by a change of scale of $|b|$ and get $\|b\|\leqq|b|\leqq M\|b\|$ with $1\leqq M<\infty$. If we fix any $i$ this must also hold in $B_i$ since $B_i$ is a closed linear subspace in $B$; so, if we let $K_i$ be the "sphere" of those points of $B_i$ with $|b_i|\leqq 1$, we see that $K_i$ is contained in the unit cube, $E_{b_i}[\sup_{j\leqq i}|b_{ij}|\leqq 1]$, and contains the cube $E_{b_i}[\sup_{j\leqq i}|b_{ij}|\leqq 1/M]$. If we let $\alpha=1/M$, we know that the points $(\epsilon_1\alpha, \epsilon_2\alpha, \cdots, \epsilon_i\alpha)$, $\epsilon_j=\pm 1$, all lie in $K_i$. Considering, for example, $b=(\alpha, \cdots, \alpha, \alpha)$ and $b'=(\alpha, \cdots, \alpha, -\alpha)$, we have $b-b'=(0, \cdots, 0, 2\alpha)$, and $(b+b')/2=(\alpha, \cdots, \alpha, 0)$. Now $|b-b'| \geqq \|b-b'\|=2\alpha$, so $\frac{1}{2}|b+b'|\leqq 1-\delta(2\alpha)$; that is, $|(\alpha, \cdots, \alpha, 0)|\leqq 1-\delta(2\alpha)$ or $|(\alpha/(1-\delta(2\alpha)), \cdots, \alpha/(1-\delta(2\alpha)), 0)|\leqq 1$. This could have been done with any choice of $\epsilon_j$, $j\leqq i-1$, so we can repeat $i-2$ more times, noting that $1-\delta(2\alpha)<1$ and that $\delta(\epsilon)$ decreases as $\epsilon$ decreases, so that $\delta[\alpha/(1-\delta(2\alpha))^j]\geqq\delta(2\alpha)$, $j=1, \cdots, i-1$; setting $b_i=(\alpha/(1-\delta(2\alpha))^{i-1}, 0, \cdots, 0)$, we have $|b_i|\leqq 1$, but $|b_i|\geqq\|b_i\|>1$ if $i$ is large enough. This contradiction proves our theorem.

Since $B$ is clearly separable if all $B_i$ are, we could apply a theorem of Clarkson to show that an example satisfying all the conditions including strict convexity exists. However we need only take $B_i$ to have an $l^{q_i}$ norm instead of an $(m)$ norm and if $q_i\rightarrow\infty$ fast enough ($q_i=i$ will do) then this new example is isomorphic to the old one while, by our lemma, it is strictly convex.

It is clear from the nature of our construction that if we had taken $B_i$ with the $l^1$ norm instead of the $(m)$ norm a similar contradiction would have appeared if we tried to fit in a uniformly convex norm (just for simplicity in the construction it is easier to take the dimension of $B_i$ to be $2^i$ in this case). Finally the finite dimensionality of the $B_i$ was used essentially only in proving reflexivity; this and the existence of subspaces of large dimension are our main requirements. Let $l^q_\mu$, $\mu$ a cardinal number $\geqq 0$, $1<q<\infty$, be the $\mu$-dimensional space with $l^q$ norm; in this notation the ordinary $l^q$ spaces would be $l^q_{\aleph_0}$. Using our example as a guide we have the following result:

THEOREM 2. *The space* $\mathfrak{P}^p\{l^{q_i}_{\mu_i}\}$ *is not isomorphic to a uniformly convex space, provided there exists a sequence* $\{i_j\}$ *of indices for which* $\lim_j \mu_{i_j}\geqq\aleph_0$ *while at the same time either* $\lim_j q_{i_j}=+\infty$ *or* $\lim_j q_{i_j}=1$.

In the first case we can show that $\mathfrak{P}^p\{l^{q_i}_{\mu_i}\}$ has a subspace isomorphic to the example first given while in the second case it contains a subspace isomorphic to the example suggested in the paragraph preceding this theorem. By the same sort of embedding argument we can clearly get the same result for $\mathfrak{P}^p\{L^{q_i}\}$ if the $q_i$ are not bounded away from one and infinity.

A recent result of Boas is that $\mathfrak{P}^p\{l^{q_i}\}$ is uniformly convex if $q_i=q$ for all $i$ and $1<q<\infty$. It is not yet known whether $\mathfrak{P}^p\{l^{q_i}\}$ can be made uniformly convex, or is already so, if the $q_i$ are bounded away from 1 and $\infty$; i.e., if there exist $m$, $M$ such that $1<m\leqq q_i\leqq M<\infty$ for all $i$.

**An application to ergodic theory.** Milman and Alaoglu and Birkhoff have use for the following lemma:

*If K is a closed, convex, non-empty subset of a uniformly convex space, then there is a unique point in K of least norm.*

From a theorem of Mazur[11] that any closed, convex set is closed under weak convergence we have the following result.

LEMMA. *If K is a closed convex non-empty set in a Banach space whose unit sphere is weakly compact,*[12] *there is at least one point in K of least norm. If B is also strictly convex, then there is just one such point.*

From this it follows that the general ergodic theorems depending on the minimal methods of proof of Alaoglu and Birkhoff hold in more spaces than the uniformly convex ones; we note the extensions of one or two of their theorems (we number them correspondingly); that the extensions are real and not merely formal follows from our existence theorem.

THEOREM 5. *If $G$ is abelian, if $B$ is strictly convex and has unit sphere weakly compact, and if the norms of the operators $T_g$ are at most one. then every element of $B$ is ergodic.*

THEOREM 6. *If the norms of the operators $T_g$ are at most one, if the unit sphere in $B$ is weakly compact, strictly convex and without sharp edges, then every element of $B$ is ergodic.*

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[11] S. Mazur, *Über konvexe Mengen in linearen normierten Raümen*, Studia Mathematica, vol. 4 (1933), pp. 70–84.

[12] The unit sphere of $B$ is weakly compact if given any sequence $\{b_n\}$ of points with $\|b_n\|\leqq 1$ there is a subsequence $\{b_{n_i}\}$ and a point $b_0$ (clearly of norm not greater than 1) for which $\lim_i f(b_{n_i})=f(b_0)$ for all $f\in B^*$.

COROLLARY. *If $B$ is any $\mathfrak{P}^p\{l^{q_i}_{\mu_i}\}$ or $\mathfrak{P}^p\{L^{q_i}\}$, $1<p<\infty$, $1<q_i<\infty$, $\mu_i$ any cardinal $\geqq 0$, then every element of $B$ is ergodic.*

Theorem 6 can be stated in a more symmetric form.

THEOREM 6′. *If the norms of the operators $T_g$ are at most one, and if the unit spheres in both $B$ and $B^*$ are weakly compact and strictly convex, then every element of $B$ is ergodic.*

By a theorem of Gantmakher and Šmulian[13] weak compactness of the unit sphere in both $B$ and $B^*$ follows from the weak compactness of either unit sphere; so this can be weakened slightly. Alaoglu and Birkhoff in a footnote remark that the condition that the unit sphere in $B$ have no sharp edges is equivalent to the condition that the unit sphere in $B^*$ be strictly convex. Note that a theorem of Milman asserts that a Banach space with a weakly compact and strictly convex unit sphere is reflexive.

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[13] V. Gantmakher and V. Šmulian, *Sur les espaces linéaires dont la sphère unitaire est faiblement compact*, Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS, new series, vol. 17 (1937), pp. 91–94.

# DIFFERENTIAL AND INTEGRAL INVARIANTS OF PLANE CURVES AND HORN ANGLES[1]

J. M. FELD

**1. Introduction.** A horn angle has been defined by E. Kasner as a figure formed by an ordered pair of oriented analytic arcs which pass through a common point in a common direction and which have contact of order $n \geqq 1$ at the common point. The order of contact is said to be the order of the horn angle. Thus, in a horn angle of the first order the arcs forming its sides have, at the vertex of the angle, the same inclination (direction) but different curvatures. Horn angles of the first order have a unique absolute conformal invariant, first found by Kasner.[2] Two relative conformal invariants of horn angles of the first order were given later in a joint paper by Kasner and Comenetz.[3] Kasner has also determined conformal invariants of the fifth order for horn angles of second order contact.[4]

Kasner and Comenetz obtained the above-mentioned invariants by the use of power series and by repeated differentiation of the Cauchy-Riemann equations. It is shown in §2 of this paper that the relative and absolute conformal invariants of horn angles of the first order can be obtained directly from two standard formulas; one of these expresses $d^2y/dx^2$ in terms of $u$ and $v$ where $x=x(u)$ and $y=y(v)$; the other expresses the Schwarzian derivative

$$\{y, x\} = \frac{d^3y}{dx^3}\Big/\frac{dy}{dx} - \frac{3}{2}\left(\frac{d^2y}{dx^2}\Big/\frac{dy}{dx}\right)^2$$

in terms of $u$ and $v$ under the same conditions. As a consequence, the invariants are presented in a form which permits of their being interpreted as invariants, not only under the conformal group, but also under a group of multiply differentiable transformations of the type: $x=x(u)$, $y=y(v)$, where $x$ is a function of $u$ alone, and $y$ is a function of $v$ alone.[5]

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[1] Presented to the Society, February 25, 1939.

[2] E. Kasner, *Conformal geometry*, Proceedings of the Fifth International Congress of Mathematicians, Cambridge, 1912, vol. 2, p. 81.

[3] E. Kasner and G. Comenetz, *Conformal geometry of horn angles*, National Academy of Sciences, vol. 22 (1936), p. 303. See also Comenetz, *Kasner's invariant and trihornometry*, American Mathematical Monthly, vol. 45 (1938), p. 28.

[4] Kasner, Transactions of this Society, vol. 44 (1938), p. 25.

[5] Transformations of this type occur, for example, in the papers: Kasner, Transactions of this Society, vol. 16 (1915), pp. 333–349; Cayley, *Collected Works*, vol. 11, p. 148.

In §3 we obtain, in terms of what we define as *higher Schwarzian derivatives*, an infinitude of differential and *integral* invariants under the inversive group. The fundamental differential inversive invariant, which is of the fifth order, was first obtained by G. Mullins[6] and later, using other methods, by Liebmann, Kubota, Morley, Patterson and Kasner.[7] The use of higher Schwarzian derivatives enables us to obtain simply and in compact form an infinitude of relative and absolute differential and integral invariants of curves.

In §4 we derive inversive invariants of horn angles of order $n \geqq 3$.

**2. Conformal invariants of horn angles of the first order.** Let the transformation

$$T: \qquad z = \phi(u), \quad \bar{z} = \bar{\phi}(\bar{u}), \quad z = x + iy, \quad \bar{z} = x - iy$$

map a region $r$ of the gaussian $z$-plane conformally on the region $r'$ of the $u$-plane. Let the analytic arcs[8] $c_1$ and $c_2$ in the region $r$ be the first and second sides respectively of a horn angle of the first order, $\alpha$, the vertex of which is at the point $P$. Let $T$ map $c_1$ and $c_2$ upon the arcs $c_1'$ and $c_2'$ respectively in $r'$ where the latter form a horn angle of the first order, $\alpha'$, with vertex at $P'$. Let $\theta$ represent the common angle of inclination at $P$ of $c_1$ and $c_2$, $k_1$ and $k_2$ their respective curvatures at $P$, while $dk_1/ds_1$ and $dk_2/ds_2$ represent respectively the values at $P$ of the derivative of curvature with respect to arc-length. The same symbols primed represent the corresponding quantities associated with $c_1'$ and $c_2'$ at $P'$.

Let $f(z, \bar{z}) = 0$ be the self-conjugate equation (in isotropic coordinates) of a curve $c$ in the $z$-plane. Likewise, let arcs $c_1$ and $c_2$ be represented by $f_1(z, \bar{z}) = 0$, $f_2(z, \bar{z}) = 0$, respectively, and let their images $c'$, $c_1'$, $c_2'$ on the $u$-plane be given by $g(u, \bar{u}) = 0$, $g_1 = 0$, $g_2 = 0$, respectively. By $d^nz/d\bar{z}^n$, $d^nz_1/d\bar{z}_1^n$, $d^nz_2/d\bar{z}_2^n$ we designate the $n$th derivatives taken along $c$, $c_1$ and $c_2$ respectively. Likewise $d^nu/d\bar{u}^n$, $d^nu_1/d\bar{u}_1^n$ and $d^nu_2/d\bar{u}_2^n$ represent respectively derivatives along $c'$, $c_1'$ and $c_2'$ in the $u$-plane.

Since

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[6] Columbia University Dissertation, 1917.

[7] H. Liebmann, *Beiträge zur Inversionsgeometrie der Kurven*, Sitzungsberichte der Bayerischen Akademie der Wissenschaften, Munich, vol. 1 (1923), p. 79. T. Kubota, *Beiträge zur Inversionsgeometrie*, Tôhoku Imperial University Science Reports, vol. 13 (1924–1925), p. 243. F. Morley, *On differential inversive geometry*, American Journal of Mathematics, vol. 46 (1926), p. 144. B. Patterson, *The differential invariants of inversive geometry*, ibid., vol. 50 (1928), p. 553. E. Kasner, Transactions of this Society, vol. 44 (1938), p. 25.

[8] This restriction can be relaxed to require only that the arcs possess derivatives of the third order.

$$\frac{dz}{d\bar z} = \left(\frac{dz}{du}\Big/\frac{d\bar z}{d\bar u}\right)\frac{du}{d\bar u} \tag{2.1}$$

we obtain

$$\begin{aligned}\frac{d^2z}{d\bar z^2} = \left(1\Big/\frac{d\bar z}{d\bar u}\right)\Bigg[\left(\frac{dz}{du}\Big/\frac{d\bar z}{d\bar u}\right)&\frac{d^2u}{d\bar u^2}\\ &+\left(\frac{du}{d\bar u}\right)^2\frac{d}{du}\left(\frac{dz}{du}\Big/\frac{d\bar z}{d\bar u}\right)\Bigg].\end{aligned} \tag{2.2}$$

Evidently the derivatives of $z$ with respect to $u$ as well as of $\bar z$ with respect to $\bar u$ are independent of the particular curve $c$; furthermore, $\alpha'$ being a horn angle of the first order, $du_1/d\bar u_1 = du_2/d\bar u_2$ at $P'$; consequently, by applying equation (2.2) to $c_i$ and $c_i'$, and subtracting the results, we get

$$\frac{d^2z_2}{d\bar z_2^2} - \frac{d^2z_1}{d\bar z_1^2} = \left[\frac{dz}{du}\Big/\left(\frac{d\bar z}{d\bar u}\right)^2\right]\left[\frac{d^2u_2}{d\bar u_2^2} - \frac{d^2u_1}{d\bar u_1^2}\right]. \tag{2.3}$$

Since $dz_1/d\bar z_1 = dz_2/d\bar z_2$ at $P$, division of equation (2.3) by equation (2.1) applied to $c_i$ and $c_i'$ results in

$$\begin{aligned}&\frac{d^2z_2}{d\bar z_2^2}\Big/\frac{dz_2}{d\bar z_2} - \frac{d^2z_1}{d\bar z_1^2}\Big/\frac{dz_1}{d\bar z_1}\\ &\qquad = \left(1\Big/\frac{d\bar z}{d\bar u}\right)\left[\frac{d^2u_2}{d\bar u_2^2}\Big/\frac{du_2}{d\bar u_2} - \frac{d^2u_1}{d\bar u_1^2}\Big/\frac{du_1}{d\bar u_1}\right].\end{aligned} \tag{2.4}$$

Consequently, *the left members of* (2.3) *and* (2.4) *represent complex relative conformal invariants of the second order of horn angles of the first order.*

When the variables in the Schwarzian derivative $\{y, x\}$ undergo a transformation $x = x(s)$, $y = y(t)$, it was proved by Cayley[9] that

$$\{y, x\} = \left(\frac{dt}{dx}\right)^2\{y, t\} - \left(\frac{ds}{dx}\right)^2\{x, s\} + \left(\frac{ds}{dx}\right)^2\{t, s\}. \tag{2.5}$$

In isotropic coordinates this becomes

$$\{z, \bar z\} = \left(1\Big/\frac{d\bar z}{d\bar u}\right)^2\left[\{u, \bar u\} - \{\bar z, \bar u\} + \{z, u\}\left(\frac{du}{d\bar u}\right)^2\right] \tag{2.5'}$$

showing the effect of a direct conformal transformation on $\{z, \bar z\}$.

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[9] A. Cayley, "On the Schwarzian derivative and polyhedral functions," *Collected Works*, vol. 11, p. 148.

From (2.5′) we find that, for the horn angles $\alpha$ and $\alpha'$ at points $P$ and $P'$,

$$\{z_2, \bar{z}_2\} - \{z_1, \bar{z}_1\} = \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)^2 \left[\{u_2, \bar{u}_2\} - \{u_1, \bar{u}_1\}\right]. \tag{2.6}$$

Consequently, *horn angles of the first order possess the complex relative conformal invariant of the third order given by the left member of* (2.6).

From the relative invariants (2.4) and (2.6) of index one and two respectively we obtain the *absolute invariant of the third order*

$$M_{12} = \frac{\left(\dfrac{d^2z_2}{d\bar{z}_2^2}\Big/\dfrac{dz_2}{d\bar{z}_2} - \dfrac{d^2z_1}{d\bar{z}_1^2}\Big/\dfrac{dz_1}{d\bar{z}_1}\right)^2}{\{z_2, \bar{z}_2\} - \{z_1, \bar{z}_1\}}.$$

A simple computation indicates that

$$\begin{aligned} &\frac{dz}{d\bar{z}} = e^{2i\theta}, \qquad \frac{d^2z}{d\bar{z}^2} = 2ie^{3i\theta}k, \\ &\frac{d^3z}{d\bar{z}^3} = 2ie^{4i\theta}\left(\frac{dk}{ds} + 3ik^2\right). \end{aligned} \tag{2.7}$$

Substitution of the values given by (2.7) in (2.4) and (2.6) shows that each of these complex relative invariants implies one real relative invariant, namely,

$$k_2 - k_1 \quad \text{and} \quad dk_2/ds_2 - dk_1/ds_1$$

respectively. In this form they were given in a joint paper by Kasner and Comenetz.[10]

Substitution from (2.7) in $M_{12}$ leads us likewise to a single real invariant, namely, Kasner's unique absolute conformal invariant of the third order of horn angles of the first order:

$$M_{12}^* = \frac{(k_2 - k_1)^2}{dk_2/ds_2 - dk_1/s_1}, \qquad M_{12} = 2iM_{12}^*.$$

Comenetz[11] has proved that if a conformal transformation, $S$:

$$x_2 = \phi(x_1, y_1), \qquad y_2 = \psi(x_1, y_1)$$

maps a region $r_1$ in the $x_1y_1$-plane on a region $r_2$ in the $x_2y_2$-plane, so that a curve $c_1$ in $r_1$ is mapped on $c_2$ in $r_2$, then

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[10] Kasner and Comenetz, National Academy of Sciences, loc. cit.

[11] American Mathematical Monthly, vol. 45 (1938), p. 86.

$$k_2 = mk_1 + m_{y_1}\cos\theta_1 - m_{x_1}\sin\theta_1$$

and

$$dk_2/ds_2 = m^2 dk_1/ds_1 + m\left[m_{x_1y_1}\cos 2\theta_1 + \tfrac{1}{2}(m_{y_1y_1} - m_{x_1x_1})\sin 2\theta_1\right],$$

where $m = 1/(J)^{1/2}$ and $J$ is the jacobian of $S$. These formulas, describing the laws of transformation of curvature and rate of change of curvature under conformal mapping, are equivalent, except for factors, to (2.2) and (2.5′) respectively. Consequently, Comenetz's formulas are obtainable by appropriately interpreting standard identities that express the second derivative and Schwarzian derivative in terms of new variables. This remark enables us to regard (2.4), (2.6) and $M_{12}$ as invariants of horn angles under a group other than the conformal group.

Explicitly, let the curves $f_1(x, y) = 0$, $f_2(x, y) = 0$ form a horn angle of the first order in a region $R$ of the $xy$-plane, $x$, $y$ being cartesian coordinates. Let the transformation $K$: $x = x(s)$, $y = y(t)$, map $R$ uniformly and continuously on a region $R'$ of the $st$-plane. We assume that $K$ is provided with at least third derivatives in $R$. Then $K$ maps the horn angle in $R$ on a horn angle of the first order in $R'$. If we now write equations (2.1) and (2.2) in terms of $x$, $y$ and $s$, $t$, thereby expressing the transformations that $dy/dx$ and $d^2y/dx^2$ undergo under $K$, just as (2.5) does for $\{y, x\}$, we obtain the result that horn angles of the first order, under a group of transformations of type $K$, have the relative[12] invariants

$$A_{12} = \frac{d^2y_2}{dx_2^2}\Big/\frac{dy_2}{dx_2} - \frac{d^2y_1}{dx_1^2}\Big/\frac{dy_1}{dx_1}, \qquad B_{12} = \{y_2, x_2\} - \{y_1, x_1\}$$

and the absolute invariant $A_{12}^2/B_{12}$.

## 3. Higher Schwarzian derivatives and inversive invariants of curves.

The inversive group in the gaussian plane is given by

$$z = (au + b)/(cu + d), \qquad z = \bar{u}, \qquad ad - bc \neq 0.$$

The first of these equations represents a subgroup $G$, containing only directly conformal inversive transformations. We first consider the transformations of $G$. Let any member of $G$ transform the curve $C$:

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[12] The term *relative* invariants as used here and elsewhere in this paper, means a quantity that is multiplied by some function which depends only upon the transformation; however, contrary to usual practice, the multiplier need not be a power of the jacobian of the transformation.

$f(z, \bar{z})=0$ into $C': g(u, \bar{u})=0$. Then, inasmuch as for such transformations $\{z, u\}=\{\bar{z}, \bar{u}\}=0$, (2.5′) reduces to

$$\{z, \bar{z}\} = \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)^2 \{u, \bar{u}\} \tag{3.1}$$

from which we gather that $\{z, \bar{z}\}$ *is a relative invariant of curves under* $G$. Noting that $d\bar{z}=(d\bar{z}/d\bar{u})d\bar{u}$, we obtain

$$\{z, \bar{z}\}d\bar{z}^2 = \{u, \bar{u}\}d\bar{u}^2$$

indicating that $\{z, \bar{z}\}d\bar{z}^2$ is an absolute invariant under $G$. Since

$$\{z, \bar{z}\} = 2ie^{2i\theta}\frac{dk}{ds}, \qquad d\bar{z} = e^{-i\theta}ds,$$

we get the real absolute invariant

$$(-1/2i)\{z, \bar{z}\}d\bar{z}^2 = -dkds.$$

If we let $d\lambda^2 = \pm dkds$, choosing the sign that makes $d\lambda^2$ positive, the integral invariant $\lambda$ represents what has come to be known as the inversive arc-length.[13] Letting $\rho$ represent radius of curvature,

$$-dkds = d\rho ds/\rho^2,$$

under which form this invariant was found, by other means, by Liebmann and Kubota.

Differentiating (3.1) twice, we obtain

$$\frac{d}{d\bar{z}}\{z, \bar{z}\} = \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)^3 \frac{d}{d\bar{u}}\{u, \bar{u}\} - 2\{u, \bar{u}\}\frac{d^2\bar{z}}{d\bar{u}^2}\Big/\left(\frac{d\bar{z}}{d\bar{u}}\right)^4 \tag{3.2}$$

and

$$\begin{aligned} \frac{d^2}{d\bar{z}^2}\{z, \bar{z}\} = {} & \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)^4 \frac{d^2}{d\bar{u}^2}\{u, \bar{u}\} \\ & - 5\frac{d}{d\bar{u}}\{u, \bar{u}\}\frac{d^2\bar{z}}{d\bar{u}^2}\Big/\left(\frac{d\bar{z}}{d\bar{u}}\right)^5 \\ & - 2\{u, \bar{u}\}\left[\frac{d^3\bar{z}}{d\bar{u}^3}\Big/\left(\frac{d\bar{z}}{d\bar{u}}\right)^5 \right. \\ & \left. -4\left(\frac{d^2\bar{z}}{d\bar{u}^2}\right)^2\Big/\left(\frac{d\bar{z}}{d\bar{u}}\right)^6\right]. \end{aligned} \tag{3.3}$$

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[13] See, e.g., Patterson, American Journal of Mathematics, loc. cit.

From (3.1), (3.2) and (3.3) we then obtain, recalling that $\{z, u\} = 0$,

$$\begin{aligned}\frac{d^2}{d\bar{z}^2}\{z, \bar{z}\}/\{z, \bar{z}\} - \frac{5}{4}\left[\frac{d}{d\bar{z}}\{z, \bar{z}\}/\{z, \bar{z}\}\right]^2 \\ = \left(1\Big/\frac{d\bar{z}}{d\bar{u}}\right)^2\Bigg[\frac{d^2}{d\bar{u}^2}\{u, \bar{u}\}/\{u, \bar{u}\} \\ - \frac{5}{4}\left(\frac{d}{d\bar{u}}\{u, \bar{u}\}/\{u, \bar{u}\}\right)^2\Bigg].\end{aligned} \tag{3.4}$$

Let the left member of (3.4) be represented by the symbol $\{z, \bar{z}\}_2$, which we shall call *the second Schwarzian derivative of z with respect to* $\bar{z}$. It follows from (3.4) that $\{z, \bar{z}\}_2$ is a relative invariant of the fifth order under the direct inversive group. In our new notation (3.4) becomes

$$\{z, \bar{z}\}_2 = \left(1\Big/\frac{d\bar{z}}{d\bar{u}}\right)^2\{u, \bar{u}\}_2. \tag{3.4'}$$

Generalizing the definition of the second Schwarzian derivative by a process of recurrence, we shall call

$$\{z, \bar{z}\}_{n+1} = \frac{d^2}{d\bar{z}^2}\{z, \bar{z}\}_n/\{z, \bar{z}\}_n - \frac{5}{4}\left(\frac{d}{d\bar{z}}\{z, \bar{z}\}_n/\{z, \bar{z}\}_n\right)^2 \tag{3.5}$$

the $(n+1)$*-th Schwarzian derivative.* We let $\{z, \bar{z}\}_1 \equiv \{z, \bar{z}\}$.

It can be shown by induction[14] that *the Schwarzian derivatives of order* $n \geqq 1$ *are all relative invariants of index two under the direct inversive group*, or that

$$\{z, \bar{z}\}_k = \left(1\Big/\frac{d\bar{z}}{d\bar{u}}\right)^2\{u, \bar{u}\}_k, \qquad k \geqq 1. \tag{3.6}$$

Consequently, *the ratios*

$$\{z, \bar{z}\}_j/\{z, \bar{z}\}_k, \qquad j \neq k; j, k = 1, 2, 3, \cdots, \tag{3.7}$$

*are absolute differential invariants of plane curves under G.* Of these, the one of lowest order, namely that of the fifth order, is

$$I_5 = \{z, \bar{z}\}_2/\{z, \bar{z}\}.$$

We proceed to express $I_5$ in terms of $k$ and derivatives of $k$ with respect to $s$.

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[14] The proof of this statement involves straightforward computation, which we shall omit.

From

$$\{z, \bar{z}\} = 2ie^{2i\theta}dk/ds$$

we obtain

$$\frac{d}{d\bar{z}}\{z, \bar{z}\} = 2e^{3i\theta}\left(i\frac{d^2k}{ds^2} - 2k\frac{dk}{ds}\right)$$

and

$$\frac{d^2}{d\bar{z}^2}\{z, \bar{z}\} = 2e^{4i\theta}\left[i\frac{d^3k}{ds^3} - 5k\frac{d^2k}{ds^2} - 6ik^2\frac{dk}{ds} - 2\left(\frac{dk}{ds}\right)^2\right].$$

Substituting these expressions in $\{z, \bar{z}\}_2$, we find

$$\{z, \bar{z}\}_2 = e^{2i\theta}\left[\frac{d^3k}{ds^3}\Big/\frac{dk}{ds} - \frac{5}{4}\left(\frac{d^2k}{ds^2}\Big/\frac{dk}{ds}\right)^2 - k^2 + 2i\frac{dk}{ds}\right] \tag{3.8}$$

and therefore

$$I_6 = 1 - \frac{i}{2}\left[\frac{d^3k}{ds^3}\Big/\left(\frac{dk}{ds}\right)^2 - \frac{5}{4}\left(\frac{d^2k}{ds^2}\right)^2\Big/\left(\frac{dk}{ds}\right)^3 - k^2\Big/\frac{dk}{ds}\right].$$

Hence, $I_6$ implies only one real invariant, namely

$$I_6^* = \frac{d^3k}{ds^3}\Big/\left(\frac{dk}{ds}\right)^2 - \frac{5}{4}\left(\frac{d^2k}{ds^2}\right)^2\Big/\left(\frac{dk}{ds}\right)^3 - k^2\Big/\frac{dk}{ds}.$$

In terms of $\rho$ and derivatives of $\rho$ with respect to $s$,

$$I_6^* = \left[-4\frac{d^3\rho}{ds^3}\frac{d\rho}{ds}\rho^2 + 4\frac{d^2\rho}{ds^2}\left(\frac{d\rho}{ds}\right)^2\rho - 4\left(\frac{d\rho}{ds}\right)^4 \right.$$
$$\left. + 5\left(\rho\frac{d^2\rho}{ds^2}\right)^2 + 4\left(\frac{d\rho}{ds}\right)^2\right]\Big/\left[4\left(\frac{d\rho}{ds}\right)^3\right]$$

in which form this invariant was first obtained by G. Mullins.[15]

Recalling that the inversive arc length $\lambda = \int(\pm dk/ds)^{1/2}ds$ and that therefore $dk/ds = (d\lambda/ds)^2$ we find that $I_6^*$ assumes the simple form

$$I_6^* = (2\{\lambda, s\} - k^2)\Big/\left(\frac{d\lambda}{ds}\right)^2.$$

From (3.6) it obviously follows that

$$\{z, \bar{z}\}_k d\bar{z}^2 = \{u, \bar{u}\}_k d\bar{u}^2, \qquad k \geqq 1.$$

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[15] Columbia University Dissertation, 1917.

Consequently,

$$\lambda_k = \int (\{z, \bar{z}\}_k)^{1/2} d\bar{z}$$

*is an integral invariant under G*. However, $\lambda_k$, being complex, implies two real invariants, one of which may be only a constant. When $k=1$ the unique non-constant invariant, as shown above, is the inversive arc-length.

Under the complete inversive group, which includes the group $G$ and the involutory transformation $z=\bar{u}$, the absolute values of the invariants referred to above are invariant.

4. **Inversive invariants of horn angles.** From (3.6) it obviously follows that curvilinear angles, and therefore horn angles of order $n$, $n \geqq 1$, have, under the direct inversive group $G$, the absolute invariants of odd order, namely $2k+1$:

$$\{z_2, \bar{z}_2\}_k / \{z_1, \bar{z}_1\}_k, \qquad k \geqq 1.$$

However, we can obtain from (3.6) certain other, less obvious horn angle invariants. Let $\alpha$ be a horn angle of order $2n+1$, $n \geqq 1$. Differentiation of (3.6) results in

$$(4.1) \qquad \begin{aligned} \frac{d}{d\bar{z}} \{z, \bar{z}\}_n = \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)^3 \frac{d}{d\bar{u}} \{u, \bar{u}\}_n \\ - 2\{u, \bar{u}\}_n \frac{d^2\bar{z}}{d\bar{u}^2} \Big/ \left(\frac{d\bar{z}}{d\bar{u}}\right)^4. \end{aligned}$$

Since $\alpha$ is of order $2n+1$, $\{z_1, \bar{z}_1\}_n = \{z_2, \bar{z}_2\}_n$, $\{u_1, \bar{u}_1\}_n = \{u_2, \bar{u}_2\}_n$ at the vertex of the horn angle; consequently, by subtracting the equations obtained by stating (4.1) for the two sides of $\alpha$ we find that

$$(4.2) \qquad \begin{aligned} \frac{d}{d\bar{z}_2} \{z_2, \bar{z}_2\}_n - \frac{d}{d\bar{z}_1} \{z_1, \bar{z}_1\}_n \\ = \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)^3 \left[\frac{d}{d\bar{u}_2} \{u_2, \bar{u}_2\}_n - \frac{d}{d\bar{u}_1} \{u_1, \bar{u}_1\}_n\right]. \end{aligned}$$

Therefore, *the left member of* (4.2) *represents a relative inversive invariant of order* $2n+2$ *and index three of horn angles of order* $2n+1$, $n \geqq 1$.

Let $\alpha$ be a horn angle of even order: $2n+2$, $n \geqq 1$. Now it follows from (3.6), by subtraction of (3.5) stated for the two sides of $\alpha$ that

$$(4.3)\qquad \begin{aligned} &\frac{d^2}{d\bar{z}_2^2}\{z_2, \bar{z}_2\}_n - \frac{d^2}{d\bar{z}_1^2}\{z_1, \bar{z}_1\}_n \\ &\qquad = \left(1 \Big/ \frac{d\bar{z}}{d\bar{u}}\right)^4 \left[\frac{d^2}{d\bar{u}_2^2}\{u_2, \bar{u}_2\}_n - \frac{d^2}{d\bar{u}_1^2}\{u_1, \bar{u}_1\}_n\right]. \end{aligned}$$

Consequently, *horn angles of order* $2n+2$, $n \geqq 1$, *have, under the direct inversive group, the relative invariant of order* $2n+3$ *and index four given by* (4.3).

BROOKLYN COLLEGE

# ON A PROPERTY OF $k$ CONSECUTIVE INTEGERS[1]

ALFRED BRAUER

S. S. Pillai[2] has just proved the following theorem: In every set of less than 17 consecutive integers there exists at least one integer which is relatively prime to all the others; there are sequences of $k$ integers for $k=17, 18, \cdots, 430$, however, which have not this property. Pillai conjectures that the same is valid for every $k \geqq 17$. I shall prove that this conjecture is true.

The method of the proof is similar to the method I applied in a joint paper with H. Zeitz[3] in proving that the following conjecture is wrong for every prime $p \geqq 43$.

*Denote by* $p_n$ *the nth prime. Then there exist at most* $2p_{n-1}-1$ *consecutive integers such that each of these integers is at least divisible by one of the primes* $p_1, p_2, \cdots, p_n$.

This conjecture was used by Legendre for his proof of the theorem of the primes in arithmetical progressions. First I prove the following.

Lemma. *Let* $\pi(x)$ *be the number of primes* $p \leqq x$. *Then we have*

$$\pi(2x) - \pi(x) \geqq 2\left[\frac{\log x}{\log 2}\right] + 2 \tag{1}$$

*for every* $x \geqq 75$.

Proof. If we put, as usual,

$$\vartheta(x) = \sum_{p \leqq x} \log p,$$

then we have

$$\begin{aligned}\pi(2x) - \pi(x) &= \sum_{x<p\leqq 2x} 1 \geqq \sum_{x<p\leqq 2x} (\log p/\log 2x) \\ &= \left\{\sum_{x<p\leqq 2x} \log p\right\}/\log 2x = \{\vartheta(2x) - \vartheta(x)\}/\log 2x.\end{aligned} \tag{2}$$

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[1] Presented to the Society, September 12, 1940.

[2] S. S. Pillai, *On m consecutive integers*, Proceedings of the Indian Academy of Sciences, section A, vol. 11 (1940), pp. 6–12.

[3] A. Brauer und H. Zeitz, *Über eine zahlentheoretische Behauptung von Legendre*, Sitzungsberichte der Berliner mathematischen Gesellschaft, vol. 29 (1930), pp. 116–125. Cf. A. Brauer, *Question concerning the maximum term in the diatomic series—proposed by A. A. Bennett*, American Mathematical Monthly, vol. 40 (1933), pp. 409–410.

It is known that[4]

$$\vartheta(2x) - \vartheta(x) \geqq \vartheta(2x-2) - \vartheta(x) > .7x - 3.4x^{1/2} - 4.5\log^2 x - 24\log x - 32. \tag{3}$$

Hence, by (1), (2), and (3), it is sufficient to prove that

$$.7x - 3.4x^{1/2} - 4.5\log^2 x - 24\log x - 32 > (2\log x/\log 2 + 2)(\log x + \log 2),$$

$$f(x) = .7x - 3.4x^{1/2} - \log^2 x(4.5 + 2/\log 2) - 28\log x - 32 - 2\log 2 > 0.$$

It is easy to see that $f(x) > 0$ holds for $x = 1024$, since $\log 1024 < 7$. Moreover we have

$$f'(x) = .7 - 1.7x^{-1/2} - \frac{9 + 4/\log 2}{x}\log x - \frac{28}{x} > 0 \quad \text{for } x \geqq 1024.$$

Hence $f(x)$ is increasing for $x \geqq 1024$ and the lemma is proved for $x \geqq 1024$.

For $75 \leqq x < 1024$ the lemma can be proved directly. For instance, it follows for $591 \leqq x < 1024$ and for $355 \leqq x < 591$ by the fact that there are 22 primes between 1024 and 1182 and 20 primes between 591 and 710. In the same way we get the lemma for $231 \leqq x < 355$, $159 \leqq x < 231$, and so on.

THEOREM. *For every $k \geqq 17$ there exists a sequence of k consecutive integers such that none of these k integers is relatively prime to the product of the others.*

PROOF. In view of the paper of Pillai, it is sufficient to prove the theorem for $k \geqq 300$. We put

$$m = \left[\frac{k}{4}\right] \geqq 75. \tag{4}$$

Let $p_1, p_2, \cdots, p_r$ be the primes in the closed interval $\{1 \cdots m\}$ and $p_{r+1}, p_{r+2}, \cdots, p_s$ the primes in the closed interval $\{m+1 \cdots 2m\}$. If we consider $k$ consecutive integers, then each of the primes

$$p_1, p_2, \cdots, p_r, p_{r+1}, p_{r+2}, \cdots, p_s \tag{5}$$

divides at least two of the $k$ integers, since each of these primes is less than $2m$, hence by (4) less than $k/2$. Therefore each of these $k$ integers which is divisible by at least one of the primes (5) is not relatively prime to all the $k-1$ other integers. Hence it is sufficient to prove that there exist sequences of $k$ integers such that for $k \geqq 300$ each of these integers is divisible by at least one of the primes (5).

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[4] See, for example, E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, vol. 1, 1909, p. 91.

We consider the simultaneous congruences

$$(6) \qquad x \equiv 1 \pmod{2}, \qquad x \equiv 0 \pmod{p_2 p_3 \cdots p_r}.$$

Let $x$ be a solution of (6). Then the integers

$$(7) \qquad x-2m,\ x-2m+2, \cdots, x-2,\ x,\ x+2, \cdots, x+2m-2,\ x+2m$$

form a sequence of $2m+1$ odd integers of the form

$$(8) \qquad x \pm 2\mu, \qquad \mu = 0, 1, \cdots, m.$$

If $\mu$ is divisible by the odd prime $p_\nu$, we have $p_\nu \leqq p_r$, since $\mu \leqq m$ because of (8). Hence we obtain from (6) that

$$(9) \qquad x \pm 2\mu \equiv 0 \pmod{p_\nu}.$$

It follows from (9) that all those integers of (7) which have not the form $x \pm 2^\tau$ with $\tau \geqq 1$ are divisible by at least one of the primes $p_2, p_3, \cdots, p_r$.

If we put

$$(10) \qquad \left[\frac{\log m}{\log 2}\right] + 1 = t,$$

then the integers of the form $x \pm 2^\tau$ with $\tau \geqq 1$ in the set (7) are the integers

$$(11) \qquad x \pm 2,\ x \pm 2^2, \cdots, x \pm 2^t.$$

By (4), it follows from the lemma and from (10) that the number of primes in the closed interval $\{m+1 \cdots 2m\}$ is

$$\pi(2m) - \pi(m) \geqq 2\left[\frac{\log m}{\log 2}\right] + 2 = 2t.$$

On the other hand the primes in this interval were $p_{r+1}, p_{r+2}, \cdots, p_s$, hence

$$(12) \qquad s - r \geqq 2t, \qquad p_{r+2t} \leqq p_s.$$

Beside the congruences (6) we now subject $x$ to the following $2t$ congruences

$$(13) \qquad \begin{aligned} x + 2^\tau &\equiv 0 \pmod{p_{r+\tau}}, \\ x - 2^\tau &\equiv 0 \pmod{p_{r+t+\tau}}, \end{aligned} \qquad \tau = 1, 2, \cdots, t.$$

These congruences always have solutions. For every solution $x$ all the numbers (7) are divisible by at least one of the primes (5), since each of the integers (11) is divisible by at least one of the primes $p_{r+1}, p_{r+2}, \cdots, p_s$ because of (13) and (12).

Hence each of the $4m+3$ consecutive integers

$$x-2m-1,\ x-2m,\ x-2m+1,\ \cdots,\ x-1,\ x,\ x+1,\ \cdots,\ x+2m+1 \tag{14}$$

is divisible by at least one of the primes (5), since

$$x - 2m - 1 \equiv x - 2m + 1 \equiv \cdots \equiv x - 1 \equiv x + 1$$
$$\equiv \cdots \equiv x + 2m + 1 \equiv 0 \pmod 2.$$

Because of (4) we have

$$k \leqq 4m + 3.$$

Therefore we can take $k$ consecutive integers from (14). None of these $k$ integers is relatively prime to the product of the $k-1$ others.

INSTITUTE FOR ADVANCED STUDY

# A CORRECTION TO "A REMARK ON THE SUM AND THE INTERSECTION OF TWO NORMAL IDEALS IN AN ALGEBRA"

TADASI NAKAYAMA

K. Asano has called my attention to an error in my paper *A remark on the sum and the intersection of two normal ideals in an algebra*.[1] Theorem 1 of the paper holds only for local simple algebras, while the remaining theorems are true as they stand.

Osaka Imperial University

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[1] This Bulletin, vol. 46 (1940), p. 469

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# THE FEBRUARY MEETING IN NEW YORK

The three hundred seventy-sixth meeting of the American Mathematical Society was held at Columbia University on Friday and Saturday, February 21–22, 1941. The attendance was approximately three hundred fifty including the following two hundred thirty-five members of the Society:

C. R. Adams, R. B. Adams, Leonidas Alaoglu, C. B. Allendoerfer, R. L. Anderson, T. W. Anderson, Lawrence Annenberg, L. A. Aroian, K. J. Arrow, F. E. Baker, S. F. Barber, Joshua Barlaz, P. T. Bateman, Felix Bernstein, E. E. Betz, Garrett Birkhoff, Archie Blake, Gertrude Blanch, R. P. Boas, Salomon Bochner, Frank Boehm, H. F. Bohnenblust, Samuel Borofsky, D. G. Bourgin, C. B. Boyer, A. T. Brauer, H. W. Brinkmann, H. C. Brodie, A. A. F. Brown, B. L. Brown, Jewell H. Bushey, J. H. Bushey, S. S. Cairns, A. D. Campbell, D. E. Christie, J. A. Clarkson, R. M. Cohn, T. F. Cope, Byron Cosby, Richard Courant, Ruth Crucet, H. B. Curry, M. D. Darkow, Norman Davids, D. R. Davis, Jesse Douglas, Arnold Dresden, R. F. Dressler, J. E. Eaton, M. L. Elveback, Paul Erdös, C. J. Everett, J. M. Feld, Aaron Fialkow, F. A. Ficken, M. I. Fields, N. J. Fine, W. B. Fite, D. A. Flanders, G. E. Forsythe, Tomlinson Fort, M. C. Foster, R. M. Foster, J. S. Frame, K. O. Friedrichs, T. C. Fry, H. P. Geiringer, Abe Gelbart, B. P. Gill, R. L. Gilman, J. W. Givens, A. M. Gleason, Michael Golomb, M. C. Graustein, C. H. Graves, M. C. Gray, S. L. Greitzer, H. M. Griffin, C. C. Grove, D. W. Hall, F. C. Hall, Marshall Hall, N. A. Hall, P. R. Halmos, Philip Hartman, C. M. Hebbert, M. H. Heins, Edward Helly, J. G. Herriot, Fritz Herzog, L. S. Hill, Einar Hille, Abraham Hillman, T. R. Hollcroft, G. M. Hopper, Harold Hotelling, S. E. Hotelling, E. A. Hoy, R. Y. Hoy, Chung-Tsi Hsu, E. M. Hull, Alexandra Illmer, R. P. Isaacs, Seymour Jablon, Nathan Jacobson, R. A. Johnson, H. A. Jordan, Mark Kac, I. N. Kagno, Shizuo Kakutani, E. R. van Kampen, Irving Kaplansky, Edward Kasner, L. S. Kennison, J. R. Kline, E. R. Kolchin, B. O. Koopman, Arthur Korn, H. N. Laden, Jack Laderman, H. G. Landau, A. W. Landers, M. K. Landers, G. A. Larew, Solomon Lefschetz, Joseph Lehner, B. A. Lengyel, Howard Levi, Madeline Levin, Marie Litzinger, E. R. Lorch, A. N. Lowan, J. C. C. McKinsey, L. A. MacColl, C. C. MacDuffee, G. R. MacLane, H. F. MacNeish, W. G. Madow, Dorothy Maharam, Dorothy Manning, W. P. Mason, Walther Mayer, A. E. Meder, F. H. Miller, H. L. Mintzer, Don Mittleman, E. C. Molina, Elizabeth Monroe, Richard Morris, Nathan Morrison, Marston Morse, Z. I. Mosesson, G. W. Mullins, F. J. Murray, C. A. Nelson, O. E. Neugebauer, C. O. Oakley, E. G. Olds, P. S. Olmstead, Oystein Ore, J. C. Oxtoby, N. G. Parke, Everett Pitcher, Harry Pollard, George Polya, E. L. Post, Walter Prenowitz, R. G. Putnam, H. A. Rademacher, C. F. Rehberg, H. J. Riblet, Moses Richardson, R. G. D. Richardson, J. S. Ripandelli, J. F. Ritt, I. F. Ritter, H. E. Robbins, S. L. Robinson, Benjamin Rosenbaum, Arthur Sard, S. A. Schelkunoff, Peter Scherk, I. J. Schoenberg, Abraham Schwartz, C. H. W. Sedgewick, L. W. Sheridan, Seymour Sherman, W. A. Shewhart, Max Shiffman, L. P. Siceloff, James Singer, L. L. Smail, M. F. Smiley, P. A. Smith, Virgil Snyder, Abraham Spitzbart, Wolfgang Sternberg, F. M. Stewart, G. R. Stibitz, J. J. Stoker, W. C. Strodt, Alvin Sugar, M. M. Sullivan, J. D. Tamarkin, Alfred Tarski, A. H. Taub, J. H. Taylor, B. J. Tepping, J. M. Thomas, R. M. Thrall, E. W. Titt, W. R. Transue, W. J. Trjitzinsky, J. W. Tukey, Oswald Veblen, Abraham Wald,

G. L. Walker, R. J. Walker, R. M. Walter, Louis Weisner, B. A. Welch, M. E. Wells, O. L. Wheeler, A. L. Whiteman, P. M. Whitman, Hassler Whitney, E. P. Wiggin, S. S. Wilks, W. H. Wise, Audrey Wishard, Jacob Wolfowitz, H. A. Wood, Leo Zippin, O. J. Zobel, Antoni Zygmund.

The opening session on Friday afternoon was a symposium on applied mathematics at which Mr. R. M. Foster presided. The symposium consisted chiefly of two addresses, one by Professor J. J. Stoker entitled *Mathematical problems connected with the bending and buckling of elastic plates* and one by Dr. W. A. Shewhart entitled *Mathematical statistics in mass production.* Discussion of the first address was led by Professor K. O. Friedrichs. Dr. Stewart Way was unable to attend and his remarks were read by the presiding officer. The discussion leaders of the second address were Professors Harold Hotelling and S. S. Wilks. Informal discussions of both addresses followed the prepared discussions.

On Saturday morning there was a session for papers in Analysis, Professor J. D. Tamarkin presiding, and another for Algebra, Topology, and Statistics, Professor Jewell Hughes Bushey presiding.

At a general session Saturday afternoon, at which President Marston Morse presided, the members of the Society voted to amend the By-Laws in accordance with the recommendation of the Council. For the convenience of members the By-Laws in revised form are published in full below.

### ARTICLE I. OFFICERS

*Section 1.* There shall be a President, three Vice Presidents, a Secretary, four Associate Secretaries, a Treasurer, and a Librarian.

*Section 2.* It shall be a duty of the President to deliver an address before the Society at the close of his term of office or within one year thereafter.

*Section 3.* The Librarian shall have charge of arrangements for the exchange of the Society's publications.

### ARTICLE II. BOARD OF TRUSTEES

*Section 1.* There shall be a Board of Trustees consisting of five trustees elected by the Society, in accordance with Article V.

*Section 2.* The function of the Board of Trustees shall be to receive and administer the funds of the Society, to have full legal control of its investments and properties, to make contracts and, in general, to conduct all business affairs of the Society.

*Section 3.* The Board of Trustees shall have the power to appoint a manager and such assistants and agents as may be necessary or convenient to facilitate the conduct of the affairs of the Society, and to fix the terms and conditions of their employment. The Board may delegate to the officers of the Society duties and powers normally inhering in their respective corporative offices, subject to supervision by the Board. The Board of Trustees may appoint committees to facilitate the conduct of the business of the Society and delegate to such committees such powers as may be necessary or convenient for the proper exercise of those powers. Agents appointed, or members of committees designated, by the Board of Trustees need not be members of the Board.

Nothing herein contained shall be construed to empower the Board of Trustees to divest itself of responsibility for, or legal control of, the investments, properties and contracts of the Society.

### ARTICLE III. EDITORIAL COMMITTEES

*Section 1.* There shall be five Editorial Committees—one of four members for the *Bulletin*, one of three members for the *Transactions*, one of three members for the *Mathematical Reviews*, one of three members for the *Colloquium Publications*, and one consisting of three representatives of the Society on the Board of Editors of the *American Journal of Mathematics.*

### ARTICLE IV. COUNCIL

*Section 1.* The Council shall consist of fifteen members at large, and the following *ex officio* members: the officers of the Society specified in Article I, the members of the Editorial Committees, ex-secretaries who have served as secretaries for ten years or more, and ex-presidents for a period of six years after the completion of their respective presidential terms.

*Section 2.* The Council shall formulate and administer the scientific policies of the Society and shall act in an advisory capacity to the Board of Trustees.

*Section 3.* All members of the Council shall be entitled to be present at any meeting of the Council and to take part in the discussion at that meeting.

*Section 4.* In the absence of the Secretary from any meeting of the Council, one of the Associate Secretaries present may be designated as Acting Secretary for the meeting, either by written authorization of the Secretary, or, failing that, by majority agreement among the Associate Secretaries present.

*Section 5.* The voting members at any meeting of the Council shall be the following: the members at large of the Council; the President; the Vice Presidents; the Secretary; the Acting Secretary for the meeting, if one is designated in the absence of the Secretary; the Treasurer; and a member of each Editorial Committee chosen annually by the committee, provided that if such voting member is absent at any meeting of the Council an alternate designated by him from among the membership of the committee shall have the power to vote.

*Section 6.* The method for settling matters before the Council at any meeting shall be by majority vote of the voting members present. If the result of such a vote is challenged at the time by any member of the Council, it shall be the duty of the Secretary—or of the Acting Secretary for the meeting—or of a member present who shall be designated by the Council for that purpose—to prepare a list of the voting members present, on the basis of which the presiding officer shall determine the true vote by a roll call.

*Section 7.* A quorum for the transaction of business at any meeting of the Council shall consist of five voting members as defined in Section 5 of this Article.

*Section 8.* Between meetings of the Council, business may be transacted by a mail vote. Voting members for this purpose shall be the members at large of the Council, the President, the Vice Presidents, the Secretary, the Treasurer, and the voting member of each Editorial Committee as designated in Section 5. An affirmative vote on any proposal by mail shall require at least a majority of the voting members, and at least three-quarters of the total number of votes transmitted to the Secretary before the time announced for the closing of the polls. If two or more voting members request postponement at the time of voting, action on the matter at issue shall be postponed until the next meeting of the Council, unless at the discretion of the Secretary the question is made the subject of a second vote by mail, in connection with which brief statements of reasons, for and against, are circulated.

### ARTICLE V. ELECTIONS AND TERMS OF OFFICERS

*Section 1.* The term of office shall be two years in the case of the trustees, the President, the Vice Presidents, the Secretary, the Associate Secretaries, and the Treasurer; three years in the case of the Librarian and the members of the Editorial Committees. The term of office for members at large of the Council shall be three years, five of the members at large retiring annually. In every case, however, the officials specified in Articles I, II, III, and IV shall continue to serve until their successors have been duly elected and qualified.

*Section 2.* The President and Vice Presidents shall not be eligible for immediate re-election to their respective offices. A member at large or an *ex officio* member of the Council shall not be eligible for immediate election (or re-election) as a member at large of the Council.

*Section 3.* Election of officers specified in Article I, trustees, members of Editorial Committees, and members at large of the Council shall be by ballot at the Annual Meeting. An official ballot shall be sent to each member by the Secretary at least one month before the Annual Meeting, and such ballots, if returned to the Secretary in envelopes bearing the name of the voter and received prior to the closing of the polls, shall be counted at the Annual Meeting. Each ballot shall contain one or more names proposed by the Council for each office to be filled, with blank spaces in which the voter may substitute other names. A plurality of all votes cast, whether cast in person or by mail, shall be necessary for election. In case of failure to secure a plurality for any office, the members present at the Annual Meeting shall choose by ballot between the two having the highest number of votes.

*Section 4.* If the President of the Society die or resign before the expiration of his term of office, the Council, may, with the approval of the trustees, designate one of the Vice Presidents to serve as President until the next Annual Meeting, when the Society shall elect a President for a two-year term. Such vacancies as may occur at any time in the group consisting of the Vice Presidents, the Secretary, the Associate Secretaries, the Treasurer, the Librarian, and the members of the Editorial Committees may be filled by the Council with the approval of the trustees.

*Section 5.* If any trustee die or resign during his tenure of office, the vacancy thus created shall be filled for his unexpired term by the Board of Trustees.

*Section 6.* If any member at large of the Council die or resign more than one year before the expiration of his term, the vacancy for the unexpired term shall be filled by the Society at the next Annual Meeting.

### ARTICLE VI. ELECTION OF MEMBERS

*Section 1.* Election of members shall be by vote of the Council.

*Section 2.* There shall be four classes of members, ordinary, contributing, institutional contributing, and sustaining.

*Section 3.* A firm, corporation, institution, association or individual interested in the support of mathematics, may be elected to sustaining membership. A sustaining member shall receive the *Bulletin*, the *Transactions*, and *Mathematical Reviews* (see Article IX), and shall have the privilege of nominating one or more persons for election by the Council to ordinary membership in the Society, the number to be determined by the Council. Such nominations shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail).

*Section 4.* A firm, corporation, institution, or association interested in the support of mathematics may be elected to institutional contributing membership. An institutional contributing member shall receive the *Bulletin* or have the privilege, for each twenty-five dollars contributed, of nominating one person to ordinary membership in the Society. Such nominations shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail).

*Section 5.* Except in the case of nominees of sustaining members and of institutional contributing members, application for admission to ordinary membership shall be made by the applicant, on a blank provided by the Secretary, and shall be approved by two members of the Society. Such applications shall not be acted upon until at least thirty days after their presentation to the Council (at a meeting or by mail), except in the case of members of other societies entering under special action of the Council.

*Section 6.* An ordinary member may become a contributing member by paying the dues for such membership. (See Article VII, Section 4.)

*Section 7.* An ordinary member or a contributing member shall receive the *Bulletin*, except in the case of members whose dues are remitted under Article VII, Section 10.

## ARTICLE VII. DUES

*Section 1.* Persons elected to ordinary membership in the Society by the Council, under the provisions of Article VI, Section 5, shall be admitted to membership upon the payment, within sixty days of the date of their election, of an initiation fee of five dollars.

*Section 2.* The annual dues of persons elected by the Council to ordinary membership under the provisions of Article VI, Section 5, shall be eight dollars, with the following exceptions: (1) during the first three years of membership, the annual dues shall be six dollars; a person shall be considered to have completed his first year of membership on January 1 following his election; (2) the amount of dues may be altered by reciprocity agreements with other societies; (3) the Council may make special rulings in exceptional cases, with the approval of the Board of Trustees. Each new member shall pay in proportion to the unexpired fraction of the year at the time of his election.

*Section 3.* Four dollars and fifty cents of the annual dues of those members who receive the *Bulletin* under the provisions of Article VI shall be allocated in payment therefor.

*Section 4.* The minimum dues for a contributing member shall be fifteen dollars per year. Members may, upon their own initiative, pay larger dues.

*Section 5.* The dues for an institutional contributing member for any year shall not be less than twenty-five dollars. Institutions may pay larger dues.

*Section 6.* The dues of a sustaining member for any year shall not be less than one hundred dollars. A sustaining member who contributes annually at least five hundred dollars shall be designated as a Patron of the Society.

*Section 7.* Persons elected to ordinary membership as nominees of sustaining members or institutional contributing members under the provisions of Article VI, Sections 3–4, shall not be required to pay an initiation fee. They shall not be required to pay dues, so long as it is agreed that they are designated by the sustaining member or institutional contributing member under these provisions. If a nominee of a sustaining member or an institutional contributing member later becomes a dues-paying member, he shall pay dues at the rate of six dollars a year for the remainder (if any) of his first three years of membership, and eight dollars a year thereafter.

*Section 8.* If the annual dues of any member remain unpaid beyond a reasonable time, the Board of Trustees shall remove his name from the list of members, after due notice.

*Section 9.* Any member not in arrears of dues may become a life member on the payment of a sum to be determined in accordance with actuarial principles. A life member shall have for life the status and privileges of an ordinary member, without further payment of dues.

*Section 10.* After retirement from active service on account of age, any member who is not in arrears of dues and with membership extending over at least twenty years may—by giving proper notification to the Secretary—have his dues remitted, on the understanding that he will thereafter receive the programs of the meetings, but not the *Bulletin.*

## ARTICLE VIII. MEETINGS

*Section 1.* The Annual Meeting of the Society shall be held between the fifteenth of December and the fifteenth of January next following. Notice of the time and place of this meeting shall be mailed by the Secretary or an Associate Secretary to the last known post office address of each member of the Society. The times and places of the Annual and other meetings of the Society shall be designated by the Council. No matter of general business shall be considered at any meeting of the Society except the Annual Meeting, without the recommendation of the Council.

*Section 2.* The President may call a meeting of the Council whenever the affairs of the Society make it desirable; he shall call a meeting of the Council on written request of five of its members.

*Section 3.* The Board of Trustees shall have its regular Annual Meeting on the first day of the Annual Meeting of the Society. Special meetings of the Board of Trustees may be called by the Chairman of the Board upon three days' notice of such meeting mailed to the last known post office address of each trustee. He shall call a meeting upon the receipt of a written request of two of the trustees. Meetings for the transaction of business may also be held by common consent of all the trustees.

*Section 4.* Papers intended for presentation at any meeting of the Society shall be passed upon in advance by a program committee appointed by or under the authority of the Council; and only such papers shall be presented as shall have been approved by such committee. Papers in form unsuitable for publication, if accepted for presentation, shall be referred to on the program as preliminary communications or reports.

## ARTICLE IX. PUBLICATIONS

*Section 1.* The Society shall publish an official organ called the *Bulletin of the American Mathematical Society.* It shall publish a journal called the *Transactions of the American Mathematical Society,* the object of which shall be to make known important researches presented at meetings of the Society. It shall publish a periodical called *Mathematical Reviews* containing abstracts or reviews of current mathematical literature. It shall publish a series of volumes called *Colloquium Publications* which shall embody in book form new mathematical developments. It shall also cooperate in the conduct of the *American Journal of Mathematics.*

*Section 2.* The editorial management of the *Bulletin, Transactions, Mathematical Reviews,* and *Colloquium Publications,* and the participation of the Society in the management of the *American Journal of Mathematics* shall be in charge of the respective Editorial Committees as provided in Article III, Section 1.

### ARTICLE X. AMENDMENTS

These By-Laws may be amended or suspended at any meeting of the Society on recommendation of the Council and by a two-thirds vote of the members present, provided notice of such proposed action and of its general nature shall have been given in the call for such meeting.

Following the business meeting Professor H. F. Bohnenblust gave an address entitled *Partially ordered linear spaces*. During the latter part of the afternoon there were two sessions, one for Geometry, Professor Virgil Snyder presiding, and one for Algebra and Analysis at which Professor E. L. Post presided.

Titles and cross references to the abstracts of papers read at this meeting follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1 to 9 were read before the section for Analysis; those numbered 10 to 18 before the section for Algebra, Topology, and Statistics; those numbered 19 to 22 in the section for Geometry; those numbered 23 to 27 in the section for Algebra and Analysis; papers numbered 28 to 54 were read by title. Paper 21 was read by Professor Kasner. Dr. Bers was introduced by Dr. Abe Gelbart, Dr. Weinstein by Dr. H. E. Robbins, Dr. Snapper by Professor J. H. M. Wedderburn, Mr. Rosenbloom by Professor J. R. Kline, Dr. Bergman by Dr. D. C. Spencer, Dr. Kober by Professor Einar Hille, and Dr. Wundheiler by Professor D. J. Struik.

1. R. P. Boas: *A note on functions of exponential type.* (Abstract 47-3-129.)

2. P. R. Halmos: *On the decomposition of measures.* (Abstract 47-3-133.)

3. Abe Gelbart: *On the behavior of a function of two complex variables in the neighborhood of an isolated essential singularity.* (Abstract 47-3-131.)

4. I. J. Schoenberg: *On absolutely convex functions.* (Abstract 47-5-245.)

5. George Polya: *On converse gap theorems.* (Abstract 47-3-139.)

6. Dorothy Maharam: *On measure in abstract sets.* (Abstract 47-3-136.)

7. Lipman Bers: *Analytic functions of two complex variables in domains bounded by two analytic hypersurfaces.* (Abstract 47-3-128.)

8. H. N. Laden: *On Kryloff-Stayermann interpolation.* (Abstract 47-3-135.)

9. Alexander Weinstein: *On the decomposition of a Hilbert space by its harmonic subspace.* (Abstract 47-5-252.)

10. Ernst Snapper: *Structure of linear sets.* (Abstract 47-3-124.)

11. P. C. Rosenbloom: *Post algebras:* I. *Postulates and general properties.* (Abstract 47-3-123.)

12. Hassler Whitney: *Geometric methods in combinatorial topology.* Preliminary report. (Abstract 47-3-180.)

13. Oystein Ore: *Transformation of sets.* (Abstract 47-3-122.)

14. Shizuo Kakutani: *A generalization of Brouwer's fixed-point theorem.* (Abstract 47-3-177.)

15. Archie Blake: *The exploratory determination of statistical distributions.* (Abstract 47-3-169.)

16. Abraham Wald: *On testing statistical hypotheses concerning several unknown parameters.* (Abstract 47-5-275.)

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T. R. HOLLCROFT,
*Associate Secretary*

# WILLIAM CASPAR GRAUSTEIN—IN MEMORIAM

William Caspar Graustein was born in Cambridge, Mass., November 15, 1888. He graduated from Harvard, Magna cum Laude in 1910. He spent the next year in the same place doing graduate work in mathematics, his main interest being then, as ever afterwards, in geometry. He received his first initiation into the higher branches of that subject by reading Klein's incomparable 1893 lectures on "Höhere Geometrie." As a matter of fact this nearly had a most disastrous effect, for when Study in Bonn heard that his prospective pupil had been reading Klein, he wrote, "So ist er ganz und gar verdorben." However, Study relented sufficiently to receive him in the autumn of 1911, thereby initiating a close personal and intellectual intimacy that lasted many years, and had the deepest influence on the whole of Graustein's subsequent career.

He took the doctor's degree, Summa cum Laude in July, 1913, and returned at once to Harvard as instructor in mathematics. He was then promoted to an assistant professorship in the newly formed Rice Institute in Houston, Texas. He joined his Cambridge friend Griffith Conrad Evans in building up there, with the warm approval of the President Edgar Odell Lovett, the most advanced mathematical centre south of Mason and Dixon's line. This lasted for four fruitful years, then came the war. Graustein was bound he would serve his country, in spite of official discouragement on account of his name and antecedents, and serve he did in the Ordnance Department at the proving ground at Aberdeen, Maryland, eventually becoming a first lieutenant.

He returned to Harvard in the autumn of 1919 to help in the congenial task of rehabilitating the Division of Mathematics, which in one year had lost three of its most valuable members through death and transfer. He worked mightily at that task till the day of his death. He was chairman of the Division from 1932 to 1937, and Assistant Dean of the Faculty of Arts and Sciences 1939–41. But he served mathematics outside his own university as well, acting for twelve years (1924–36) as an Associate Editor, and for five years (1936–41) as one of the Editors, of the Transactions of the American Mathematical Society, and chairman of the very busy organizing committee of the proposed 1940 International Mathematical Congress in 1938. He was killed in a motor accident, January 22, 1941.

Graustein's doctoral thesis, written with Study, was entitled "Eine reelle Abbildung analytischer komplexer Raumkurven." For a number of years Study had been interested in the geometry of the complex domain. In three-space we may have varieties depending analytically on any number from one to five of real parameters. But it is always heartening to have something real to represent complex elements, and many writers had occupied themselves with this problem of representation. Study had recently flirted with one of the methods, which was based on an idea initiated by Poncelet, but more closely associated with the writing of a vituperative Frenchman, F. Maximilien Marie. It consists in replacing each complex point by an ordered pair of real points, the closest pair of the elliptic involution whose double points are the given point and its conjugate. This conjugate is represented by the same pair in opposite order. Analytically the point with rectangular Cartesian coordinates $x^i$ is represented by the ordered pair

$$y^i = \frac{x^i + \bar{x}^i}{2} + i\,\frac{x^i - \bar{x}^i}{2}; \qquad \omega^i = \frac{x^i + \bar{x}^i}{2} - i\,\frac{x^i - \bar{x}^i}{2}.$$

When it comes to representing simple loci like lines, circles and planes, there are a distressing number of special cases which must be handled separately; the isotropic elements cause trouble. The points of a complex space curve depend on two real parameters, and will be represented by the points of two real surfaces connected by an analytic relation. Graustein shows that:

A. They are translation surfaces.

B. There is a first and a second system of curves on the one so related to a first and a second system on the other that at corresponding points the tangent to the first curve on either surface is parallel to that of the second curve on the other.

C. The relation is area-preserving, but not conformal.

Graustein retained his interest in complex geometry for a number of years. In Volume 16 of the Transactions he takes up the question of when pairs of complex elements in euclidean three-space can be carried into one another by real motions. The criterion depends on certain real invariants, as the squared distance from a complex point to its conjugate. In the twenty-fourth volume of the same journal he generalizes the Marie representation. What is the most general representation where a complex point $(z)$ is represented by a real ordered pair $(x)(y)$ it being required that the conjugate point $(\bar{z})$ shall be represented by $(y)(x)$, a real point shall be represented by itself counted twice, the functions involved shall be analytic, and the relations involved shall be invariant under some specified group. Two years later in Volume 26 of the Annals of Mathematics he relates the generalized method of representation to the main problem of his thesis.

A complex curve is represented by two real surfaces in one-to-one correspondence with correspondence of parallelism. This idea led Graustein to make a further study of the relation of two surfaces when the normals at corresponding points are parallel, and in Volume 23 of the Transactions he published a paper on "Parallel Maps of Surfaces." If we have a one-to-one correspondence between two surfaces, neither of which is developable, of such a sort that normals at corresponding points are parallel, they have the same spherical representation. If a point $P'$ corresponds to a point $P$, there is a projective relation among the tangents at $P'$ connecting each with that which is parallel to the tangent at $P$ corresponding to the given one. When the relation is non-parabolic, that is to say, has two self-corresponding elements, the projective invariant $I$ of this relation is fundamental in the transformation. Graustein shows that it is equal to the cross ratio which the pair $PP'$ make with the corresponding focal points in the congruence of lines connecting corresponding points. This beautiful result leads to many interesting theorems. The author wrote five other short papers dealing with mapping and applicability.

The development of the tensor calculus of Ricci and Levi-Civita introduced a modification in differential geometry as profound as that produced nearly a century before by the publication of Gauss' "Disquisitiones Generales." Many geometers threw themselves entirely into the new work; Graustein was more cautious, he recognized the advantages in the new notations, new points of view and new techniques, especially when more than three dimensions were involved. But what attracted him most was the invariant or covariant character of the new processes, and that led him to the idea of developing methods on the more classical lines, which seemed to him to go really deeper into the geometrical questions involved. The first result of this study was his "Méthodes Invariantes dans la Géométrie Infinitésimale," crowned by the Belgian Academy of Sciences in 1929. A year later appeared in this Bulletin, Volume 36, an address delivered by request before this Society, "Invariant Methods in Classical Differential Geometry."

What sort of things are invariant under the transformations of differential geometry? First of all Jacobians are. A vector normal to a surface is not concerned with the choice of parametric curves on the surface, and the three components of such a vector are proportional to the three Jacobians $\partial(x^i x^j)/\partial(uv)$. Again we have three fundamental quadratic differential forms, that which gives the directions of the isotropic lines, that which gives the asymptotic lines, and that which gives the isotropic directions on the spherical representation. Their algebraic invariants will give geometric invariants of the surface. Beyond these there is an invariant process called directional differentiation. This was, I think, first developed by Lamé. Graustein uses it with telling effect. The directional derivative of a point function $f$ in the direction of a given element of arc is

$$\lim_{\Delta s\to 0}\frac{\Delta f}{\Delta s}=\frac{\partial f}{\partial s}\cdot$$

The great trouble is that the order of differentiation is material when we perform the process twice

$$\frac{\partial^2 f}{\partial s\partial s'}\neq\frac{\partial^2 f}{\partial s'\partial s}\cdot$$

He turns the difficulty by introducing a new operator $\nabla$. If $s$ and $s'$ are arcs on an orthogonal system of curves,

$$\frac{\nabla f}{\nabla s}=\frac{\partial f}{\partial s}+\frac{1}{\sigma'}f,\qquad \frac{\Delta f}{\Delta s'}=\frac{\partial f}{\partial s'}-\frac{1}{\sigma}f$$

where $1/\sigma$ is the geodesic curvature of $s$ and $1/\sigma'$ that of $s'$. It then appears that

$$\frac{\nabla}{\nabla s}\,\frac{\partial f}{\partial s'}=\frac{\nabla}{\nabla s'}\,\frac{\partial f}{\partial s}\cdot$$

The Gauss and Mainardi-Codazzi equations are easily found in terms of these new symbols.

It is clear that such invariant methods, though first developed for the case of euclidean measure in three-space, can be widely generalized. Graustein did just this in the thirty-sixth volume of the Transactions. Here we have a forty-page article on the geometry of Riemann spaces. The basis of directional differentiation is a mixed tensor given by the equation

$$\frac{\partial}{\partial s^q}\,\frac{\partial f}{\partial s^p}-\frac{\partial}{\partial s^p}\,\frac{\partial f}{\partial s^q}=B_{ij}k\frac{\partial f}{\partial s^k}\cdot$$

Another development of these directional methods appeared two years earlier in the same journal, in an article entitled "Parallelism and Equidistance in Differential Geometry." Consider two curves $C$ and $C'+\Delta C'$ on a surface. What shall we mean by saying that they are equidistant with regard to a one-parameter family of curves $C'$? Clearly that the arcs of the latter bounded by the first two are all equal. But when they are not equal we should like to develop some measure of their spreading. If the curves $C$ are given by the values of the function $f$ Graustein defines as the "distantial spread" of the curves $C$ with regard to the curves $C'$

$$\frac{1}{b'} = -\frac{\partial}{\partial s}\log\left|\frac{\partial f}{\partial s'}\right|.$$

I mention in passing a personal regret that the author found it necessary to mar such a pretty conception with such a cacophonous name. Analogous to this is the angular spread, defined by the equation

$$\frac{1}{a} = \frac{1}{\rho} + \frac{d\omega}{ds}$$

where $\omega$ is the properly measured angle between that curve $C$ on which the distances are measured and the various curves $C'$ while $1/\rho$ is the geodesic curvature. The two types of spreads are closely connected, and have many pretty properties. $1/\rho$ vanishes in the case of Levi-Civita parallelism. The two are connected by interesting relations.

I will give it as my personal opinion that Graustein's work dealing with invariant methods was his most important contribution to mathematical science. His last published paper, in the forty-seventh volume of the Transactions, was somewhat different. It gave a complete and beautiful solution of the difficult problem of harmonic minimal surfaces. This had been only incompletely handled by previous writers.

Any notice of Graustein's life work should include some mention of his textbooks. The first of these, "Plane and Solid Analytic Geometry" written in collaboration with W. F. Osgood appeared in 1921. It is a thoroughly usable introductory text, with adequate exercises. It has never seemed to me, however, that the book was entirely at peace with itself. When a distinguished analyst and an enthusiastic geometer work together, if both are good teachers, the result may well be a good book, but it may also lack the inner consistency which it would have possessed had either author written it alone. No such charge can be brought against Graustein's "Introduction to Higher Geometry" which first saw the light in 1930 after many years of most careful preparation. Soon after W. E. Byerley returned to Harvard in 1876 he introduced a course in "Modern

Methods in Analytic Geometry" drawing its inspiration from the lectures by Evan W. Evans that he had heard while he was at Cornell. That course became at once one of the fixtures in the Harvard mathematical curriculum, successive teachers modified it to suit their own ideas and Graustein gave it repeatedly. This excellent book represented the results of his experience. The first three quarters are devoted to a careful presentation of the projective geometry of points, lines, and conic sections, combining geometric and algebraic methods easily and naturally. Then fifty pages are given to the circle, and the geometry of inversion. This is prettily done. I cannot help feeling, however, that it should have been postponed to the end of the book, as the author seems to have introduced it because he felt, and rightly, that no well trained geometer should be ignorant of these things, rather than because it fitted naturally with the rest of the work. The last chapter of forty pages, deals all too briefly with the projective geometry of three-dimensional space.

Graustein's last textbook "Differential Geometry" appeared in 1935. This also represents the results of many years of teaching an introductory course in the subject. It is classical and conservative in spirit, great use is made of vector methods, and Study's vector notation. Near the end we have a twenty page introduction to the absolute differential geometry. It is interesting to contrast this book with the recent one on the same subject by Eisenhart, published just five years after. One has the impression that Graustein introduced some ideas of absolute geometry as a matter of conscience, he did not like the idea that a student who had finished the book should have no knowledge whatever of the more modern technique. Eisenhart introduces tensor notation as soon as he can and feels himself cramped as long as he is confined to three dimensions.

It is not the function of this Bulletin to carry long reviews of textbooks, but it is eminently proper to bring to the memory of the members of our Society beautifully written works of a great teacher and great friend of all who teach or learn mathematics.

## WILLIAM C. GRAUSTEIN

1. Thesis: *Eine reelle Abbildung analytischer komplexer Raumkurven*, 1913.
2. *The equivalence of complex points*, Transactions, January 1915, pp. 34–44.
3. Review: *Dickson's Linear Algebras*, Bulletin, July 1915, pp. 511–522.
4. *On the geodesics and geodesic circles on a developable surface*, Annals, March 1917, pp. 132–138.
5. *On two related transformations*, Journal, July 1917, pp. 233–240.
6. *Note on isogenous complex functions*, Bulletin, July 1918, pp. 473–477.
7. *Parallel maps of surfaces*, Transactions, April 1922, pp. 298–332.
8. *Representations of a complex point*, Transactions, December 1922, pp. 245–254.

9. *Spherical representation of conjugate systems*, Annals, December 1922, pp. 89–98.
10. *Note on a certain type of ruled surface*, Bulletin, October 1923, pp. 341–344.
11. *Determination of a surface*, Journal, October 1923, pp. 294–314.
12. *Applicability with preservation of both curvatures*, Bulletin, January 1924, pp. 19–23.
13. *Isometric W-surfaces*, Transactions, April 1924, pp. 176–204.
14. *A necessary and sufficient condition that two surfaces be applicable* (with B. O. Koopman), Transactions, July 1924, pp. 369–372.
15. *The scientific work of Joseph Lipka*, Bulletin, July 1924, pp. 352–356.
16. *Real representations of analytic complex curves*, Annals, September 1924, pp. 131–143.
17. Review: *Eisenhart's Transformation of Surfaces*, Bulletin, October 1924, pp. 454–460.
18. Review: *Coolidge's Complex Geometry*, Bulletin, July 1925, pp. 451–456.
19. *Semi-parallel maps of surfaces*, Annals, March 1926, pp. 271–278.
20. *An invariant of a general transformation of surfaces*, Bulletin, July 1926, pp. 357–364.
21. *A geometrical method of solving the biquadratic equation*, Monthly, May 1928, pp. 236–238.
22. *A geometrical problem in which order of choice is important*, Monthly, October 1928, pp. 412–415.
23. Review: *Weatherburn's Differential Geometry*, vol. 1, Bulletin, November 1928, pp. 785–786.
24. Belgian memoir (1929): *Méthodes Invariantes dans la Géométrie Infinitésimale des Surfaces*, pp. 1–96.
25. *Note on a certain type of parabola*, Bulletin, November 1929, pp. 619–622.
26. *The linear element of a Riemannian $V_n$*, Journal, April 1930, pp. 351–356.
27. *Invariant methods in classical differential geometry*, Bulletin, August 1930, pp. 489–521.
28. *On the average number of sides of polygons of a net*, Annals, January 1931, pp. 149–153.
29. Review: *Weatherburn's Differential Geometry*, vol. 2, Bulletin, November 1931, pp. 796–797.
30. *Parallelism and equidistance in classical differential geometry*, Transactions, July 1932, pp. 557–593.
31. *The geometry of Riemannian spaces*, Transactions, July 1934, pp. 542–585.
32. *Applicability with preservation of both curvatures*, Duke Journal, June 1936, pp. 177–191.
33. *A new form of the four-vertex theorem*, Monatshefte für Mathematik und Physik, vol. 43, Vienna, 1936, pp. 381–384.
34. *Extensions of the four-vertex theorem*, Transactions, January 1937, pp. 9–23.
35. Review: *Petermann and Hagge's Gewachsene Raumlehre*, Monthly, June–July 1937.
36. *The four-vertex theorem of a certain type of space curves* (with S. B. Jackson), Bulletin, October 1937, pp. 737–741.
37. *Harmonic minimal surfaces*, Transactions, March 1940, pp. 173–206.

"Plane and Solid Analytic Geometry" (with W. F. Osgood), 1921.
"Introduction to Higher Geometry," 1930.
"Differential Geometry," 1935.

J. L. COOLIDGE

# BOOK REVIEWS

*Bibliography on Orthogonal Polynomials.* By J. A. Shohat, Einar Hille, and J. L. Walsh. Washington, D. C., National Academy of Sciences, 1940. 9+203 pp. $3.00.

This bibliography is concerned principally with Hermite, Legendre, Laguerre, Jacobi, Tchebycheff, and related polynomials. These polynomials, which enter into many phases of pure and applied mathematics, happen to be both important and interesting. Since the time of Legendre (1752–1833) new applications have been requiring new advances in the theory of the polynomials, and new advances in the theory have been finding new applications. Hence it is inevitable that the literature of the subject has grown to vast proportions and will continue to grow. One who, by reason of preference or necessity, wishes to become acquainted with the whole theory or with some part of it is embarrassed not only by the extensiveness of the literature but also by the fact that so many different persons have made contributions in so many different books and periodicals. It is fortunate that the National Research Council of the National Academy of Sciences has collaborated with the three authors to bring forth this bibliography.

The book begins with an alphabetical list of complete titles of 332 periodicals. These periodicals are thereafter referred to by number. This necessitates repeated reference to the list of titles, but it eliminates confusion which results from abbreviations.

The second part of the book gives a code index. Some Roman capitals represent special classical orthogonal polynomials (OP); for example H represents Hermite OP. Other Roman capitals specify the domain of orthogonality and the character of the weight-function. Lower case Greek letters refer to topics, properties, applications, etc.; subheadings are lower case Roman letters and then Arabic numbers. For example $\alpha$ refers to general properties of OP; $\alpha d$ to zeros of OP; $H\alpha d$ to zeros of Hermite OP; and $H\alpha d4$ to bounds for the zeros of Hermite OP. The symbol $\beta$ refers to expansions of functions; $\beta a1, \beta a2, \cdots, \beta a19$ refer to 19 different classes of functions ($L$, $L^2$, $L^p$, continuous, bounded variation, etc.) to be expanded; and $\beta b1, \cdots, \beta b19$ refer to properties of expansions. For example $\beta b14.1, \cdots, \beta b14.9$ refer to 9 different types of convergence of expansions such as uniform, absolute, almost everywhere, and in mean; and $\beta b15.1, \cdots, \beta b15.6$ refer to summability of expansions. The symbol $\gamma$ refers to general series of OP not necessarily resulting from expansion of functions; $\delta$ to mechanical quadratures; $\epsilon$ to interpolation; $\zeta$ to approximations; $\eta$ to the moment problem; $\theta$ to probability and statistics; and the code continues until $\omega$ is reached.

The third and main part of the book (pp. 25–184) lists 1952 articles by about 600 authors. The full title of each article is given exactly as it appears with the article except in the case of non-Latinic alphabets, when a translation is given. Following each title is an abstract in code. For example, the fourth entry under E. Hille is:

A class of reciprocal functions. [36] (2) 27 (1926) 427–64.

$$\begin{gathered}\mathrm{H}{:}\alpha\mathrm{b4-b7-b10-b13-c-d1-d3-d4-d7-f-f3-g-i-j-l1}\\ \mathrm{m1.1-m1.2-n2-}\beta\mathrm{a1-a5-b12-b14.1-b14.2-b14.3-b14.4}\\ \mathrm{-b14.8-b14.9-b15.1},\ \theta\mathrm{a2},\ \kappa\mathrm{b},\ \mu,\ \pi,\ \rho.\end{gathered}$$

It takes only a short time to find that the [36] refers to the Annals of Mathematics and to decode the abstract to obtain a clear and precise statement of the topics discussed in the paper.

The fourth part of the book lists 89 books and 39 theses containing theory of OP. Some of these are abstracted and some are not.

The fifth and last part of the book is a topical index which lists works on (i) Hermite OP; (ii) Jacobi OP; (iii) Laguerre OP; (iv) Legendre OP; (v) Applications to theory of approximation, probability and statistics, and mathematical physics; (vi) Series in OP; (vii) Interpolation, mechanical quadratures; (viii) Moment problem; (ix) OP in the complex domain; and (x) Tables and graphs. One feels that these lists are long; there are for example more than 500 papers on series in OP of which the three authors of this bibliography are collectively responsible for 29. Checking over 500 abstracts to find, for example, papers on uniform convergence of series in Hermite polynomials sounds like a formidable task; but actually this task is very much simplified by the code form of the abstracts. One simply looks for H$\beta$b14.1 without being required to read hundreds of pages of verbose abstracts.

So far as the reviewer can determine, the bibliography is accurate and is complete in the sense that it contains all titles which it certainly should contain. There are naturally many books and papers with application to OP which are not cited; for example some on general theory of orthogonal functions which apply to polynomials as well as to other orthogonal functions; some on trigonometric and power series; and some on Sturm-Liouville theory. But to start listing all these and other works which bear upon some ramification of the theory of OP would lead to bibliography without end, and one must feel that the authors have shown good judgment in delimiting their bibliography.

The standard of excellence and usefulness of this book by three capable authors is such that the book will be a valuable adjunct to libraries and to the research equipment of those who work with *OP*. It is appropriate that the book should be used as a model for future bibliographies in other subjects.

RALPH P. AGNEW

*Theoretical Hydrodynamics*. By L. M. Milne-Thomson. London, Macmillan, 1938. 552 pp. $11.25.

A very comprehensive mathematical treatment of the theory of fluid motion is contained in this text, which presents the lectures by the well known author on this subject to the junior members of the Royal Corps of Naval Constructors at Greenwich. The material presented consists mainly of the theory of the perfect fluid motion, for the first time based consistently on vector notation throughout the text, which thus becomes very concise and brief in the details of mathematical deduction. The great advantages of this form of mathematical writing for this field are obvious to anyone who surveys the wealth of material given in this text. In justice to the author not all the saving in space should be attributed to the use of vector notation; however, instead a considerable amount can be ascribed to the experienced and able use of the descriptive text which is brief but complete in all details. However, the physical interpretations given seem to be treated somewhat too briefly for engineers, and experimental and practical applications are outside of the scope of this theoretical treatment.

A brief review of the contents will bring out the structure of the book. Chapter I contains elementary problems of great variety based on Daniel Bernoulli's theorem. The mathematical tools are introduced in Chapter II on vector analysis, which is followed by the discussion of general properties of fluid motion and such phases of two-dimensional flow as can be treated without recourse to the complex variable. Chapter V introduces the latter and opens up the main part of the book, Chapters VI–XIV, dealing comprehensively with two-dimensional motion in all its aspects from the standpoint of the complex variable and conformal mapping. In addition to the chapter headings of streaming motion, aerofoils, sources and sinks, moving cylinders, theorem of Schwarz and Christoffel, the wake, rectilinear vortices, we find also jets and currents and waves treated extensively in this part. The succeeding four chapters are taken up with three-dimensional phenomena and related mathematical problems under the headings of Stokes' stream function, spheres and ellipsoids, solid moving through a liquid, vortex motion. The last chapter is devoted to the introduction of the equations of motion for viscous fluids and concludes with a brief description of boundary layer theory.

While this book can be recommended as a text in classical hydrodynamics for advanced graduate student engineers, it will take a more important place as a reference work and as a source of many original approaches in the manner of presentation for those teaching the subject. The over five hundred exercises ranging from easy to very difficult should prove very interesting since many were taken from the official examinations for Constructor Lieutenants at the Naval College and for the degree of M. Sc. at the University of London.

A. T. IPPEN

*Convergence and Uniformity in Topology.* By J. W. Tukey. Annals of Mathematics Studies, no. 2. Princeton, University Press, 1940. 9+90 pp. $1.50.

The extension of metric methods to non-metrizable topological spaces has been a principal development in topology of the past few years. This has occurred in two directions: one through a rebirth of interest in Moore-Smith convergence due to results of Garrett Birkhoff, and the other through the concept of uniform structure due to André Weil. In this pamphlet these ideas and their interrelations are given a full and detailed treatment. Many of the results are new. This is likewise true of the point of view and much of the mechanism.

Chapters I and II are concerned with set theory, partially ordered sets, Zorn's lemma, and directed sets. In particular a *stack* (=the set of finite subsets of a given set, ordered by inclusion) is a directed set. Directed sets are classified into cofinally equivalent types, and these are found to be partially ordered. Chapter III introduces the *phalanx* (a function from a stack to a topological space $T$). It is proved that the topology of $T$ is describable by the convergence of its phalanxes. Chapter IV considers compactness and equivalent properties in terms of phalanxes. The *biggest* compactification of a space is defined and constructed from ultraphalanxes. Chapter V is concerned with coverings of a space (by open sets), and the equivalence of the existence of families of coverings to the existence of metrics and pseudo-metrics. This leads naturally to the notion of *uniform structure* (Chapter VI). A uniform structure is a family $\{U\}$

of coverings of $T$ such that, for any two, $U_1$, $U_2$, there is a third, $U_3$, such that, for each $u_0 \in U_3$, the sum of the $u$'s in $U_3$ meeting $u_0$ lie in a set of $U_1$ and in a set of $U_2$. Such a structure exists if and only if $T$ is completely regular. This notion generalizes the notion of metric. A given space may have various inequivalent uniformities; a compact space has only one. The concept of completeness is extended, and it is proved that any space with a uniform structure may be completed. Chapter VII carries over these notions from a collection of spaces to their product. An extension of Tychonoff's embedding theorem is proved. Chapter VIII presents a number of examples.

An extensive symbolism makes the reading difficult—especially so when meeting one of the numerous misprints or omissions.

N. E. STEENROD

*Formal Logic.* By Albert A. Bennett and Charles A. Baylis. New York, Prentice-Hall, 1939. 17+407 pp.

It is well known that logic, which for a long time remained in practically the same form in which Aristotle left it, has lately been undergoing a revolution. The cause of this phenomenon is, essentially, the impact of mathematics upon logic. It has been realized for some time that, by using mathematical methods, the statement of logical principles can be greatly simplified; likewise their scope can be extended so as to take in types of reasoning—such as those dealing with relations—which were only awkwardly handled by the traditional methods. The result has been the creation of an entirely new formal framework of great power and elegance.

As in many such revolutions a long time has had to elapse between the discovery of the new ideas and their appearance in elementary textbooks. Although the revolution in logic has been going on for nearly a century, yet it is only within about the last decade that textbooks of general logic have shown appreciably the effects of the new ideas. But since 1930 there have appeared a number of books of that character in which this effect has been marked. These include books by Stebbing, Cohen and Nagel, Eaton, Chapman and Henle, and a recent book by Churchman. None of these books contains a complete revamping of the subject, but every one is something of a compromise between the old and the new; nevertheless the amount of information concerning the new doctrine is considerable.

The present book belongs to the category just described. It aims "to provide a simple and clear survey of the field of formal logic synthesizing classical and modern developments into a unified treatment." It is the joint work of a mathematician and a philosopher, both of whom identified themselves with the new movement to the extent of becoming officers in the Association for Symbolic Logic. Under these circumstances a deeper penetration by the new ideas is to be expected.

The book begins with a chapter on the nature of logic. This is extremely well done; there is nothing comparable to it in any of the other books at the reviewer's disposal. A feature of it is the inclusion of examples of relatively complicated reasoning processes to be carried out informally. This is followed by a chapter on logical symbolism, in which the nature of symbols, and also of propositions, is discussed. After this follow two chapters containing a philosophical discussion of individuals, concepts, relations, classes, etc. The next two chapters are devoted to the traditional analysis of propositions and syllogisms. Boolean algebra, with a class interpretation, is then taken up in a chapter entitled "Formal Non-syllogistic Class Inference." After a chapter on classification, definition, and descriptions there follow three chapters, with a distinctly modern flavor, on the propositional algebra, the theory of propositional functions, and the formulation of logical deductive systems. The book closes with two chapters on probability and scientific method, occupying together about fifty pages.

The above outline will give an idea of the scope and general plan of the book. Let us now consider a few criticisms.

The reviewer is somewhat disappointed in regard to the treatment of the traditional logic. The procedures of the Boolean class algebra are simpler than those of the traditional logic and include them as special cases. Consequently, with the introduction of the former logic the latter has become practically useless. Yet in spite of this the authors devote a considerable portion of their book to developing the traditional logic along essentially traditional lines; in particular they establish the valid moods by giving a table showing the possible conclusions in each of 64 possible cases, and then saying that the reader may check the table by means of the Veine diagrams. There should be a Society for the Prevention of Cruelty to Students to stop this sort of thing!

The foregoing must not be supposed to mean that traditional logic has no uses whatever. There are in fact two main uses, neither of them practical. In the first place a book like this will be studied largely by philosophers; and in philosophy it seems that the history of the subject is especially important. From this point of view Aristotelian logic will be interesting to philosophers for a long time to come. Again, the derivation of traditional logic on the basis of an underlying Boolean algebra is a useful illustration of the technique of the latter. The first of these purposes will be equally well served, and the second vastly better, if traditional logic is treated after Boolean algebra, rather than before. Then the syllogistic rules can be derived, e.g. in ways suggested by the reviewer in *Mind*, vol. 45 (1936), pp. 209–216.

In connection with the above criticism there is another related one. The traditional hypothetical syllogism, dilemma, etc., are treated by the authors in the chapter on syllogisms. It is evident, however, that these topics have to deal with propositions as unanalyzed wholes, and should therefore be treated in an earlier part of the discussion. It would be useful to have all the relations between propositions as such, including those discussed here in Chapter 2, treated together under the heading Propositions. So far Churchman is the only author who has done this (see his Chapter 2).

The discussion should be preceded by a general discussion of significance of formal methods. The present authors postpone this to Chapter 11, at which time the student has become sufficiently sophisticated to tackle the difficulties of applying this idea to the calculus of propositions. But an idea of the nature of formal reasoning, as exemplified in the class calculus, can be presented to the reader at a much earlier stage; just as a quite adequate idea of the nature of logic as such is presented in Chapter 1. Such a discussion is the natural sequel to Chapter 1 as presented; and the reviewer was conscious of a hiatus at that point. A discussion of the sort stated is necessary in order to answer the question of how we can use formal methods to establish logical theorems, such as would be necessary for a development of the theory of the syllogism according to the above lines.

Let us now turn to a criticism of a different kind. In the discussion of concepts and classes the authors introduce at the start two levels of equivalence (and hence of inclusion), viz., strict or logical equivalence, and material or factual equivalence. The distinction is in keeping with a conceptualist-idealist point of view which the authors maintain throughout the work. The reviewer sympathizes with this point of view, and recognizes that for some purposes it may be necessary to consider simultaneously not only two, but even three or more such levels of abstraction. But if such distinctions are introduced into the calculus of classes the result is a mathematical system of great complexity. It seems better, when setting up the formal calculus, to introduce it first with only equality relation; to point out that this equality relation may be interpreted in either way; and to reserve the consideration of a calculus with two or more equality (or inclusion) relations until a later stage of the inquiry. Generalizations of this kind are not considered in any other elementary text.

To sum up, we have in this book by Bennett and Baylis a textbook on logic, designed for presentation to beginners, and intended as an introduction to modern mathematical, as well as to traditional formal logic. This is a difficult expository problem; and one for which a thoroughly satisfactory solution has not yet been found by anyone. That the book should, under these circumstances, be something of a compromise, is perhaps inevitable. The reviewer has criticized it from an ideal point of view, with reference to the goal to be attained—which, by the way, is of some importance for mathematics; these criticisms are to be taken not as pointing out defects in this book but as suggesting ways in which the next approximation to the goal can be improved. The text is one of great merit; most of the criticisms here made would apply to any similar book the reviewer knows of.

HASKELL B. CURRY

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*The Theory of Group Characters.* By D. E. Littlewood. Oxford University Press, 1940. 8+292 pp.

The theory of group representations and group characters has already been treated very recently in the comprehensive expositions of Hermann Weyl's *The Classical Groups*,[1] and F. D. Murnaghan's *The Theory of Group Representations*. We now have a third treatment. All three differ not only in emphasis but also in the spirit[2] of their approach to the subject.

Littlewood's book is intended by him to give "a simple and self-contained exposition of the theory[3] in relation to both finite and continuous groups, and to develop some of its contacts with other branches of pure mathematics such as invariant theory and the theory of symmetric functions." Thus the first fifty-two pages of his text are devoted to an exposition designed to make it self-contained. Chapter I consists mainly of a discussion of the classical canonical form of a matrix under similarity transformations, the properties of unitary, orthogonal and real orthogonal matrices, the reduction of Hermitian matrices under unitary transformations, and the definition of direct product. Chapter II presents the concept of an algebra and its regular representations, the consequent definition of trace, and the further topics necessary for an understanding of the property that a semi-simple algebra over the complex field may be expressed as a direct sum of total matric algebras. The third chapter includes the topics of permutation groups, the relation of the number of partitions of an integer $n$ to the number of classes in the symmetric permutation group on $n$ letters, and the representations of finite groups by permutation groups. Finally the discussion of the semi-simplicity of a group algebra and the basic theory of group representations are presented in the beginning of Chapter IV.

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[1] Reviewed in this Bulletin, vol. 46 (1940), pp. 592–595.

[2] Cf. Footnote 4.

[3] Of group *characters*, not of group representations.

The somewhat sketchy character of this first part of Littlewood's book would seem to have been dictated by its brevity and to result necessarily in proofs which are neither as complete[4] nor as conclusive as is desirable. Nonetheless, simpler and shorter modern proofs (of the results on matrices and linear algebras which are developed) do exist and could have been employed. They were hardly to be expected, however, in an exposition which uses *spur* for *trace* and *trace* for a special *spur*, which employs *Frobenius algebra* where nearly all others use *group ring* or *group algebra*, which returns to *simple* matric algebra after fourteen years of *total* matric algebra, which presents the theory of group representations so that a major mining operation is necessary to unearth the fundamental *Schur lemma*, and which ignores completely nearly all work of the past fifteen years on matrices and linear algebras.

The remaining portion of the text comprises nearly five-sixths of it, and is devoted to the properties of group characters themselves. They are defined in Chapter IV and a beginning is made there also of the subject matter of Chapter IX, which is a study of the relations between the characters of a group and those of its subgroups. In Chapter V the formula of Frobenius for the characters of the symmetric group is derived and the properties of primitive characteristic units and the Young tableaux are treated. Chapter VI contains a discussion of immanants and $S$-functions.[5] The author remarks that the $S$-functions may be regarded as generalizations of the coefficient functions in Taylor series and so his Chapter VII consists of a study of relations between $S$-functions and special series. Chapter VIII presents a method for the numerical computation of the characters of the symmetric permutation group. The main topic of Chapter IX is stated above but we call attention to Littlewood's interpretation of the topic as a study also of the problem of determining the structure of a group with given characters by an analysis of the characters. The last two chapters connect the theory of group characters for finite groups with the theory of continuous groups, and the book closes with an appendix consisting of tables of the characters of the symmetric groups on $m \leqq 10$ letters and of transitive subgroups.

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[4] For example, the demonstration given of the primitive result that a matrix satisfies its characteristic equation uses the similarity theory to prove the result for matrices with distinct characteristic roots and is completed by the statement "if $A$ is any matrix whatsoever we can find a matrix $Z$ such that $A+\mu Z$ has all its characteristic roots distinct and satisfies its characteristic equation. We now take the limit as $\mu$ tends to zero, whence $A$ satisfies its characteristic equation."

[5] Richard Brauer has noted the fact, in his review of this book in the Mathematical Reviews, vol. 2 (1941), p. 3, that these functions were obtained by I. Schur in his Berlin thesis of 1941 as the characters of the integral rational representations of the full linear group. See also Brauer's review for certain questions he raises about the validity of Littlewood's discussion of the characters of a group relative to those of its subgroups.

Littlewood's book is thus primarily an exposition in which the major emphasis is on the theory of group characters for its own sake rather than as a tool for other theories. This is contrary to the attitude of the original investigators in the field, and does result in a book which is notable for the immense number of formulae and formidable computations it contains. The author's justification of his attitude lies perhaps in the remark with which he opens his preface: "Since the discovery of group characters by Frobenius at the end of the last century the development of the theory has been so spectacular and the theory has shown such powerful contacts with other branches of mathematics, both pure and applied, that the inadequacy of its treatment by text-books is rather surprising." The reviewer is inclined to regard as exaggeration the words "spectacular," "powerful" and "surprising," and as unjust the word "inadequacy," but hopes with the author that the special properties of group characters which he derives may have some future interesting applications.

A. A. ALBERT

*The Theory of Group Representations.* By Francis D. Murnaghan. Baltimore, The Johns Hopkins Press, 1938. 369 pp.

Books on group representations come out in waves, it seems. In short succession, three new books on the subject appeared: Murnaghan's Theory of Group Representations, Weyl's Classical Groups, and D. E. Littlewood's Theory of Group Characters. Perhaps, it is worthwhile to compare the general point of view of these new volumes with the books published during the preceding periods in the history of the theory.

First, group representations were treated as a special chapter of the theory of finite groups, furnishing a powerful method for the study of these groups. The books of Burnside, Blichfeldt, Dickson, Miller, and Speiser belong to this period. Of course, in those days no self-respecting physicist could be seen near the shelves of libraries housing works on group theory. This was the realm of mathematicians, interested in the subject because of its great mathematical importance and its inner beauty.

Suddenly, with the birth of quantum mechanics and modern nuclear physics, everything was changed. There were no self-respecting theoretical physicists anymore who would not know about representations, Schur's lemma, characters, the orthogonality relations, and so forth. The books of this period (Wigner's Gruppentheorie, Weyl's Theory of Groups and Quantum Mechanics, van der Waerden's Gruppentheoretische Methode in der Quantenmechanik) were dedicated half to group theory and half to quantum mechanics, treating the former subject as far as it was needed for the applications to physics. There is no need to describe the immense progress accomplished in this work. Very likely, the books mentioned will be counted among the classics of science.

It is not surprising that the opening of this new field was accompanied by a change of existing values. Certain special groups, the symmetric group and the rotation group suddenly came into prominence. It would not be true to say that they reenacted the story of Cinderella. Such mathematicians as Frobenius, A. Young, I. Schur, Weyl had studied them in detail, but apparently nobody thought of writing books on them, before the applications to physics were found. Of course, the mathematical work was obliterated by the spectacular success of the applications.

What we see today, seems a natural reaction. The mathematicians feel that a theory which admits such applications deserves to be put in the form of books for its own sake. Of course, the needs of physics do not exhaust the mathematical riches. There are more general cases which can be treated, connections to other mathematical theories which can be studied, new questions which can be asked. In this situation, it is hardly an accident that several mathematicians felt the need for a new book on group representations and proceeded to write it, independently of each other. The mathematical theories were treated in all detail, the physical applications were not given. Of course, the three books of Murnaghan, Weyl, and Littlewood differ widely, owing to the different interests of the authors, and their different points of view.

Professor Murnaghan describes the object and scope of his book with the following words: "We have attempted to give a quite elementary and self-contained account of the theory of group representations with special reference to those groups (particularly the symmetric group and the rotation group) which have turned out to be of fundamental significance for quantum mechanics (especially nuclear physics)."

The book begins with a short discussion of the group concept, of linear spaces, and of matrices. No systematical treatment of these subjects is attempted, while, in the later text, the knowledge of the reader is widened, whenever the need arises. We have a representation of a group $G$ by linear transformations of a fixed "carrier space" $S$, if with every element $A$ of $G$ a linear transformation $M_A$ of $S$ is associated such that we have $M_A M_B = M_{AB}$ for any two group elements $A$, $B$ of $G$. Chapter II deals with the question of reducibility. If all the linear transformations of a representation $F$ leave a subspace $S_1$ of the carrier space $S$ invariant, then the representation is reducible. It breaks up into two representations $F_1$ and $F_2$, where $F_1$ consists of the linear transformations induced by $F$ in $S_1$, and $F_2$ of those induced by $F$ in the projection space of $S$ modulo $S_1$. We have complete reducibility, when $S$ is the direct sum of two invariant subspaces $S_1$ and $S_2$; in this case, $F$ is completely determined by the two lower dimensional representations $F_1$ and $F_2$. (It may be remarked that the terminology of the book is not the customary one; what Murnaghan terms "completely reducible" is usually denoted as decomposable while the words "completely reducible" are used in another connection.) Then the fundamental theorem of Burnside is proved, which gives a necessary and sufficient condition for reducibility of representations; after that follows the generalization of this theorem by Frobenius and Schur. In order to handle the question of complete reducibility in important cases, Auerbach's theory of bounded representations is developed. The third chapter is concerned with the representations of finite groups and their characters. Each representation is characterized by its character, i.e., the trace of the linear transformation $M_A$ which represents the general element $A$ of $G$. The main properties of these group characters are derived. This is about as much as is given of the general theory, the greater part of the book is dedicated to the study of special groups, and it is on these special groups that the emphasis of the book is placed. Two types of finite groups are treated in detail, the symmetric and the alternating permutation groups. An important method for dealing with finite groups is the forming of the average of a function of a general group element. In order to extend this method to compact continuous groups, it is necessary to replace the summation by an integration over the group manifold. Particular attention has been devoted to this method of group integration for the unitary and the orthogonal groups. The treatment of the representations of the full linear group includes all continuous representations. In the case of the orthogonal group, not only the one-valued representations but also the two-valued spin-representations are thoroughly studied. In all these cases, many interesting properties of the group characters are derived, e.g., the analysis of the direct product of two representations is given. This is one of Professor Murnaghan's own fields of research. Two short chapters on crystallographic groups and on the Lorentz group and semivectors conclude the book.

This description of the contents will show that the "meet" of Murnaghan's book and Weyl's "Classical Groups" is smaller than one might have expected. Actually, on several occasions, friends of the reviewer expressed regret that two so similar books appeared at the same time. It is therefore necessary to emphasize that they are written for quite different classes of readers, with different ultimate aims. The two authors have different mathematical temperaments, their tastes differ. Professor Murnaghan has kept the modern algebraic view more or less in the background, and sometimes is less insistent on general results than, perhaps, the algebraist would like to see. This should not be understood as a criticism of the book. The avowed aim of the author was to make the theory of representations of the most important groups accessible in a relatively elementary way, and to write a self-contained book, without borrowing much from other theories. It is not the connection of the theory with other parts of algebra and mathematics in which Murnaghan is interested. The need to study the representations of certain groups has arisen, for instance in physics, and he wants to find out as much as possible about these representations, in the most direct way.

Professor Murnaghan's book is written with the clarity which could be expected from the author. All the details are worked out with great care. So the book should prove of high value to mathematicians and physicists who want to become acquainted with the material. Very little special knowledge is required of the reader. The volume can be warmly recommended.

There are a number of misprints, but hardly any of them are of a serious nature. Professor Murnaghan authorizes me to say that a list of misprints is available and will be supplied on request.

RICHARD BRAUER

# ANNOUNCEMENT OF THE PROJECT FOR THE COMPUTATION OF MATHEMATICAL TABLES

FEDERAL WORKS AGENCY
WORK PROJECTS ADMINISTRATION
NEW YORK, N. Y.

Sponsored by the Bureau of Standards, Washington, D. C., Dr. Lyman J. Briggs, Director, and Dr. Arnold N. Lowan, Technical Director of the Project. Administrative Staff: Mr. Murray Pfeferman and Mr. Milton Abramowitz. Technical Staff: Dr. Gertrude Blanch, Dr. Norman Davids, Mr. William Horenstein, Mr. Jacob L. Miller, Miss Ida Rhodes, Dr. J. Rosenthal, and Mr. Herbert E. Salzer.

The Project for the Computation of Mathematical Tables has been in operation since January 1, 1938. The aim of this project is to compute Mathematical Tables of fundamental importance in Mathematics, Physics, Chemistry, Engineering, Statistics and related sciences. Lists of (A) Tables Published, (B) Tables in Process of Reproduction, (C) Tables for Which Manuscripts are Completed, (D) Tables for Which Computations are Completed, (E) Tables for Which Computations are in Progress and (F) Tables under Consideration, are given in this announcement.

The project would welcome suggestions for the computation of tables of interest in pure and applied Mathematics as well as information regarding computational work in progress elsewhere. Communications should be addressed to Oliver A. Gottschalk, Acting Administrator, Work Projects Administration, 70 Columbus Avenue, New York City.

Requests for copies of published tables should be addressed to Dr. Lyman J. Briggs, Director, National Bureau of Standards, Washington, D. C.

## *A. Tables Published*

1. Table of the First Ten Powers of the Integers from 1 to 1,000. 80 pp. (1939).
2. Tables of the Exponential Function $e^x$. 535 pp. (1939).

$x = [-2.5000(0.0001)1.0000; 18D]$,[1] $x = [1.0000(0.0001)2.5000; 15D]$,
$x = [2.500(0.001)5.000; 15D]$, $x = [5.00(0.01)10.00; 12D]$.

3. Tables of sin $x$ and cos $x$ for Radian Arguments. 275 pp. (1940).

$x = [0.000(0.001)25.000; 8D]$, $x = [0(1)100; 8D]$,
$x = [0.00001(0.00001)0.00009; 15D]$, $x = [0.0001(0.0001)0.0009; 15D]$,
$x = [0.001(0.001)0.009; 15D]$, $x = [0.01(0.01)0.09; 15D]$,
$x = [0.1(0.1)0.9; 15D]$, $x = [0.00000(0.00001)0.01000; 12D]$.

There is also included a conversion table between radians and degrees.

4. Tables of sin $x$, cos $x$, sinh $x$, cosh $x$ for Radian Arguments. 405 pp. (1940).

$x = [0.0000(0.0001)1.9999; 9D]$, $x = [0.0(0.1)10.0; 9D]$.

There is also included a conversion table between radians and degrees.

5. Tables of Planck's Radiation and Photon Functions. Published in the Journal of the Optical Society of America, vol. 30, pp. 70–81 (1940).

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[1] The figures in the square brackets give the range and the interval of the argument, and the number of decimal places ($D$) or significant figures ($S$) in the entries.

6. Table of $Si(x)$, $Ci(x)$, $Ei(x)$ and $-Ei(-x)$, where

$$Si(x) = \int_0^x \frac{\sin t}{t}\,dt, \qquad Ci(x) = \int_\infty^x \frac{\cos t}{t}\,dt,$$

$$Ei(x) = \int_{-\infty}^x \frac{e^t}{t}\,dt, \qquad -Ei(-x) = \int_x^\infty \frac{e^{-t}}{t}\,dt,$$

$$x = [0.0000(0.0001)2.0000;\ 9D].$$

7. Table of $Si(x)$, $Ci(x)$, $Ei(x)$ and $-Ei(-x)$,

$$x = [0.000(0.001)10.000;\ 9S,\ 10S].$$[1]

8. Tables of Grid Coordinates (American Polyconic Projection) at 5 minute intervals of latitude and longitude for latitudes from 7° to 28°. (For the War Department.)

### *B. Tables in Process of Reproduction*

1. Table of $\log_e N$, $N = [1(1)100{,}000;\ 16D]$.
2. Table of $\log_e x$, $x = [0.0000(0.0001)10.0000;\ 16D]$.
3. Table of the Probability Functions:

$$\frac{2}{\pi^{1/2}}e^{-x^2}, \qquad \frac{2}{\pi^{1/2}}\int_0^x e^{-t^2}dt,$$

$$[x = 0.0000(0.0001)1.0000;\ 15D], \qquad x = [1.000(0.001)5.736;\ 15D].$$

4. Table of the Probability Functions:

$$\frac{1}{(2\pi)^{1/2}}e^{-x^2/2}, \qquad \frac{1}{(2\pi)^{1/2}}\int_{-x}^x e^{-t^2/2}dt,$$

$$x = [0.0000(0.0001)1.0000;\ 15D], \qquad x = [1.000(0.001)7.800;\ 15D].$$

5. Tables of Section Moduli and Moments of Inertia for Structural Members used in Naval Architecture. (For the Bureau of Marine Inspection and Navigation.)
6. Table of the Electronic Functions

$$G = \frac{1}{(1-\beta^2)^{1/2}}, \qquad V = \frac{m_0c^2}{e}(G-1) \quad \text{and} \quad H\rho = \frac{m_0c^2}{e}\beta G$$

for $\beta$ ranging from 0 to 0.99999 99990 at various intervals.

7. Table of arc tan $x$ for

$$x = [0.000(0.001)7.000(0.01)50.00(0.1)300.0(1)2000(10)10000;\ 12D].$$

8. Table of Five-Point Lagrangian Interpolants for arguments ranging between 0 and 2, intervals of 0.001. (For the War Department.)

### *C. Tables for which Manuscripts are Completed*

1. Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for complex arguments, $z = \rho e^{i\theta}$, for $\rho$ ranging from 0 to 10 at intervals of 0.01 and $\theta$ ranging from 0° to 90° at intervals of 5°, to 10 decimal places.
2. Table of the definite integrals:

$$A(k, n) = \int_0^1 x^k \sin n\pi x\,dx \quad \text{and} \quad B(k, n) = \int_0^1 x^k \cos n\pi x\,dx \quad \text{for} \quad k = 0,$$

$$1, 2, \cdots, 10 \quad \text{and} \quad n = 1, 2, \cdots, 100 \text{ to 15 decimals.}$$

3. Table of $\int_0^x J_0(t)dt$ from 0 to 10 at intervals of 0.01 to 10 places.
4. Table of exact values of five-point Lagrangian Interpolants. (For the War Department.)

### *D. Tables for which Computations are Completed*

1. Table of the fractional powers $x^y$ for both $x$ and $y$ ranging from 0 to 1 at intervals of 0.01 and for other values of $x$ and $y$ to 15 decimal places.
2. Tables of Stellar Functions for "Point-Source" Models.

### *E. Tables for which Computations are in Progress*

1. Bessel Functions of the fractional orders $\pm 1/3$, $\pm 2/3$, $\pm 1/4$, $\pm 3/4$ for real and pure imaginary arguments ranging from 0 to 10 at intervals of 0.01 to 10 decimal places.
2. Table of $Q_n(x) = (\pi/2x)^{1/2} J_{n+1/2}(x)$ for $n = 1, 2, 3, \cdots, 20$ and $x$ ranging from 0 to 10 at intervals of 0.01 to 10 decimal places.
3. Table of the Bessel Functions $Y_0(z)$ and $Y_1(z)$ for the same complex arguments as in *C*.1.
4. Tables of tan $x$ and cot $x$ for $x$ ranging from 0 to 2 radians at intervals of 0.0001.
5. Table of $Q_n(x) = (\pi/2x)^{1/2} J_{n+1/2}(x)$ for $n = -21, -20, \cdots, 0$ and $x$ ranging from 0 to 10 at intervals of 0.01 to 10 decimal places.
6. Tables of $Si(x)$ and $Ci(x)$ for $x$ ranging from 10 to 100 at intervals of 0.01.
7. Various hydraulic tables based on Kutter's and Manning's formulae. (For the War Department.)
8. Table of the Associated Legendre Functions of the First and Second Kind, for $n$ ranging between 1 and 10, and $m$ between 0 and 4; for arguments $x$ and $ix$ where $x$ ranges between 0 and 10 at intervals of 0.1. Also corresponding values for half-integral values of $n$, and values of the functions for arguments in degrees. (Tabulation suggested by the National Defense Research Committee.)
9. Tables of Grid Coordinates of Northwestern Extension of U. S. (American Polyconic Projection. For the War Department.)
10. Table for Map Projections of Northwestern Extension of U. S. (American Polyconic Projection. For the War Department.)
11. Tables of Length of Meridional Arc at one-minute intervals. (For the War Department.)

### *F. Tables under Consideration*

1. A 12 place table of Inverse Circular and Hyperbolic Functions other than arc tan $x$.
2. Extensive tables of Elliptic Integrals and Elliptic Functions.
3. Table of the Gamma Function for complex arguments, $x+iy$, for both $x$ and $y$ ranging from 0 to 5 at intervals of 0.05.
4. Table of the Error Functions for complex arguments.
5. The zeros of the Legendre Polynomials up to the 16th order to 15 decimal places and the weight coefficients for Gauss' Mechanical Quadrature Formula.
6. Tables of the Confluent Hypergeometric Functions for selected values of the parameters.
7. Table of the integral $\int_0^x Y_0(t)dt$.
8. Tables of three-point, four-point, and six-point Lagrangian Interpolants.
9. Table of the first ten powers of the reciprocals of the integers from 1 to 1000.
10. Table of solutions of Mathieu's Differential Equation.

## NOTES

There is a mistake in the naming of the Committees to Select Hour Speakers on page 179 of the March Bulletin. These committees are: For the Annual and Summer Meetings, Professors J. R. Kline, Chairman, L. M. Graves and M. H. Ingraham; For Eastern Sectional Meetings, Professors T. R. Hollcroft, Chairman, C. R. Adams and Einar Hille. Those for the Western and Far Western Sectional Meetings are correct as announced.

The Mathematical Association of America plans a series of publications called The Herbert Ellsworth Slaught Memorial Papers. These papers will be primarily of expository character, probably between forty and eighty printed pages in length.

The Sixth International Congress for the Unity of Science will be held at the University of Chicago September 2–6, 1941.

A symposium including an address by Professor G. A. Bliss of the University of Chicago on *Functions of lines in ballistics* and an address by Professor Marston Morse of the Institute for Advanced Study on *Some new methods and problems in minimal surface theory* was held at the University of Cincinnati on November 9, 1940.

A symposium on mathematical economics was held at the University of Notre Dame February 28 and March 1, 1941. The following addresses were given: *Functionals in the theory of economics*, Professor H. T. Davis, Northwestern University; *The theory of technological unemployment*, Professor Oscar Lange, University of Chicago; *Problems of valuation and rate-making, in public enterprises*, Professor Harold Hotelling, Columbia University; *The dynamics of commodity prices*, Dr. C. F. Roos, Institute for Applied Economics, New York City; *Statistical cost-curves and price policy*, Professor Joel Dean, University of Chicago; *The determination of weight-functions for dynamic economics*, Dr. Franz Alt, Institute for Applied Economics; *Unexpected effects of certain taxes*, Professor Harold Hotelling; *On the solvability of the Walrasian system of equations*, Dr. Abraham Wald, Columbia University; *The geometrical and arithmetical determination of the index of prices*, Professor Karl Menger, University of Notre Dame.

Professor Lawrence Crawford of South African College, Cape Town, has been elected president of the Royal Society of South Africa.

At the Annual Meeting of the Mathematical Association of America Professor R. W. Brink of the University of Minnesota was elected president, Professor R. E. Langer of the University of Wisconsin first vice president, and Professor B. H. Brown of Dartmouth College second vice president.

Professor Einar Hille of Yale University has been elected a member of the Royal Physiographical Society of Lund, Sweden, as successor of Professor Vito Volterra.

Associate Professor A. A. Albert of the University of Chicago delivered the William Lowell Putnam lecture at Harvard University in March. He spoke also at Princeton University, the Institute for Advanced Study and Brown University.

The following eighty-nine doctorates, with mathematics or mathematical physics as a major subject, were conferred during 1940 in universities in the United States and Canada; the major subject is mathematics unless otherwise specified. The university, month in which the degree was conferred, minor subject (other than mathematics), and the title of the dissertation are given in each case if available.

P. H. Anderson, Illinois, June, minor in economics, *Distributions in stratified sampling.*

L. A. Aroian, Michigan, February, *A study of R. A. Fisher's t-distribution and the related F-distribution.*

Fred Assadourian, New York, June, minor in physics, *Intransitive Abelian almost-translation groups of almost-periodic functions.*

A. A. Aucoin, Louisiana State, May, *An algebraic treatment of diophantine analysis.*

R. A. Beaumont, Illinois, June, minor in astronomy, *Projections of non-Abelian groups upon Abelian groups containing elements of infinite order.*

E. G. Begle, Princeton, June, *Locally connected spaces and generalized manifolds.*

J. O. Blumberg, Pittsburgh, August, *Fundamental regions in* $S_6$ *for the simple quaternary* $G^4_{60\phi}$, *Type* II.

J. I. Bohnert, Pittsburgh, August, *Conical surface roulettes.*

A. A. F. Brown, Princeton, June, *Jordan fields on groups.*

G. W. Brown, Princeton, April, *The reduction of a certain class of composite statistical hypotheses.*

R. H. Bruck, Toronto, June, *The general linear group in a field of characteristic p.*

CALCUTTA CENTRAL

E. F. Canaday, Duke, June, *The sum of the divisors of a polynomial.*

R. F. Clippinger, Harvard, June, *Matrix products of matrix powers.*

R. H. Cole, Wisconsin, June, *The expansion problem associated with an ordinary linear differential equation and boundary conditions applying at a set of colinear points.*

R. H. Cook, Iowa State, December, major in applied mathematics, minors in physics and mathematical physics, *Evaluation of error in approximation by function methods.*

W. E. Cox, Ohio State, June, *On commutative normal matrices and unitary equivalence of matrices.*

A. B. Cunningham, West Virginia, June, minor in physics, *Non-involutorial space transformations associated with a $Q_{1,2}$ congruence.*

Norman Davids, New York, June, minor in physics, *Minimal surfaces spanning closed manifolds and having prescribed topological position.*

Bernard Dimsdale, Minnesota, July, minor in physics, *Lacunary orthogonal polynomials.*

W. D. Duthie, Princeton, June, *Segments in ordered sets.*

John Dyer-Bennet, Harvard, June, *A free algebra with three operations.*

C. J. Everett, Wisconsin, March, *Rings as groups with operators.*

Sheng-Chin Fan, Michigan, June, *Integration with respect to an upper measure function.*

H. T. Fleddermann, Louisiana State, May, *Density properties of sets.*

Sister Elizabeth Frisch, Catholic University, June, minors in physics and biology, *Determination of a set of independent relations characterizing a certain system consisting of a conic and quartic curve.*

Holly C. Fryer, Iowa State, July, major in mathematical statistics, minors in mathematics and genetics, *An analysis of group differences arising from a Poisson distribution of observations obtained from irradiation experiments.*

R. E. Gaskell, Michigan, June, *A problem in heat conduction and an expansion theorem.*

A. M. Gelbart, Massachusetts Institute of Technology, June, minor in physics, *Growth properties of functions of two complex variables.*

J. H. Giese, Princeton, June, *Conformally flat hypersurfaces of spaces of constant curvature.*

P. W. Gilbert, Duke, June, minors in physics and philosophy, *N-to-one mappings of linear graphs.*

B. E. Gillam, Missouri, June, *A new set of postulates for euclidean geometry.*

H. E. Goheen, Stanford, June, minor in physics, *On the primitive groups of classes* 4P *and* 5P.

Katharine E. Hazard, Chicago, August, *Index theorems for the problem of Bolza in the calculus of variations.*

M. H. Heins, Harvard, June, *Studies in the theory of functions analytic and single-valued in a multiply-connected region.*

A. D. Hestenes, Harvard, June, *On the transformation* $x'^{-c}e^{x'} = x^{-c}e^{x+1}$ *and the related linear* $\phi$*-difference equations of G. D. Birkhoff.*

J. F. Heyda, Illinois, June, minor in physics, *Uniqueness properties over sets of positive linear measure of functions of a complex variable.*

F. B. Hildebrand, Massachusetts Institute of Technology, June, minor in physics, *Solution by polynomial approximation of singular integral equations arising in static field theory.*

L. C. Hutchinson, Massachusetts Institute of Technology, June, minor in physics, *On the classification of the trivector.*

R. F. Jackson, Harvard, June, major in mathematical physics, *Reciprocity theorems in the theory of atomic spectra.*

F. P. Jenks, Notre Dame, June, *A new set of postulates for Bolyai-Lobachevsky geometry.*

J. L. Kelley, Virginia, June, *A study of hyperspaces.*

Fred Kiokemeister, Wisconsin, June, *The parastrophic criterion for the factorization of primes.*

H. L. Langhaar, Lehigh, June, *Steady flow in the transition length of a cylindrical conduit.*

J. H. Levin, Chicago, August, *Minima of double integrals with respect to unilateral variations, and applications to subharmonic functions.*

P. E. Lewis, Illinois, June, minor in physics, *Characters of Abelian groups.*

D. T. McClay, Harvard, June, *Analytic families of somas.*

A. W. McGaughey, Cincinnati, June, *Lacunary double Fourier series.*

Ingo Maddaus, Michigan, February, *Types of "weak" convergence of linear normed spaces.*

Dorothy Maharam, Bryn Mawr, June, *The notion of measure on abstract sets.*

A. V. Martin, Duke, June, minor in philosophy, *Monotone transformations of non-compact two-dimensional manifolds.*

A. B. Mewborn, California Institute of Technology, June, minor in physics, *Contributions to the general geometry of paths.*

C. E. Miller, Toronto, June, *On finite collineation groups of a given degree.*

W. L. Mitchell, Wisconsin, July, *Topological rings and infinite matrices.*

J. C. Montgomery, Yale, June, *Relations between the roots of a polynomial and the roots of its first derivative.*

A. M. Mood, Princeton, June, *The distribution theory of runs.*

C. W. Moran, Illinois, June, minor in physics, *Asymptotic theory of linear differential equations singular in several parameters.*

W. L. Morris, Duke, June, minor in physics, *Kinetic-harmonic systems.*

H. T. Muhly, Johns Hopkins, June, *Valuations and infinitely near algebraic loci.*

C. G. Mumford, Duke, June, *Topics in double Fourier series.*

J. P. Nash, Rice Institute, June, *A class of continuous functions and convergence criteria for their Fourier series.*

Albert Neuhaus, Chicago, March, *Products of normal semi-fields.*

H. E. Newell, Wisconsin, June, *On the asymptotic forms of the solutions of an ordinary linear matric differential equation in the complex domain.*

K. L. Nielsen, Illinois, June, minor in astronomy, *General boundary value problems for linear differential equations.*

J. M. H. Olmsted, Princeton, June, *Lebesgue theory on a Boolean algebra.*

R. S. Pate, Illinois, June, minor in astronomy, *Systems with operations which are not single-valued.*

H. V. Price, Iowa, August, minors in applied mathematics and education, *Analyses of a group of pre-tests for students of first year college mathematics.*

M. O. Reade, Rice Institute, June, *Generalizations to space of the Cauchy and Morera theorems.*

C. J. Rees, Pennsylvania, February, *Elliptic orthogonal polynomials.*

F. D. Rigby, Iowa, August, minors in applied mathematics and physics, *Topologies in distributive lattices.*

C. V. Robinson, Missouri, June, *Contributions to distance geometry:* (1) *Congruence order of euclidean and spherical subsets.* (2) *Helly theorems on the sphere.*

N. N. Royall, Brown, June, *On Laplace transforms of multiply monotonic functions.*

J. M. Sachs, Chicago, December, *Theory of surfaces in conjugate parameters.*

W. J. Schart, Ohio State, August, *Conditions for solutions of certain differential equations which have specified properties.*

Edith R. Schneckenburger, Michigan, February, *On 1-bounding monotonic transformations which are equivalent to homeomorphisms.*

K. C. Schraut, Cincinnati, June, *Convergence factor theorems for double and multiple series having unbounded partial sums.*

I. E. Segal, Yale, June, *Ring properties of certain classes of functions.*

C. E. Shannon, Massachusetts Institute of Technology, June, minor in electrical engineering, *Algebra for theoretical genetics.*

R. W. Shephard, California (Berkeley), August, I. *Incidence of taxation studied in a simplified economic system.* II. *A note on the length of production.*

Seymour Sherman, Cornell, June, minor in philosophy, *A comparison of linear measures in the plane.*

S. S. Smith, Chicago, August, *Motion of a spheroid and a sphere under mutual gravitation.*

Abraham Spitzbart, Harvard, June, *Approximation in the sense of least pth powers by polynomials with auxiliary conditions of interpolation.*

D. W. Starr, Illinois, June, minor in political science, *The Schrödinger wave equation from the point of view of singular integral equations.*

B. M. Stewart, Wisconsin, July, *Left-associated matrices with elements in an algebraic domain.*

G. B. Thomas, Cornell, June, *Regular ternary quadratic forms.*

V. J. Varino, Wisconsin, July, *An extension of the theory of matrices with elements in a principal ideal ring.*

L. F. Walton, California (Berkeley), August, *On ideal numbers.*

J. S. de Wet, Princeton, June, *On the connection between the spin and statistics of elementary particles.*

A. L. Whiteman, Pennsylvania, February, *Additive prime number theory in real quadratic fields.*

Orla V. Wood, California (Berkeley), May, *On the linear homogeneous differential equation of the second order with four regular singular points.*

Max Wyman, California Institute of Technology, June, minor in physics, *General differential geometry with two types of linear connection.*

The following doctorates were conferred in 1939, but were not included in the list in the preceding volume of this Bulletin (vol. 46, pp. 391–397).

J. F. Daly, Princeton, October, *On the unbiased character of likelihood-ratio tests for independence in normal systems.*

G. N. Garrison, Princeton, October, *Quasi-groups.*

Andrew Sobczyk, Princeton, October, *Projections in Minkowski and Banach spaces.*

M. E. Wescott, Northwestern, August, minor in economics, *A set of Newton polynomials analogous to Laguerre's polynomials.*

G. W. Whaples, Wisconsin, *On the structure of modulus with a commutative algebra as operator domain.*

Assistant Professor J. E. Eaton of Hofstra College has been promoted to an associate professorship.

Dr. K. C. Schraut of the University of Dayton has been promoted to an assistant professorship.

Assistant Professor Saunders MacLane of Harvard University is on leave this term at the University of Chicago.

The following appointments to instructorships are announced: Cornell University: Mr. W. H. Durfee; Fenn College (Cleveland): Mr. C. W. Topp; Harvard University: Dr. I. E. Segal; University of Michigan: Dr. Sam Perlis.

Mr. H. W. Macrosty, president of the Royal Statistical Society, died January 19, 1941, at the age of seventy-six years.

Professor Francesco Palatini of the Technical Institute of Turin died June 2, 1940, at the age of seventy-four years.

Professor Emeritus E. S. Loomis of Baldwin-Wallace College died December 11, 1940.

Professor Emeritus D. C. Miller of Case School of Applied Science died February 22, 1941, at the age of seventy-four years. He was a member of the Society from 1923 to 1936.

The death of Mr. J. O. Watts of Queen's University in Kingston, Ontario, has been announced.

# ABSTRACTS OF PAPERS

## SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

## ALGEBRA AND THEORY OF NUMBERS

### 182. Reinhold Baer: *A unified theory of projective spaces and finite abelian groups.*

It is the object of this investigation to develop in detail a theory which contains as special cases both projective geometry and the theory of finite abelian groups. The primary abelian operator groups lend themselves in quite a natural manner for this purpose. Those partially ordered sets that are exactly the systems of all the admissible subgroups of suitable primary abelian operator groups are determined, and it is shown that the group and its operators are completely determined by the system of admissible subgroups. Those questions which are fundamental in the two theories are discussed, such as the basis-theorem, the relations between dualities and bilinear forms, and so on. (Received March 12, 1941.)

### 183. Richard Brauer: *On the connection between the ordinary and the modular characters of groups of finite order.*

Let $G$ be a group of finite order $g=p^a g'$, where $p$ is a fixed prime and $(p, g')=1$. If $\mathfrak{Z}$ is a character of $G$, then for every element $A$ of an order prime to $p$ the value $\mathfrak{Z}(A)$ is a linear combination of the modular group characters of $G$. The coefficients are rational integers, the decomposition numbers of $G$. In this paper it is shown that the value $\mathfrak{Z}(A)$ of $\mathfrak{Z}$ for elements $A$ of an order divisible by $p$ can be expressed by means of the modular characters of certain subgroups $N_i$ of $G$. The matrix $Z$ of all ordinary group characters of $G$ then appears as a product $DX$ of two square matrics. The matrix $X$ contains in its rows the values of the modular characters of $G$ and of the $N_i$ while the coefficients of $D$ are integers of the field of the $p^a$th roots of unity. A number of properties of the coefficients of $D$ are given. (Received March 18, 1941.)

### 184. R. H. Bruck and T. L. Wade: *Bisymmetric tensor algebra.* I.

This paper lays a basis for a study of the linear associative algebra of bisymmetric tensors (H. Weyl, *The Classical Groups*, p. 98) which gives a realization of a semisimple algebra. The ordered product of two tensors $A^{(i)}_{(j)}$ and $B^{(i)}_{(j)}$ is defined by $A^{(i)}_{(m)}B^{(m)}_{(j)}=C^{(i)}_{(j)}$, where $(i)=i_1\cdots i_p$. The unit tensor $\delta^{(i)}_{(j)}=\delta^{i_1}_{j_1}\cdots\delta^{i_p}_{j_p}$ may be decomposed into a direct sum of immanent tensors (T. L. Wade, abstract 47-1-10). For a tensor $A^{(i)}_{(j)}$, a determinant, adjoint, and inverse are defined with the aid of a tensor $\delta^{(i_1)\cdots(i_N)}_{(j_1)\cdots(j_N)}$ which constitutes a further generalization of the familiar generalized Kronecker delta; here $N=n^p$. Also the concepts of rank and of rank tensor are introduced. Explicit algebraic construction is given of the factors of the determinant in the case of a bisymmetric tensor. The determinant of a Kronecker product (Murnaghan, *The Theory of Group Representations*, pp. 68–69) is a special case of the determinant defined in this paper. (Received March 27, 1941.)

### 185. R. H. Bruck and T. L. Wade: *Bisymmetric tensor algebra.* II.

If an idempotent numerical tensor $C_{(j)}^{(i)}$ has rank $r$ (as defined in part I), its rank tensor, an absolute numerical tensor, factors into the product of two relative tensors. For a certain invariant subalgebra of the algebra of bisymmetric tensors, $C_{(j)}^{(i)}$ is the unit, and for this subalgebra the rank tensor plays the role of the generalized Kronecker delta for $p=1$ (see part I). $C_{(j)}^{(i)}$ may be thought of as the idempotent numerical tensor corresponding to an operator $c$, or as an immanent tensor corresponding to an $\epsilon$, where $c$ and $\epsilon$ are used in the sense of H. Weyl (*The Classical Groups*, pp. 120–127). In either case a determinant, adjoint, and inverse are defined, and the relation between the two cases is made explicit. In connection with this work the homomorphic correspondence between the group-ring of the symmetric group on $p$ letters and the algebra of absolute numerical tensors is established and used considerably. (Received March 27, 1941).

### 186. D. M. Dribin: *A theorem on sets of ideals in solvable extensions of algebraic number fields.*

The following theorem is proved: Let $k$ be an algebraic number field and $K|k$ a normal extension with solvable group. Let $\Pi$ be any set of prime ideals in $k$ which decompose into prime ideals of relative degree one in $K$ and which are all equivalent in $k$ in the ordinary (broad) sense. Then if $\Pi_K$ is the set of all prime ideal divisors in $K$ of the ideals in $\Pi$, all the ideals of $\Pi_K$ are equivalent (in the ordinary sense) in $K$. (Received March 14, 1941.)

### 187. Fred Kiokemeister: *The parastrophic criterion for the factorization of primes.*

The parastrophic form $\pi(\xi)$ of a Frobenius algebra is defined as the determinant of the parastrophic matrix in a given basis of the algebra. In particular the parastrophic form may be computed with respect to the basis of a maximal domain of integrity $K$ of an algebraic number field. Let $p$ be a rational prime number. Then the prime ideal factorization $(p)=\mathfrak{p}_1^{\alpha_1}\mathfrak{p}_2^{\alpha_2}\cdots\mathfrak{p}_s^{\alpha_s}$ of $p$ in $K$ is paralleled by the factorization of $\pi(\xi)$ modulo $p$ into a product of powers of irreducible factors. This result is obtained through a study of the relationship between the ideal structure of a linear algebra and the zeros of $\pi(\xi)$. (Received February 14, 1941.)

### 188. C. C. MacDuffee: *Products and norms of ideals.*

In an integral domain of a Frobenius algebra over the quotient field of a principal ideal ring, the number $h$ of ideal classes is finite. To every left (or right) ideal $\mathfrak{a}$ correspond $h$ matrices $A_1, A_2, \cdots, A_h$, one for every ideal class of the domain. The first matrix $A_1$, corresponding to the principal class, is a matrix which determines a minimal basis of $\mathfrak{a}$. If $B_1, B_2, \cdots, B_h$ correspond similarly to the ideal $\mathfrak{b}$ which belongs to the $i$th ideal class, then the matrix $C_1=A_iB_1$ corresponds to the product ideal $\mathfrak{a}\times\mathfrak{b}$. Thus ideal multiplication can always be accomplished by means of matric multiplication. Call $|A_i|=N_i(\mathfrak{a})$ the $i$th norm of $\mathfrak{a}$, and $N_1(\mathfrak{a})$ the principal norm. It is known that the principal norm of a product is not always equal to the product of the principal norms. It is here proved, however, that the principal norm of $\mathfrak{a}\times\mathfrak{b}$ is always equal to the product of the $i$th norm of $\mathfrak{a}$ by the principal norm of $\mathfrak{b}$. (Received March 31, 1941.)

### 189. Saunders MacLane and O. F. G. Schilling: *A formula for the direct product of cross product algebras.*

Let $N$ be a separable normal extension of a field $L$ with Galois group $\Gamma$. Suppose that $K$ is a normal subfield of $N$ belonging to the group $\Delta$. The authors prove the formula $(N, \Gamma, F)\times(K, \Gamma/\Delta, G)=(N, \Gamma, FG^*)$ where $F$, $G$ are factor sets belonging to $N$, $K$, respectively. The factor set $G^*$ of $N$ is obtained by "$\Delta$-symmetric extension" of $G$ to $N$. (Received March 20, 1941.)

### 190. Saunders MacLane and O. F. G. Schilling: *Group extensions characterizing complete fields.*

Let $F$ be a field which is relatively complete with respect to a valuation $\Phi$ with residue class field $\mathcal{J}$. The authors define the "dominant" $F'$ of $F$ as the factor group of the multiplicative group of $F$ modulo the group of units $u$ with $\Phi(1-u)>0$. The group $F'$ is a group extension of the multiplicative group of $\mathcal{J}$ by the value group. Suppose that $K$ is a finite separable extension of $F$ whose degree over the unramified field is prime to the characteristic of $F$. The dominant $K'$ of $K$ is then a "linked" extension of $F'$. The authors prove that the extensions $K\supset F$ are in 1-1 correspondence with the linked extensions $K'\supset F'$. The type number defined by Albert (Annals of Mathematics, (2), vol. 41 (1940), pp. 678–679) is essentially an integer describing a cyclic extension $K'\supset F'$. (Received March 20, 1941.)

### 191. Saunders MacLane and O. F. G. Schilling: *The principal divisor theorem for function fields.*

Let $F$ be a field of algebraic functions of one variable. The authors investigate the structure of extensions $K\supset F$ in which the divisors of a given class group $B$ of $F$ become principal. An unramified extension $K\supset F$ in which exactly the divisors of $B$ become principal is termed a Hilbert class field for $B$. In general there exist several "minimal" normal Hilbert class fields belonging to a group $B$. The Galois groups of such separable normal fields are abelian if an only if (i) the order of $B$ is prime to the characteristic of $F$, and (ii) the characters of $B$ can be realized in $F$. The methods of proof differ radically from the ones used in the theory of algebraic number fields. (Received March 20, 1941.)

### 192. I. M. Niven: *Equations in quaternions.*

It is proved that the equation $x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_n=0$, with real quaternion coefficients, has a root in the algebra of real quaternions when $n$ is odd. Also, the number of roots of this equation may be finite or infinite, and if finite, does not exceed $(2n-1)^2$. (Received March 20, 1941.)

### 193. I. M. Niven: *Sums of nth powers of quadratic integers.*

The first problem is to find those quadratic fields all of whose integers are expressible as sums of $n$th powers. This is solved completely, criteria being given in terms of $n$ and the basis of each field. The second problem is to determine whether any given quadratic integer is expressible as a sum of $n$th powers. Necessary and sufficient conditions are again given; in this case however, integers of real quadratic fields are not treated when $n$ is even. (Received March 5, 1941.)

### 194. R. S. Pate: *Rings with multiple-valued operations.*

A set $R$ of elements forming a hypergroup with respect to addition and having a distributive associative multiplication such that $ab$ is a subset of elements of $R$ is defined to be a ring. If $R$ is an additive group every product contains the same number $n$ of elements, and $n$ is a divisor of the order of the group. A subring $A$ of $R$ such that $AR\subset A$ is an ideal. To a fixed element $q$, to every element $r$ of $R$ may be associated an arbitrary set of corresponds $x_i$ and $y_i$ such that $r+x_i\supset q$ and $r\subset y_i+q$. A $Q$ ideal contains $q$, and, if it contains $r$, contains all corresponds of $r$. Usually, the necessary and sufficient condition that a set of ideals form a lattice is that they be $Q$ ideals. For a proper correspondence $Q$, the classical decomposition of ideals may be derived. An analogue of an indeterminate domain may be defined and a result derived similar to the Gauss lemma. It can be shown that the "indeterminate domain" of a ring with a "well-behaved" multiplication has this same multiplication only if it is an ordinary ring. (Received March 17, 1941.)

### 195. Sam Perlis: *A characterization of the radical of an algebra.*

The following theorem is proved: If $F$ is any field and $A$ is an algebra over $F$ with a unity element, the radical of $A$ consists of all elements $h$ such that $g+h$ is regular for every regular $g$. If $A$ does not have a unity element, one may be adjoined without altering the radical. (Received March 14, 1941.)

### 196. Everett Pitcher and M. F. Smiley: *Transitivities of betweenness.*

The importance of the transitivity "*abc*, *adc*, and *bxd*→*axc*" in the theories of lattices and of metric spaces leads the authors to investigate all possible five point transitivities of betweenness on a line whose conclusions state that *just one* relation holds and whose hypotheses consist of three relations. Thirty-three distinct postulates are found. Of these, only ten fail to be equivalent to combinations of the fundamental properties (1), (2), and (3) of Huntington and Kline (Transactions of this Society, vol. 18 (1917), p. 305). A discussion of the two possible weak transitivities on four points is included. The influence of each of these transitivities, as well as the fundamental ones of Huntington and Kline, when applied to a definition of betweenness in lattices (this Bulletin, abstract 47-5-201) is investigated. Eight of the five point transitivities are shown to hold in metric ptolemaic spaces, while only one of them is valid in every metric space. (Received March 31, 1941.)

### 197. H. J. Riblet: *Certain theorems for symmetric differential functions.*

Symmetric differential functions, in which the $n$ quantities $y_1, \cdots, y_n$ are not all distinct, are considered. It is shown that the symmetric function theorem still holds although in different form. For the case in which $y_1, \cdots, y_n$ are all distinct, an improved bound is given for the power of the discriminant which occurs in the expression of any symmetric differential function in terms of the elementary symmetric functions and their derivatives. (Received March 31, 1941.)

### 198. W. M. Scott: *On matrix algebras over an algebraically closed field.*

The matrix algebra $A$ is considered as an $(A, A)$ module, that is to say, as an abelian group with $A$ as a right and left operator system. By taking $A$ in reduced form, a set of parts $C_{ij}$ is obtained, of which the $C_{ii}$ are the irreducible constituents of $A$. These parts $C_{ij}$ are called the simple parts of $A$, as each forms a simple $(A, A)$ module. Each nonzero simple part is isomorphic to a composition factor group of the module $A$. A second set of simple $(A, A)$ modules, the elementary modules of $A$, is defined by use of the Cartan basis system employed by C. Nesbitt in his study of the regular representations of algebras (Annals of Mathematics, (2), vol. 39 (1938), pp. 634–658). A (1-1) correspondence exists between the elementary modules and the composition factor groups of $A$. The simple parts of $A$ are expressible as linear combinations of these elementary modules, and a similar statement may be made concerning simple parts of representations of $A$. A classification of the simple parts of $A$, by use of chains, gives a decomposition of $A$ by elementary transformations into parts, each of which defines a directly indecomposable invariant subalgebra of $A$. (Received March 25, 1941.)

### 199. H. A. Simmons: *Classes of maximum numbers associated with symmetric equations in n reciprocals.* IV.

The equation considered here is $\sum_{i=1}^{n}(1/x_i)+\sum_{i=1}^{m}a_i(\prod(x))^{-i}=b/a$, $a=(c+1)b-1$, in which $n$ is an integer $>1$, $b$, $c$, $m$ are arbitrary positive integers, $\prod(x)=x_1x_2\cdots x_n$, and the $a_i$ are non-negative integers. Using the terminology of the author's article with W. E. Block (Duke Mathematical Journal, vol. 2, p. 317), the results of the present paper are as follows: The largest product of the numbers in any $E$-solution of the equation is the product of the $w_i$ of the Kellogg solution $w$ of the equation; this product is not attained by the numbers in any $E$-solution except $w$. With certain restrictions on the $a_i$ $(i=2, 3, \cdots, m)$, the Kellogg solution has the two remarkable properties that were established for the Kellogg solutions employed in the article referred to above. (Received March 12, 1941.)

### 200. H. A. Simmons: *Use of matrices in solving linear Diophantine equations.*

Given the equation $\sum_{i=1}^{n}a_ix_i=k$, $n>1$, in which the $a_i$ are integers different from zero and $k$ is an integer. It is known that this equation has a solution (in integers) if and only if $(a_1, a_2, \cdots, a_n)$ divides $k$. The purpose of this paper is to express in matricial form the substitutions that are used in solving the equation by the method of substitutions (as given, say, in E. Cahen, *Théorie des Nombres*, vol. 1, chaps. 9, 10). Each matrix here used is nonsingular and every matrix employed except the last one has determinant $\pm 1$. A special order of carrying out the desired matric multiplications is advantageous. The method used in this paper is usually briefer than even the more refined recent methods that employ continued fractions. (Received March 12, 1941.)

### 201. M. F. Smiley: *Betweenness in general lattices.* Preliminary report.

V. Glivenko (*Contributions à l'étude des systèmes de choses normées*, American Journal of Mathematics, vol. 59 (1937), pp. 941–956, and *Géométrie des systèmes de choses normées*, American Journal of Mathematics, vol. 58 (1936), pp. 799–828) studied the relations between the properties of the lattice operations and metric betweenness in a metric lattice. He showed that $c$ was metrically between $a$ and $b$ if and only if $(a\cap c)(b\cap c)=c=(a\cup c)(b\cup c)$. In particular, he characterized distributive lattices as those in which betweenness is "transitive," that is, whenever $c$ is between $x$ and $y$ and both $x$ and $y$ are between $a$ and $b$, then $c$ is between $a$ and $b$. Starting with this criterion for betweenness as a definition to be applied to general lattices, this result is obtained without the use of a metric. It is also shown that the transitive property of Pasch which holds in any metric space, namely, $b$ between $a$ and $c$ with $c$ between $a$ and $d$ implies that $b$ is between $a$ and $d$ (see K. Menger, *Untersuchungen über allgemeine Metrik*, Mathematische Annalen, vol. 100 (1928), pp. 78–80), characterizes modular lattices. (Received February 7, 1941.)

### 202. B. M. Stewart: *Left-associated matrices with elements in an algebraic domain.*

The $n$ by $n$ matrices $A$ and $B$ with elements in an algebraic domain of order $k$ are said to be left associates if there exists a unimodular matrix $P$ with elements in the domain such that $PA = B$. If the domain is a principal ideal ring, a necessary and sufficient condition that $A$ and $B$ be left associates is that $A$ and $B$ have the same Hermite normal form. For a domain whose class number is greater than one the presence of non-principal ideals prevents such a direct solution. But if each element of $A$ is replaced by its $k$ by $k$ second matric representation there is produced an enlarged matrix $A'$ of order $kn$ by $kn$ with elements in the rational domain, so that for $A'$ the Hermite form is well-defined and easily found. This paper investigates the possibility that a necessary and sufficient condition that $A$ and $B$ be left associates is that the corresponding enlarged matrices $A'$ and $B'$ be left associates. The necessity is shown, and the sufficiency is proved for nonsingular matrices, for matrices whose column class is the principal class, and for matrices of rank 1. Use is made of the results of Steinitz concerning the equivalence problem $PAQ = B$. (Received March 8, 1941.)

### 203. H. P. Thielman: *Groups of linear fractional transformations.*

G. A. Miller (American Journal of Mathematics, vol. 22, p. 185) has shown that two operations $s_1$ and $s_2$, each of period two, can be selected in such a way as to give a product $s_1s_2$ of any desired period. In the present paper this theorem is applied to linear fractional transformations of the form $T = (ax+b)/(cx+d)$. If $T$ is of period two it must be of the form $S = (ax+b)/(cx-a)$, $|a| + |c| \neq 0$. Two transformations $S_1 = (a_1x+b_1)/(c_1x-a_1)$ and $S_2 = (a_2x+b_2)/(c_2x-a_2)$ are selected so that the product will be of any given period $n$. The dihedral group is then formed by extending the cyclic group generated by $S_1S_2$ by one of the elements $S_1$ or $S_2$. If in $S_1$ $c_1 = 0$, and in $S_2$ $a_2 = 0$, the groups of subtraction and division are obtained (G. A. Miller, Quarterly Journal of Pure and Applied Mathematics, 1906, pp. 80–87; E. J. Finan, American Mathematical Monthly, 1941, pp. 3–7). Simple representations of all dihedral groups are obtained. The results are consequences of the functional equations satisfied by the coefficients of $S_1$ and $S_2$. (Received March 13, 1941.)

### 204. R. M. Thrall: *On Young's semi-normal representation of the symmetric group.*

The main body of the paper is devoted to a new (and shorter) derivation of Young's semi-normal orthogonal form for the irreducible representations of the symmetric group $S_m$ on $m$ letters. The procedure is such as to admit an immediate extension which yields a specific unitary form for the irreducible representations of the alternating group $A_m$. This new derivation is based on a definition of the semi-normal idempotents of the group ring $R_m$ of $S_m$, directly in terms of the regular Young diagrams ("standard tableaux") associated with the partitions of $m$. (Received March 11, 1941.)

### 205. G. L. Walker: *The elementary divisors of a direct product of matrices.*

If $A=(a_{ij})$ and $B=(b_{\alpha\beta})$ are square matrices, the direct product (or Kronecker product) of $A$ and $B$ is $A\otimes B=(a_{ij}B)$, that is, having elements $a_{ij}b_{\alpha\beta}$ arranged in rows by lexicographic order on $i$ and $\alpha$ and in columns by lexicographic order on $j$ and $\beta$. Known results on the characteristic roots of $A\otimes B$ are extended to the determination of the elementary divisors of $A\otimes B$ in terms of the elementary divisors of $A$ and of $B$. The problem is first reduced to the case of $A$ and $B$ having the single elementary divisors, say $(\lambda-\rho)^m$ and $(\lambda-\sigma)^n$ respectively. Then $A\otimes B$ has elementary divisors: (i) $(\lambda-\rho\sigma)^{m-n-1+2t}$ where $t=1, 2, \cdots, n$ if $\rho\sigma\neq 0$ and $m\geqq n$; (ii) $\lambda^m$ repeated $n$ times if $\rho=0$, $\sigma\neq 0$; (iii) $\lambda^k$, $\lambda^k$ where $k=1, 2, \cdots, n-1$ and $\lambda^n$ repeated $m-n+1$ times if $\rho=\sigma=0$ and $m\geqq n$. (Received March 31, 1941.)

### 206. Morgan Ward: *Finite operators of formal power series.*

Let $R$ be a ring of formal power series $A=\sum a_nx^n$ with coefficients in a commutative field. A finite operator $\omega$ transforms $A$ into another series $\omega A=\sum a_n'x^n$ where $a_n'$ is a function of $a_0, a_1, \cdots, a_{n-1}, a_n$ alone. The ring of all such operators is studied and in particular the subring of linear operators. The results include as special cases a previously developed "calculus of sequences" (Ward, American Journal of Mathematics, vol. 58 (1936), pp. 255–266), a large part of the symbolic calculus of Blissard, Lucas and Bell and Hurwitz' theory of integral power series. (Received March 8, 1941.)

### 207. Louis Weisner: *Polynomials whose roots lie in a sector.*

From two polynomials $A(z)=\sum a_kz^k$ and $B(z)=\sum b_kz^k$, form the composition-polynomials $f(z)=\sum a_kb_kz^k$ and $g(z)=\sum k!a_kb_kz^k$. The author proves that if the roots of $A(z)$ are real while those of $B(z)$ lie in a sector $S$ with vertex at the origin and aperture less than or equal to $\pi$, then the roots of $f(z)$ and $g(z)$ lie in $S$ or the sector vertical to $S$. When $S$ is the positive or negative real axis, these results reduce to well known theorems of Malo and Schur. The author shows further that when the roots of $A(z)$ are negative, the roots of $f(z)$ and $g(z)$ lie in $S$. (Received March 13, 1941.)

### 208. L. R. Wilcox: *Extensions of semi-modular lattices.*

Let $L$ be a semi-modular lattice in which $a, b\in L$ implies the existence of $c\in L$ with $a+c=a+b$, $(a, c)\perp$, and in which there are no points (L. R. Wilcox, Annals of Mathematics, (2), vol. 40 (1939), pp. 490–505). Restricted dual-ideals are sets $S\subset L$ such that $a\in S$, $b\geqq a$, implies $b\in S$, and $a, b\in S$, $a$ not parallel to $b$, implies $ab\in S$. All restricted dual-ideals form a lattice $L'$ in which $L$ is order-isomorphically embedded. It is proved in this paper that the restricted dual-ideals determined by two-element sets $[b, c]$ form a complemented modular sublattice $\bar{L}$ of $L'$, which contains the image of $L$. Properties of $\bar{L}$ are studied. (Received March 14, 1941.)

### 209. M. A. Woodbury: *On ordered groups.*

A group $G$ is said to be ordered if it is simply ordered and the group operation preserves order. If the group $G$ has the property that it contains an element $u$ such that for every $g \in G$ there exists a natural number $n$ for which $u^n > g$, then $G$ is homomorphic (preserves the group operation and $\leqq$) to a subgroup of the additive group of real numbers. If $G$ has the further property that for any $g \in G$ preceded by the identity there is a natural number $n$ for which $g^n > u$, then the homomorphism becomes an isomorphism. These two properties are equivalent to the Archimedean postulate. The Archimedean property is necessary for an isomorphism as is shown by an example of Reidemeister (*Grundlagen der Geometrie*, pp. 40–41) of a non-Archimedean ordered quasi-field. (Received March 14, 1941.)

### 210. Leonard Carlitz: *Some interpolation formulas connected with polynomials in $GF(p^n)$.*

The interpolation formulas discussed are of two types, namely, formulas arising from interpolation at $x^{p^{ni}}$, and secondly, interpolation at arbitrary $M(x)$ in $GF(p^n, x)$. This paper is closely connected with a previous paper (this Bulletin, abstract 47-1-80). (Received April 1, 1941.)

### 211. Leonard Carlitz: *The coefficients of the reciprocal of certain series.*

Given the "linear" function $f(t) = \sum_{i=0}^{\infty} (A_i/F_i) t^{p^{ni}}$, where the $A_i$ are polynomials in $GF(p^n, x)$, define $\beta_m$ by means of $t/f(t) = \sum_{m=0}^{\infty} (\beta_m/g_m) t^m$. In this paper various properties of $\beta_m$ are discussed. Use is made of an identity of the form $f(xt) - xf(t) = \sum_{i=1}^{\infty} \alpha_i f^{p^{ni}}(t)$ and also of the inverse function of $f(t)$. (Received April 1, 1941.)

### 212. L. W. Griffiths: *The minimum number of variables in universal functions of polygonal numbers.*

The universal functions of polygonal numbers of order $m+2$, in which the sum of the coefficients is not greater than $m+2$, were determined in an earlier paper (Annals of Mathematics, (2), vol. 31 (1930), pp. 1–12). In each such universal function the number $n$ of variables was at most $m+2$. In this paper a universal function is exhibited for each $m$ which involves the smallest number of variables appearing in any of these universal functions for that $m$, and the value of this minimum $n$ is determined in terms of $m$. (Received February 28, 1941.)

### 213. D. H. Lehmer: *On the values of certain Hurwitzian continued fractions.*

Euler was the discoverer of a number of remarkable regular continued fractions representing certain simple combinations of $e$ and its rational powers as, for example, $(e+1)/(e-1) = [2, 6, 10, 14, \cdots]$ or $(e^2+1)/(e^2-1) = [1, 3, 5, 7, \cdots]$. In this paper we find the value of continued fraction $[b, a+b, 2a+b, \cdots]$, whose partial quotients form a general arithmetic progression, in terms of quotients of Bessel functions of imaginary argument. Only in case $2b/a$ is an odd integer is the continued fraction simply expressible in terms of rational powers of $e$. All continued fractions like Euler's $(e-1)^{-1} = [0, 1, 1, 2, 1, 1, 4, 1, 1, 6, \cdots]$, whose quotients are periodic except for a subsequence in arithmetic progression, are also evaluated as Bessel quotients, as well as certain continued fractions whose partial quotients are made up of two or more arithmetic progressions. Semiregular continued fractions whose numerators are all equal to $-1$ likewise may be evaluated in terms of Bessel quotients of real arguments. (Received March 8, 1941.)

### 214. C. D. Olds: *On some arithmetical identities.*

In this note the author considers certain arithmetical identities of the type: $\sum \cos [2(a+b)\pi/p] = \sum \cos [2(a-b)\pi/p]$, $b \neq a$, where $a$ and $b$ range successively over all the quadratic residues of the given prime number $p$. Some applications are indicated. (Received March 8, 1941.)

### 215. Max Zorn: *Transcendental p-adic numbers related to roots of unity.*

Elementary remarks about some sequences which converge $p$-adically. (Received March 10, 1941.)

## ANALYSIS

### 216. E. F. Beckenbach: *On functions having subharmonic logarithms.*

Two new characterizations of functions having subharmonic logarithms are given. One is expressed in function-theoretic terms, while the other is given by a generalized isoperimetric inequality. (Received March 10, 1941.)

### 217. Stefan Bergman: *The method of the minimum integral and the analytic continuation of functions.*

The measures of geometrical objects which occur in the theory of $CT$'s (conformal transformations) and $PT$'s (pseudo-conformal transformations) can be expressed as functions of minima $\lambda_{\mathfrak{B}}(t)$ of integrals $\int_{\mathfrak{B}} |h|^2 dx_1 dy_1 \cdots dx_n dy_n$, $z_k = x_k + iy_k$, where $h$ runs through all functions analytic in $\mathfrak{B}$ and subjected to certain conditions at the point $(t)$. If $\mathfrak{S} \subset \mathfrak{B}$ then $\lambda_{\mathfrak{S}}(t) \leqq \lambda_{\mathfrak{B}}(t)$, and since the equation $ds^2_{\mathfrak{B}}(z) = |dz|^2 / \lambda_{\mathfrak{B}}(t)$ defines a metric invariant with respect to $CT$'s the lemma of Schwarz-Pick (see Comptes Rendus de l'Académie des Sciences de l'URSS, vol. 16 (1937), p. 11) is obtained. On the other hand $\lambda_{\mathfrak{B}}(t)$ can be expressed with the aid of a system of functions orthogonal in $\mathfrak{B}$. Thus, in the case of Schlicht $CT$'s one obtains certain refinements of the lemma of Schwarz-Pick. The introduction of the concept of B-area enables the author to generalize this technique to $PT$'s. Finally, if one supposes that functions $w_k(z_1, z_2)$, $k = 1, 2$, of the $PT$ satisfy certain conditions on a surface bounding a segment $\mathfrak{a}$ of the boundary then $w_1$, $w_2$ are regular in $\mathfrak{a}$, and the $PT$ $(w_1, w_2)$ can be extended analytically through $\mathfrak{a}$ outside its original domain of definition. Applying then the method of the minimum integral for the extended domain, the author obtains results concerning distortion on the boundary for $PT$'s. (Received April 1, 1941.)

### 218. Lipman Bers: *On a generalized harmonic measure.*

Suppose $D$ is a domain in $R_n$, $B$ its (bounded) boundary, $B'$ the set of the regular boundary points of $D$, $u(P)$ $(P \in D)$ a bounded harmonic function (b.h.f.). There exists a function $\mu_u(P, e)$ (generalized harmonic measure) which is a b.h.f. of $P$ for every fixed Borelian set $e \subset B$, a completely additive set function for every fixed point $P \in D$ and satisfies the condition $\lim [\mu_u(P, e) - u(P)] = 0$, $\lim \mu_u(P, B-e) = 0$ $(P \to R \in B')$ for every open set $e \subset B$, $R \in e$. If $f(Q)$ is a continuous function, $v(P) = \int_B f(Q) d\mu_u(P, e_Q)$ is the solution of the following problem (a generalization of the Dirichlet problem): determine a b.h.f. $v$ such that $\lim [v(P) - f(R)u(P)] = 0$ $(P \to R \in B')$. If $f$ is a bounded Baire function, $v$ is a b.h.f. which is "representable by $u$." The bounded harmonic functions which possess a positive lower bound and which are representable by a given function $u$ form a "representation class" $C(u)$. Every function of $C(u)$ is representable by any other function of this class and only by these functions. In a given domain there exists either only one or continuously many different representation classes. (Received April 1, 1941.)

### 219. P. B. Burcham: *Some inclusion relations in the Hausdorff difference matrix.*

The matrix $(\Delta^m c_n)$, where $\{c_n\}$ is a regular Hausdorff moment sequence, has been defined as the difference matrix (Transactions of this Society, vol. 48 (1940), pp. 185–207). In this paper it is established that corresponding row sequences of the matrices $(\Delta^m a_n)$ and $(\Delta^m b_n)$ define equivalent methods of summation provided that the sequences $\{a_n\}$ and $\{b_n\}$ define the equivalent methods of summation, Cesàro summability $(C, \gamma-\alpha)$ and hypergeometric summability $(H, \alpha, 1, \gamma)$, $\Re(\alpha, \gamma, \gamma-\alpha)>0$. Additional results are obtained concerning the column and diagonal sequences of these two matrices. In the case that $\{a_n\}$ and $\{b_n\}$ define the Cesàro methods $(C, \alpha)$ and $(C, \beta)$, $\Re(\alpha, \beta)>0$, it is found that all of the diagonal sequences of the associated difference matrices define equivalent methods of summation. (Received February 24, 1941.)

### 220. Richard Courant: *On a new type of variational problems, with physical demonstrations.*

The classical calculus of variations concerns joining extremals between two points, or spanned within a closed curve, or problems of a similar type. There are, however, problems of equilibrium in physics that suggest a more general type of variational problems, of which the following is an example: Given $n$ points in the plane; find a simply connected graph of extremal arcs joining the $n$ points and having stationary length in the variational metric. A famous classical problem by Steiner gives the simplest illustration. There is a connection between these problems and isoperimetric problems with inequalities as subsidiary conditions whereby in particular any ordinary variational problem appears as a limiting case of an isoperimetric problem. The position and the treatment of these general problems carries over to more dimensions; for example: Given in three dimensions a closed graph (say the system of edges of a cube); desired a system of minimal surfaces bounded by the graph and having common boundary lines to be determined, such that the total area becomes stationary. The problems, their solution, and their interdependence can be illustrated by simple soapfilm experiments. (Received March 31, 1941.)

### 221. E. L. Crow: *The expansion problem associated with an ordinary differential equation of first order in which the coefficient is a quadratic polynomial in the parameter.*

Consider the differential equation $dy(x, \lambda)/dx=p(x, \lambda)y(x, \lambda)$, where $x$ is a real variable and $\lambda$ is a complex parameter. The expansion problem associated with this equation and a two-point boundary condition has been discussed by R. E. Langer (Transactions of this Society, vol. 25 (1923), pp. 155–172) for the case in which $p(x, \lambda)$ has a finite number of simple poles as a function of $\lambda$. Convergence was established by expressing the series of characteristic solutions as a sum of residues at the characteristic values. However, the method fails when $p(x, \lambda)$ has a double pole, in particular when $p(x, \lambda)$ is of the form $a(x)\lambda^2+b(x)\lambda+c(x)$. The present paper approximates the circumstances of the latter problem by replacing one of the boundary points on the real axis by two neighboring points in the complex plane. The variable $x$ is allowed to range over a simply-connected region including the three boundary points. An adaptation of a method developed by Langer (Transactions of this Society, vol. 46 (1939), pp. 151–190) for use in the complex domain is employed to establish the uniform convergence of simultaneous expansions of two arbitrary functions. (Received March 14, 1941.)

### 222. R. J. Duffin: *Möbius transforms and Fourier transforms.*

Let $g(x)$ be the Fourier cosine transform of $f(x)$. It is shown that formally $f(x)=(2\pi)^{1/2}\sum_{n=1}^{\infty}\mu_n/nx\sum_{m=1}^{\infty}g(2\pi m/nx)$. Here $\mu_n$ is the familiar Möbius symbol having the values 0, 1, $-1$. The application of suitable summability methods validates this inversion formula for a wide class of functions. It is sufficient to assume that $f(x)$ be absolutely integrable. Conditions may also be imposed on $g(x)$; for instance, the formula is valid if $g(x)$ is of bounded total variation. (Received March 13, 1941.)

### 223. Benjamin Epstein: *Asymptotic problems associated with a Laplace-Mellin integral equation.* Preliminary report.

In this paper the author studies the integral equation $\psi(s)=\int_A^{\infty}\sum_{n=0}^{\infty}(\lambda+n)^{-s}f(\lambda)d\lambda$, $A\geqq 0$, $\text{Re}\, s>1$, $f(\lambda)$ $L$-integrable. In the first half of his paper he proves Abelian and Tauberian theorems for this equation relating the asymptotic properties of $\psi(s)$ as $s\rightarrow 1^+$ with those of $f(\lambda)$ as $\lambda\rightarrow\infty$. The technique utilized is the repeated application of Tauberian and Abelian theorems for the Laplace integral equation. The second half of the paper is devoted to a moment problem analogous to the Stieltjes moment problem over the infinite interval. For example, the following question is considered: Suppose $s$ runs through the monotone non-decreasing set of real numbers $\sigma_n$, $n=1, 2, \cdots$, with limit point infinity; then are there conditions on the size of $\psi(\sigma_n)$ and the density of $\sigma_n$ which assure uniqueness of solution? Analytic criteria for this problem are found. The chief tools utilized are theorems by Carleman and Junnila for functions analytic in a half-plane and a theorem of Boas (Transactions of this Society, vol. 46, pp. 142–150). (Received March 11, 1941.)

### 224. H. H. Goldstine: *The parametric problem of the calculus of variations in general analysis.*

The well known problem of the calculus of variations is here extended so that the integrand is defined on a region of the spaces of points $(t, x_p, r_p)$, where $p$ ranges over an arbitrary class $P$. Analogues of the usual necessary conditions, and at least some of the sufficiency conditions, are obtained. The principal point of departure from the results for the problem in finitely many dimensions is in the discussion and character of extremals. (Received March 11, 1941.)

### 225. W. J. Harrington and J. B. Rosser: *A study of certain functions auxiliary to Brun's method in number theory.*

In 1937 J. B. Rosser made a preliminary report on *An improvement of Brun's method in number theory* (this Bulletin, abstract 43-3-142). Certain upper and lower bounds stated in that report were obtained by approximating certain functions by very extensive computations. In the present paper these functions are studied and much superior methods of evaluation are devised. The functions consist of infinite series, the terms of which are multiple integrals. The principal aim of the paper is to establish certain identities which express the sums of the series in closed form. Preliminary to proving these identities, one has to prove convergence, continuity, and existence of derivatives. These matters take up the bulk of the paper. In the report mentioned above, the results stated were relevant only to a particular application of Brun's method. The functions studied in this paper are taken to be sufficiently general so that the evaluations obtained can be used in the entire range of application of Brun's method. This paper prepares the way for the publication of the paper of which the preliminary report was made in 1937. (Received March 26, 1941.)

### 226. J. F. Heyda: *A uniqueness property of general monogenic functions.*

Let $f(\alpha)$ be the class of functions defined by $f(\alpha)=\iint q(x, y)(z-\alpha)^{-1}dxdy$ ($\alpha$ in $E$) integrated over the complement $C(E)$ of a closed set $E$ in a bounded domain $K$ of the $\alpha$-plane. The function $q(x, y)$ is real and continuous in $\overline{K}$ and vanishes at points of $E$; moreover $z=x+iy$, where $(x, y)$ is a point in $C(E)$. The form of the functions $f(\alpha)$ is suggested by the non-analytic part of the derivative of $F(\alpha)$, where $F(\alpha)$ is general monogenic in $E$ (as defined by W. J. Trjitzinsky, Annales de l'École Normal Supérieure, vol. 55, pp. 119–191) and is represented there by $F(\alpha)=h(\alpha)+\iint_{C(E)}\log(z-\alpha)q(x, y)dxdy$. Suppose $E$ contains a closed interval $I$ and $\Gamma$ is a closed linear set of points in $I$ having positive measure. The following uniqueness problem is considered: what must be the "rarefication" of the set $C(E)$, or, alternately, how fast must $|q(x, y)|\rightarrow 0$ as $\rho\rightarrow 0$ ($\rho$ is the distance of $(x, y)$ from the frontier of $C(E)$) in order that knowledge of the functional values of $f(\alpha)$ on $\Gamma$ will determine uniquely the values of $f(\alpha)$ on $\Gamma'$, where $I\supset\Gamma'\supset\Gamma$ and measure $\Gamma'>$measure $\Gamma$. (Received February 28, 1941.)

### 227. Dunham Jackson: *Generalization of a theorem of Korous.*

An elementary treatment of the convergence of series of orthogonal polynomials is greatly facilitated if the polynomials of the orthonormal set are known to be uniformly bounded on the domain of orthogonality, or on a part of it where convergence is to be proved. A demonstration due to J. Korous (see G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, vol. 23, p. 157) shows in a few lines that the orthonormal polynomials corresponding to a weight function $\rho\sigma$ on an interval are thus bounded, if the polynomials for weight $\rho$ have the desired property, and if the factor $\sigma$ satisfies a Lipschitz condition and has a positive lower bound on the entire domain of orthogonality. The purpose of this note is to show that the argument of Korous can be extended so as to apply under fairly general conditions to orthogonal polynomials in two real variables on an algebraic curve, and in particular to orthogonal trigonometric sums, which can be regarded as orthogonal polynomials on a circle. (Received February 6, 1941.)

### 228. H. Kober: *On a theorem of Schur and on fractional integrals of purely imaginary order.*

Schur's theorem ideals with linear transformations on $L_2$ to $L_2$ involving a kernel $K(x, y)$, homogeneous of degree $-1$, such that $K(x, 1)x^{-1/2}\in L_1(0, \infty)$. It is shown that two types of fractional integrals of order $\alpha$ belong to this class of transformations for $\Re(\alpha)>0$ and tend to bounded limits in $L_2(0, \infty)$ as $\Re(\alpha)\rightarrow 0$. The inversion and group properties of such fractional integrals of purely imaginary order are studied.

The results admit of partial extensions to $L_p$. Characteristic values and functions are also determined. (Received February 13, 1941.)

### 229. R. E. Lane: *The first three rational operations.*

This paper establishes the foundation for an approach to the number system in a manner designed to postpone the need for infinite processes such as Dedekind cuts. In contrast to the Cantor and Dedekind approaches, it considers numbers as being generated from the unit-elements of a sequence of rational operations analogous to addition and multiplication. Postulates are laid down for such a sequence of operations, and a number of theorems are derived for the first three rational operations: intersection, union, and addition. (Received March 4, 1941.)

### 230. Lincoln La Paz: *Double integral variation problems with prescribed transversality coefficients.*

H. A. Simmons has recently published an interesting derivation of the transversality relationship for the variable limit problem of the calculus of variations for $n$-tuple integrals (Transactions of this Society, vol. 36 (1934), pp. 29–43). It is the purpose of the present note to formulate and solve an inverse problem suggested by this transversality relationship. Attention is restricted to the double integral case but the argument made and the conclusions drawn are easily extended to $n$-tuple integrals. Transversality coefficients are defined for a regular double integral problem. Necessary and sufficient conditions for a pair of functions to be transversality coefficients of such a problem are obtained and the integrand function of the most general variation problem associated with a pair of transversality coefficients is determined. (Received March 13, 1941.)

### 231. Hans Lewy: *Fractional potentials.*

In the theory of the logarithmic potential $\log r$, Laplace's formula gives the distribution of mass in terms of the potential. The author derives an analogous explicit formula for potentials $r^{-\beta}$ with $\beta$ constant and $0<\beta<2$. Under certain rather general conditions, for a given function $U$, a mass distribution is determined whose potential is $U$. Similar formulae hold in spaces of arbitrary dimension. They can be utilized in the construction of the conductor potential by reducing this problem to that of an integral equation of second kind. The restriction of $\beta$ to negative numbers may be dropped. (Received February 25, 1941.)

### 232. A. T. Lonseth: *Dirichlet's principle for certain nonlinear elliptic equations.*

Courant's "Dirichlet principle" method of solving the Dirichlet problem for self-adjoint linear partial differential equations of elliptic type (*Methoden der Mathematischen Physik*, vol. 2, Berlin, 1937) is extended to nonlinear equations of type $\Delta u = 1/2\ \partial P(x, y; u)/\partial u$, where $\Delta$ is the Laplacian differential operator and $P(x, y; u)$ is non-negative, analytic in its three arguments, and convex in $u$. The (unique) solution automatically minimizes the integral whose first variation it nullifies. An alternative proof of the minimizing property is obtained. Previous existence proofs for this type of nonlinear elliptic equation are due to Bieberbach (method of successive approximations) and to Lichtenstein (method of Ritz). Analyticity of $P(x, y; u)$ may be replaced by less stringent conditions, and $\Delta$ by a more general self-adjoint linear elliptic operator. (Received March 12, 1941.)

### 233. L. H. Loomis: *A converse to the Fatou theorem.*

Let $f(z)$ be analytic and bounded in $|z|<1$ and let $F(z)$ be an indefinite integral of $f(z)$. Then $F(z)$ is continuous on the closed circle $|z| \leqq 1$ and defines a boundary function $F(t)$, $t=e^{i\theta}$. The proof of the Fatou theorem consists of showing that $f(z)$ has a radial limit equal to the derivative $F'(t)$ wherever $F'(t)$ exists. In this note the author proves conversely that wherever the radial limit $f(e^{i\theta})=\lim_{r\to 1} f(re^{i\theta})$ exists the derivative $F'(t)=F'(e^{i\theta})$ exists and has the same value. Thus the boundary function $f(t)$ is identically equal to the derivative of its indefinite integral, in the sense that the two functions exist at precisely the same points and have there the same values. An example is constructed to show that the hypothesis of boundedness cannot be materially weakened. (Received April 1, 1941.)

### 234. L. H. Loomis: *A simple proof of the Fatou theorem.*

The Fatou theorem in its simplest formulation can be stated as follows: If $f(x)$ is analytic and bounded in the unit circle $|z|<1$ then the radial limit $\lim_{r\to 1} f(re^{i\theta})$ exists for almost all $\theta$. The standard proof of this theorem is based on the fact that a function of a real variable having a Lipschitz constant has a derivative almost everywhere. The manipulations involved center around the Poisson integral formula. The present proof assumes the same basis but attains essential manipulatory simplification by using the Cauchy integral formula instead of the Poisson integral and substituting the half-plane for the circle. (Received April 1, 1941.)

### 235. E. R. Lorch: *The spectrum of linear transformations.*

Let $T$ be a bounded linear transformation in a complex Banach space, and let $C$ be a simple closed rectifiable curve lying entirely within the resolvent set $R$ of $T$. Then the integral $(1/2\pi i)\int_C d\zeta/(\zeta I-T)$ exists and represents a projection $P$ reducing $T$. $P=0$ if and only if every point interior to $C$ lies in $R$. $P=I$ if and only if every point exterior to $C$ lies in $R$. If $C'$ represents another curve in $R$ and $P'$ its associated projection, then $PP'=P$ if $C$ lies in the interior of $C'$; $PP'=0$ if $C$ and $C'$ lie exterior to each other. This leads to the construction of a homomorphism between an algebra of sets on the one hand and an algebra of projections reducing $T$ on the other—this mapping giving the spatial manifold counterparts of a fundamental class of spectral sets. (Received February 13, 1941.)

### 236. K. L. Nielsen: *Concerning boundary value problems for linear differential equations when the boundary conditions are given by Stieltjes integrals.*

This paper deals with the boundary values problem formulated by $L(x, \lambda; y(x, \lambda)) = f(x)$ and the boundary conditions $M_i(y) \equiv \sum_{k=1}^{n} \int_c^d y^{(k-1)}(t, \lambda) d\alpha_{i,k}(t) = 0$, $(i=1, \cdots, n)$, where $L(x, \lambda; y(x, \lambda))$ is the differential polynomial $\sum_{k=0}^{n} \lambda^{H(n-k)} a_{n-k}(x, \lambda) y^{[k]}$, the operators $M_i$ are linearly independent, the involved integrals are in the sense of Stieltjes and the $\alpha_{i,k}(x)$ are functions of bounded variation. Tamarkin (Mathematische Zeitschrift, vol. 7 (1927), pp. 1–54) presented developments under the assumption that the roots of the characteristic equation are distinct and that $H=1$. Trjitzinsky (Acta Mathematica, vol. 67 (1936), pp. 1–50) has obtained an asymptotic representation of the solution of the differential equation for $x$ in the real interval $(c, d)$ and the parameter $\lambda$ in certain regions extending to infinity for the case where the roots of the characteristic equation are not distinct and $H$ is allowed to exceed unity. The author takes Trjitzinsky's results and obtains the restrictions which are necessary on the $\alpha_{i,k}(x)$ and states a theorem determining the values of $\lambda$ for which the nonhomogeneous boundary value problem is possible when $(c, d)$ is the interval for which the Trjitzinsky existence theorem holds. (Received March 31, 1941.)

### 237. C. D. Olds: Expansion of an arbitrary function in terms of certain polynomials.

Let $\{P_i(z)\}$ be a set of polynomials which satisfies an equation of the type: $Q(z)/n! = \sum_{i=0}^{n} \alpha_i P_i(z)$. (When $\{P_i(z)\}$ are the Bernoulli polynomials, for example, $Q(z) = z^n$.) This property is used to show that under certain conditions an arbitrary function $f(z)$, regular in the unit circle, can be expanded in the form $f(z) = \sum_{i=0}^{\infty} \beta_i P_i(z)$. Moreover, for certain of these polynomials the expansion is shown to be valid by summing the series directly. In the case of the Bernoulli polynomials use is made of certain results concerning their maximum values due to D. H. Lehmer (American Mathematical Monthly, vol. 47 (1940), pp. 533–538). (Received March 8, 1941.)

### 238. G. H. Peebles: *On the behavior of series of polynomials orthogonal with respect to a weight function of changing sign.*

Most of the properties of polynomials orthogonal with respect to a non-negative weight function, which are needed to show convergence of the formal expansion of a function, are lost, if the weight function changes sign in the interval of orthogonality. The partial sums corresponding to a weight function $\rho(x)$ which changes sign a finite number of times are related very simply, however, to the partial sums corresponding to $\pi(x)\rho(x)$, where $\pi(x)$ is a polynomial or reciprocal of a polynomial such that $\pi(x)\rho(x)$ is non-negative on the interval of orthogonality. This relation is used to study the behavior of the partial sums of the expansions of functions in terms of polynomials orthogonal with respect to a weight function of changing sign. (Received March 14, 1941.)

### 239. Tibor Radó: *On convergence in length and convergence in area.*

The purpose of the paper is to complete and to extend various results in the literature on convergence in length and convergence in area. The specific tool is the following inequality. On a bounded measurable set $E$ of a euclidean space, let there be given two three-dimensional vector-functions $x_1$, $x_2$. Then $[\int \|x_1 \times x_2\|]^2 \leq [\int \|x_1\| \cdot \|x_2\|]^2 - [\int x_1 \cdot x_2]^2$, where the integrals are taken over $E$, provided only that the integrals involved exist in the Lebesgue sense. This inequality, which is an immediate consequence of the identity $(x_1 \times x_2)^2 = x_1^2 x_2^2 - (x_1 \cdot x_2)^2$, is a generalization of the inequality (7) in Adams and Lewy, *On convergence in length*, Duke Mathematical Journal, vol. 1 (1935), pp. 19–26. (Received March 28, 1941.)

### 240. W. C. Randels: *On Bessel's inequality in abstract spaces.*

Bochner (Fundamenta Mathematicae, vol. 20, p. 262) has defined Fourier series for functions whose values are in abstract spaces, and has pointed out that Bessel's inequality need not hold. In this paper examples of spaces in which the inequality does hold are given and Bochner's counterexample is extended. (Received March 3, 1941.)

### 241. C. E. Rickart: *Integration in a convex linear topological space.*

A theory of integration is developed for multi-valued functions $F(\sigma)$ defined over a $\sigma$-field $\mathfrak{M}$ of subsets of an abstract set $M$ and with values in a convex linear topological space $\mathfrak{X}$. The value of the integral is a closed set in $\mathfrak{X}$. The fundamental properties obtained by Kolmogoroff (Mathematische Annalen, vol. 103 (1930), pp. 654–696) for $\mathfrak{X}$ the space of real numbers are shown to carry over to the more general situation. The method of definition of the integral is an extension of that used by R. S. Phillips (Transactions of this Society, vol. 47 (1940), pp. 114–145). A notion of convergence for subsets of $\mathfrak{X}$ similar to the Hausdorff convergence for subsets of a metric space plays a central role throughout. If the integral is restricted to be single-valued, then a very general convergence theorem involving a notion of approximate convergence is obtained. The integral contains an integral of G. B. Price (Transactions of this Society, vol. 47, pp. 1–50) as well as the Kolmogoroff and Phillips integrals. (Received March 14, 1941.)

### 242. Raphaël Salem: *On some properties of symmetrical perfect sets.*

I. Let $P$ be a symmetrical perfect set of measure zero constructed in $(0, 2\pi)$ by the dyadic process; $E_p$ the set of $2^p$ equal intervals obtained after $p$ operations; $F_p$ the continuous non-decreasing function $(F_p(0)=0,\ F_p(2\pi)=1)$ equal to $k/2^p$ in the $k$th interval contiguous to $E_p$, and linear in each interval of $E_p$; $F$ the limit of $F_p$. If $c_n$ is the Fourier-Stieltjes coefficient of $dF$ with respect to $e^{nix}$ and $E(p)$ the measure of $E_p$, the series $\sum E^2[a \log n]\cdot|c_n|^2$ converges for $a>3/2 \log 2$, its sum depending only on $a$. Thence follow some properties of sets of absolute convergence for trigonometrical series. II. The following theorem is proved: if $N$ is any set of absolute convergence for trigonometrical series, the sum of $N$ and of any finite set is also a set of absolute convergence. (Received March 6, 1941.)

### 243. Raphaël Salem: *On trigonometrical series whose coefficients do not tend to zero.*

If $\sum \rho_n \cos(nx-\alpha_n)$, with $\rho_n \geqq 0$, $\limsup \rho_n > \alpha > 0$, converges in a set $E$, and if the point $x=0$ belongs to $E$, then there is a sequence of integers $\{n_k\}$ such that $\sin n_k x$ tends to zero for every $x$ belonging to $E$. If $E$ is perfect and if $F$ is any non-decreasing function constant in every interval contiguous to $E$ but increasing from one interval to another, the Fourier-Stieltjes cosine-coefficients of rank $2n_k$ of $dF$ tend to the greatest possible limit $(F(2\pi)-F(0))/\pi$ as $k\rightarrow\infty$. If $E$ is a perfect set such that this condition is satisfied, then there exists a trigonometrical series whose coefficients do not tend to zero and which converges "almost everywhere" in the set $E$. (Received March 6, 1941.)

### 244. Henry Scheffé: *Linear differential equations with two-term recurrence formulas.*

Linear differential equations with two-term recurrence formulas (defined below) occur very often in applied mathematics. The equation $\sum_{j=0}^{n} p_j(x)y^{(j)}=0$ is assumed to have coefficients analytic at some $x=x_0$. Following Frobenius, a formal substitution $y=\sum_{\nu=0}^{\infty} c_\nu \xi^{\nu+r}$ is made where $\xi=x-x_0$ if $x_0\neq\infty$, $\xi=1/x$ if $x_0=\infty$. The formula obtained by equating to zero the coefficient of $\xi^{\mu+r}$ for integral values of $\mu$ is called the recurrence formula relative to the point $x_0$. For the existence of a two-term recurrence formula, that is, one of the type $f(\nu, r)c_\nu+g(\nu, r)c_{\nu-h}=0$, for $\nu\geqq h$, a positive integer, necessary and sufficient conditions on the coefficients $p_j(x)$ are obtained for some neighborhood of $x_0$. It is then further assumed that the $p_j(x)$ are given elsewhere by their analytic continuations. Transformations $z=a\xi^b$, $w=\xi^c y$ are exhibited which carry the differential equation into a generalized hypergeometric equation with indices $p\leqq q+1$. Interesting corollaries follow. (Received March 12, 1941.)

### 245. I. J. Schoenberg: *On absolutely convex functions.*

1. A function $f(x)$ is said to be *completely monotone* for $x>0$ if (1) $(-1)^n f^{(n)}(x)\geqq 0$ $(x>0;\ n=0, 1, 2, \cdots)$. The following theorem shows that these requirements are redundant. Theorem: $f(x)$ is completely monotone for $x>0$ if it satisfies, for $x>0$, the following conditions: (2) $f(x)\geqq 0, f'(x)\leqq 0, f''(x)\geqq 0$, and (3) $(-1)^n f^{(n)}(x)\geqq 0$ for infinitely many values of $n$ (no matter how distantly spaced). Moreover, the third condition (2) may be replaced by the following weaker one: $\liminf_{x\to+\infty} f'(x)>-\infty$. 2. A function $f(x)$ is said to be *absolutely convex* for $-1<x<1$ if it satisfies in that range the inequalities $f^{(2n)}(x)\geqq 0$ $(n=0, 1, 2, \cdots)$. S. Bernstein has shown that $f(x)$ is analytic and regular in the domain bounded by the circle $|x|=1/4$ and the four tangents to it through the points $x=1$, $x=-1$. It is shown here that $f(x)$ is analytic within the unit circle $|x|<1$, which is the best result possible. (Received February 7, 1941.)

### 246. H. M. Schwartz: *Sequences of Darboux-Stieltjes integrals.*

The results obtained for sequences of Stieltjes integrals (abstract 47-1-43) are extended to sequences of upper and lower Darboux-Stieltjes integrals taken in the sense of Lebesgue (*Leçons sur l'Intégration*, 2d edition). (Received March 1, 1941.)

### 247. W. R. Scott: *A lemma on the Weierstrass E-function.* Preliminary report.

Let there be given an integrand $f(x^1, x^2, x^3, X^1, X^2, X^3)$ which we abbreviate $f(x, X)$. We assume that $f$ satisfies the usual differentiability and homogeneity conditions. Let there be given further a set of values $x_0$, $X_0$ such that (1) $f(x_0, X_0)>0$, (2) $f(x_0, X)\geqq 0$ and $E(x_0, X_0, X)\geqq 0$ for every vector $X$, where $E$ is the Weierstrass $E$-function. Let $Y_0$ denote the vector with components $\partial f/\partial X^i$, $i=1, 2, 3$, taken at $(x_0, X_0)$. Then there exists a $\delta=\delta(x_0, X_0)>0$ such that $f(x_0, X)\geqq\delta$ for every unit vector $X$ such that the scalar product $XY_0$ is positive. This lemma is then applied to answer various questions concerning the lower semi-continuity of simple and double integrals suggested by the following papers; E. J. McShane, *Integrals over surfaces in parametric form*, Annals of Mathematics, (2), vol. 34 (1933); *Semi-continuity of integrals* . . . , Duke Mathematical Journal, vol. 2 (1936); T. Radó, *On the semi-continuity of double integrals in parametric form*, to appear in Transactions of this Society. (Received March 12, 1941.)

### 248. W. T. Scott and H. S. Wall: *Gronwall summability.*

It is shown that a necessary and sufficient condition for a Hausdorff mean $[H, c_n]$ to be a Gronwall mean $(f, g)$ (T. H. Gronwall, *Summation of series and conformal mapping*, Annals of Mathematics, (2), vol. 33 (1932), pp. 101–117) is that $[H, c_n]$ be the product of $(E, \beta)$ by $(C, \alpha)$, that is, $c_n=\beta^n/C_{n+\alpha,n}$, $(n=0, 1, 2, \cdots)$, where $0<\beta\leqq 1$, $\alpha>0$. The $(f, g)$-mean which is identical with this $[H, c_n]$-mean has $f(w)=\beta w/[1-(1-\beta)w]$, $g(w)=(1-w)^{-\alpha-1}$. It is also shown that a method of summation considered by W. A. Mersman (this Bulletin, vol. 44 (1938), pp. 667–673) is identical with $(f, g)$ with $f(w)=[1-(1-w)^{1/2}]/[1+(1-w)^{1/2}]$, $g(w)=(1-w)^{-1}$. (Received February 25, 1941.)

### 249. J. A. Shohat: *On the best polynomial approximation for functions possessing derivatives.*

By virtue of the characteristic property of the polynomial of best approximation, of degree less than or equal to $n$, to $f(x)$ on a given interval, it may be also considered as a Lagrangean interpolation polynomial for $f(x)$. In this way are obtained very simply general theorems on best approximation in case $f^{(n+1)}(x)$ exists, also on the distribution of the points of deviation. (Received February 3, 1941.)

### 250. L. H. Swinford: *On Abel's equation.*

The substitution $y=\alpha+\beta z^{1/2}/\gamma+z^{1/2}$ allows one to replace Abel's equation $y'+g(x)y^2+f(x)y^3=0$ by a system of two first order equations, by which means new cases of integrability are found. (Received March 7, 1941.)

### 251. Otto Szász: *On convergence and summability of trigonometric series.*

A new type of necessary or sufficient condition has been given recently for the convergence of a Fourier series at a point. This combines a continuity property of the function with an order condition on the coefficients, and is generalized in this paper. Moreover, instead of a Fourier series, more general trigonometric series are considered, associating for example the termwise integrated series with a function. The results are closely connected with two summability methods introduced by Riemann and Lebesgue. (Received March 5, 1941.)

### 252. Alexander Weinstein: *On the decomposition of a Hilbert space by its harmonic subspace.*

It is shown that a lemma playing a central part in a recent paper of H. Weyl on the method of orthogonal projection in potential theory (Duke Mathematical Journal, vol. 7 (1940), pp. 411–444, Lemma 2) is an almost immediate consequence of a problem of the unified theory of eigenvalues of plates and membranes considered some years ago in a joint paper of N. Aronszajn and the author (Comptes Rendus de l'Académie des Sciences, Paris, vol. 204 (1937), p. 96). The proof based on these results does not require any special construction or computation. (Received February 7, 1941.)

## APPLIED MATHEMATICS

### 253. G. E. Hay: *The finite displacement of thin rods.*

The finite displacement of thin rods has been considered by G. Kirchhoff, who introduced approximation based on the thinness of the rod in a rather unsatisfactory manner. In the present paper the method of the tensor calculus is employed, and there is introduced a systematic method of approximation which involves the expansion of the fundamental equations as power series in a dimensionless parameter $\epsilon$ and permits a theoretical solution of the problem to any desired degree of accuracy. Finally, application of the theory is made to the problem of "straightening" certain thin rods by means of systems of forces applied to the ends. (Received March 13, 1941.)

### 254. A. E. Heins: *On the transformation theory of the solution of partial differential equations.* Preliminary report.

The regularity condition of the Fourier transform of a function defined over a finite domain, and zero outside of this domain, plays an important role in pure mathematics. This condition also plays an important role in the transformation theory of the solution of linear partial differential equations of mathematical physics. By observing this regularity condition, boundary conditions which do not enter into a particular problem may be eliminated. With the aid of the finite Fourier transform the author obtains solutions of the two-dimensional wave equation and the equation of heat conduction under prescribed initial and boundary conditions and the two-dimensional Laplace equation under prescribed boundary conditions. The boundaries are taken to be either circles or rectangles. The treatment of these problems with other contours is now in progress. (Received March 10, 1941.)

### 255. A. S. Householder: *A theory of steady-state activity in nerve-fiber networks.* II. *The simple circuit.*

Let the neuron $N_i$ have the origin $s_i$ and terminus $s_{i+1}$, subscripts being reduced modulo $n$. These neurons constitute a simple circuit. A constant stimulus $y_i$ applied to $N_i$ at $s_i$ is supposed to cause the production at $s_{i+1}$ of a stimulus $z_{i+1}$ according to the equation $z_{i+1}=\alpha_i(y_i-h_i)$, where $h_i>0$, $\alpha_i=0$ if $y_i \leqq h_i$, $\alpha_i=a_i>0$ or $<0$ if $y_i>h_i$. If $S_i$ is a constant stimulus applied from outside the circuit at $s_i$, then $y_i=S_i+z_i$. Then (1) the $S_i$ do not necessarily suffice to determine the $\alpha_i$, and hence, the $y_i$, unless $a_1a_2 \cdots a_n<1$. If $(\alpha_1, \cdots, \alpha_n)$ and $(\alpha_1', \cdots, \alpha_n')$ are distinct and both consistent with some $(S_1, \cdots, S_n)$, and if some $\alpha_i=0$ and also some $\alpha_j'=0$, then (2) $\alpha_i=0$ implies $\alpha_i' \neq 0$; (3) $\alpha_i=\alpha_j'=0$ implies $a_ia_{i+1} \cdots a_{j-1}<0$; (4) $\alpha_i=\alpha_j=0$ implies $a_ia_{i+1} \cdots a_{j-1}>0$. (Received March 8, 1941.)

### 256. William Mersman: *Heat conduction in an infinite composite solid.*

The problem of one-dimensional heat conduction in a doubly infinite composite solid, with given initial temperature distribution, is solved uniquely. The classical solution for a semi-infinite solid with surface temperature known is used to obtain an integral equation, which is solved formally by means of the Laplace transformation. The solution is then established and proved unique by classical methods. (Received March 8, 1941.)

### 257. I. S. Sokolnikoff: *Deflection of non-isotropic elastic plates.* Preliminary report.

The general solution of the equation for the deflection $w$ of a non-isotropic plate with one plane of elastic symmetry, $\alpha_{11}\partial^4w/\partial x^4+2\alpha_{12}\partial^4w/\partial x^3\partial y+2\alpha_{12}\partial^4w/\partial x^2\partial y^2+2\alpha_{31}\partial^4w/\partial y^3\partial x+\alpha_{22}\partial^4w/\partial y^4=p(x, y)$, can be written in terms of two analytic functions $F_1(z_1)$ and $F_2(z_2)$ of the independent complex variables $z_1=x+k_1y$ and $z_2=x+k_2y$, where the complex numbers $k_i$ depend on the elastic constants $\alpha_{ij}$. The determination of the functions $F_i$ can be reduced to the search for certain analytic functions of a complex variable $\zeta$ related to $z_1$ and $z_2$ via a pair of mapping functions $z_i=\omega_i(\zeta)$. The development of the theory is analogous to the treatment of certain two-dimensional problems in isotropic theory initiated by N. Muskhelišvili (Mathematische Annalen, vol. 107 (1932), pp. 282–312). Some applications of the theory to a clamped elliptic plate are made. The formulas specialize to the known results of the theory of isotropic and orthotropic plates. (Received March 8, 1941.)

### 258. R. M. Sutton: *An instrument for drawing confocal conics.*

A simple device is demonstrated for drawing families of orthogonal ellipses and hyperbolae. On a spring roller is wound a fishline in two strands so that as the line is pulled out both strands elongate at the same rate and are held under tension by the roller. Confocal hyperbolae are described by holding chalk against the cord without slippage; confocal ellipses are drawn in the usual manner by slipping the chalk within a loop of fixed length. The eccentricity of either kind of curve is changed with ease, and the distance between focal points can be altered quickly. The instrument offers convenience, flexibility, and rapidity of operation. (Received March 28, 1941.)

### 259. C. J. Thorne and J. V. Atanasoff: *The application of a general functional approximation method to thin plate problems.*

The approximate solutions to two problems of thin plate theory are obtained by a general functional method. This method includes, indeed unifies into a connected whole, many of the standard approximation methods and many more. In particular it includes the methods of Trefftz, Boussinesq or "least square," and Rayleigh-Ritz. The first problem represents that of a centrally loaded, square, clamped plate. The solution to this problem illustrates convergence to boundary values as well as the differential equation. The solution agrees accurately with the results obtained by others. The second problem is that of an infinite, corner loaded plate supported by an elastic foundation. A solution containing seventeen constants is obtained. (Received March 31, 1941.)

### 260. Alexander Weinstein: *On the vibrations of a clamped plate under tension.*

This problem, of importance for acoustic reception, has been solved in terms of Bessel's functions in the elementary case of a circular plate by W. G. Bickley, Philosophical Magazine, vol. 15 (1933), pp. 776–797. It will be shown here, *for a plate of any shape*, that a convergent sequence of *lower bounds* for the frequencies can be computed by using the solutions of the membrane problem for the same domain. In the case of a rectangular plate the resulting equations involve only trigonometric and hyperbolic functions and can be more easily solved numerically than the equations for the circular plate. *Upper bounds* for the frequencies can be computed by the Rayleigh-Ritz method. However, lower bounds are more important in the present case. (Received April 1, 1941.)

## GEOMETRY

### 261. Reinhold Baer: *Homogeneity of projective planes.*

It is the main object of this note to show that the theorems of Desargues and Pappus are valid in a projective plane if and only if there exist two one-parameter families of dualities of this plane which meet certain requirements (like existence of axis and center). (Received March 12, 1941.)

### 262. Y. K. Bal: *Study on graphs.* Preliminary report.

This is an abstract study of generalized polygons and their properties invariant under the independent rigid motion of their vertices. A relation, called "next," is introduced, in general by: I. If $A\leftrightarrow B$ then $B\leftrightarrow A$, II. $A\leftrightarrow A$ cannot hold; in particular by: II′. If $A\leftrightarrow B\leftrightarrow C$ then not $A\leftrightarrow C$. I and II′ give II. (a) $S_n$ is not null. (b) If

$A \in S$ there are $n$ and only $n$ elements next to it. (c) If $A$, $B \in S_n$ there are $(A_1, \cdots, A_n) \in S_n$ for which $A_1 \leftrightarrow A_2 \leftrightarrow \cdots \leftrightarrow A_r \leftrightarrow B$. $n$ is finite and $r$ is finite or zero. $|S_n|$ is the number of elements in $S_n$. Definition of $S_{n,p}$: For distinct elements $A_1 \leftrightarrow A_2 \leftrightarrow \cdots \leftrightarrow A_q$ and not $A_i \leftrightarrow A_j$ when $1 < |i-j| < p-1$. If $A_i \leftrightarrow A_p$ whenever $|i-j| = p-1$, $S_{n,p}$ is shown by $S'_{n,p}$. It is proved: $|S'_{n,4}| = 2n$ and $S'_{n,5}$ cannot exist. For distinct elements $A_1 \leftrightarrow A_2 \leftrightarrow \cdots \leftrightarrow A_r \leftrightarrow A_1$ is a cycle $C_n$. A cycle is either $S_1$ or $S_2$. If $|C_n| = |S_n|$ it is a boundary cycle. The order of $S_n$ is the number of distinct $b$-cycles not differing by circular permutation. The order of $S'_{n,4}$ is shown equal to $(n!)^2/n$. For $A_1$ and $B_1$, $A_1 \leftrightarrow \cdots \leftrightarrow A_{m_1} \leftrightarrow B_1$, then the smallest $m_i$ is the distance $(A_1, B_1)$ with $(A, A) = 0$. It is shown that $|S_n|$ is finite and by further study $S_n$'s are classified. (Received March 31, 1941.)

### 263. Nathaniel Coburn: *Frenet formulas for curves in unitary space.*

The object of this paper is to develop a set of Frenet formulas for any curve $X_1$ whose equations are functions of a real parameter ($t$) and which lies imbedded in a unitary space of $n$ dimensions $K_n$. The derivation proceeds analogously to that of the Frenet formulas for a curve $V_1$ which lies imbedded in a Riemannian space of $n$ dimensions, $V_n$. However, in the Riemannian case, the curve possesses $(n-1)$ independent curvatures; in the unitary space, the curve possesses $(2n-1)$ independent curvatures. Furthermore, in Riemannian space, the curvature matrix of $V_1$ in $V_n$ is skew-symmetric. Similarly, the curvature matrix of $X_1$ in $K_n$ possesses Hermitian symmetry. Finally, corresponding to the similar theorem in Riemannian space, it is shown that "a complex curve $X_1$ in a unitary space $K_n$ is uniquely determined when: (1) an initial point; (2) an initial orientation of the $n$-uple of normals; and (3) the $(2n-1)$ independent curvatures are given." The paper concludes with a study of geodesic $X_1$ in a unitary $K_n$ with semi-symmetric connection. It is shown that these geodesics are characterized by the first Frenet curvatures and the orientation of the $n$-uple of normals. (Received February 11, 1941.)

### 264. V. G. Grove: *The transformation T of congruences.*

In a projective space of three dimensions two congruences are said to be related by a transformation $T$ if their lines are in one-to-one correspondence, their developable surfaces correspond, and each congruence possesses transversal surfaces whose tangent planes at the points of intersection with a line of that congruence pass through the corresponding line of the other congruence. The transformation $T$ is of two types. In the first, called the asymptotic type, the curves on the transversal surfaces of the congruences corresponding to developables of these congruences are asymptotic curves. If one of the congruences of the pair is a $W$-congruence, the other is also. In the second or conjugate type the curves on the transversal surfaces corresponding to the developables form conjugate nets in relation $F$. If these nets are in the relation of a transformation $K$ of Koenigs, they are in the relation of Eisenhart. Associated with the conjugate case there exists a one-parameter family, or pencil, of congruences such that each focal point of a line of a variable member of the pencil lies on a line. If one congruence of this pencil is a $W$-congruence, all congruences of the pencil are $W$. (Received March 8, 1941.)

### 265. Edward Kasner and John De Cicco: *Geometry of dual-velocity systems.*

A dual-velocity system consists of the $\infty^2$ curves such that the osculating circles of the $\infty^1$ curves tangent to any line $l$, constructed at the elements of $l$, are tangent to another line $L$. Any such set may be considered to be analogous to the velocity systems as developed by Kasner. Any line transformation, not preserving all parallel pencils of lines, converts exactly one dual-velocity system into a dual-velocity system. The group preserving all dual-velocity systems is $X=\phi(x)$, $Y=y\psi(x)+\chi(x)$. This is the contact group leaving invariant the set of all dual-isothermal families. Examples of dual-velocity systems are equitangential, dual-natural, $\Delta$, and dual-$\Gamma$ families. Characterizations of these are obtained by the correspondence between the lines $l$ and $L$ mentioned above. Any dual-velocity system contains exactly $\infty^2$, $\infty^1$, one, or zero dual-isothermal families. Finally, the invariant theory of dual-velocity systems under both the dual-isothermal and equilong groups is developed. A dual-analogue of natural family has been discussed in an earlier paper. (Received February 11, 1941.)

### 266. L. J. Savage: *Distance spaces.*

By a distance space is meant a set $M$ of elements $p, q, \cdots$ over which is defined a real valued function $D(p, q)$ such that $D(p, q)=D(q, p)$, and $D(p, p)=0$. $D(p, q)$ may be thought of as the square of the distance from $p$ to $q$. This concept is some generalization of metric space. A vector space $V$ over which a scalar product $x\cdot y$ is defined can be considered as a distance space by setting $D(x, y)=(x-y)\cdot(x-y)$. For every distance space $M$ there is a "smallest" scalar product space $V(M)$ in which $M$ is imbeddable. An interesting class of distance spaces is that of differentiable manifolds over which a $D(p, q)$ is so defined as to be suitably differentiable when considered as a function of the coordinates. Because of the remark about scalar product spaces these differentiable distance manifolds can be handled much like differentiable submanifolds of euclidean space. In particular the concepts of regularity, tangent-flat, and first and second fundamental form, can be extended to them. Finally there are theorems connecting the possibility of imbedding such manifolds into euclidean and pseudo-euclidean spaces with certain restrictions on the second fundamental form. (Received March 13, 1941.)

### 267. R. K. Wakerling: *On the rational loci of $\infty^1$ $(\rho-1)$-spaces in $r$-space.*

The representation upon a $\rho$-space of the hypersurface $W_\rho^\nu$ in $S_r$, which is the rational locus of $\infty^1$ $(\rho-1)$-spaces, is investigated in this paper. The hyperplane sections of $W_\rho^\nu$ are represented in $S_\rho$ by a system of hypersurfaces $V_{\rho-1}^\nu$ passing through a given $(\rho-2)$-space $\nu-1$ times, and having in common $\sigma$ simple points. Some of the properties of $W_\rho^\nu$ are discussed together with those of its projection upon a $(\rho+1)$-space. A special case of the transformation between two $r$-spaces is given, and the paper is concluded with a brief note on rational ruled surfaces of order $r-1$ in $S_r$. (Received March 8, 1941.)

## STATISTICS AND PROBABILITY

### 268. A. H. Copeland: *If.*

If one attempts to apply Boolean algebra to the theory of probability, he discovers that it is inadequate for the treatment of conditional probabilities, selections, and observations—all three of which are of prime importance in the modern theories of probability and statistics. In a number of recent formalizations of the theory of probability an additional operator (or logical constant) "if" has been introduced in order to handle conditional probabilities. Selections and observations were not introduced into the formal schemes but were merely discussed as interpretations. In a previous paper the author showed that by a proper definition of "if" selections and observations could be formalized. However, this operator was defined in terms of certain extraneous concepts, namely, probabilities and order relations. In the present paper "if" is defined only by its relation to the Boolean operators, "and," "or," and "not." This more natural definition enables us to study the modified Boolean algebra per se and to investigate the structure of and the relations between its elements. (Received March 31, 1941.)

### 269. G. B. Dantzig: *Variance of the error of a mean computed by grouping.*

To compute the mean of a large number of observations, the range of the data is often divided into equal intervals and the mean is then computed as if each observation fell at the midpoint of the interval containing it. The error introduced by this short cut comes from two sources: (1) the size of the interval $h$ and (2) the amount of off-centerness $t$, that is, the amount of shift required to bring the nearest mid-point to the origin. Several authors have noted that the average value over all $t$ of the Sheppards' corrected moments (functions of $h$ and the grouped moments) yields the true moments of the distribution. It is suggested here that the value of $t$ should be selected at random, so that it is equally likely to have any value between 0 and $h$. Then the first grouped moment (now validly considered as a random variable) has for expected value the true first moment, and for variance $\frac{1}{2}h^2 \cdot \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(x)p(y)B_2[(x-y)/h]dxdy$, where $p(x)$ is the density of distribution and $B_2(\theta)=\theta^2-\theta+1/6$ is periodic of period one (second Bernoulli polynomial). If the amount of off-centerness is randomly chosen, Sheppards' corrections are always applicable and the error approximable by square root of the variance. (Received March 10, 1941.)

### 270. G. E. Forsythe: *Cesàro summability of random variables.*

$\{X_k\}$ and $\{Y_k\}$ are sequences of independent real-valued random variables. $\{X_k\}$ is otherwise arbitrary, while $\{Y_k\}$ is a normal family (Lévy) with all $E(Y_k)=0$. $\|a_{nk}\|$ is the Cesàro $C_\alpha$ matrix $(0<\alpha<\infty)$. Using theorems of Feller and Gnedenko, necessary and sufficient conditions are given for the existence of constants $\{d_{nk}\}$ so that both $\sum_{k=0}^{\infty}a_{nk}(X_k-d_{nk})$ and $a_{nk}(X_k-d_{nk})\rightarrow 0$ in probability as $n\rightarrow\infty$, uniformly in $k\leqq n$. If $\alpha\geqq 1$, $d_{nk}$ may always be chosen independent of $n$ and $\alpha$. With natural definitions it then follows that for summability of $\{X_k\}$ to 0, the $C_\alpha$ methods get stronger as $\alpha$ increases. This increase in strength is non-trivial for $0<\alpha<1$, but is essentially trivial for $\alpha\geqq 1$. Using another Gnedenko theorem, a necessary and sufficient condition is given for $C_\alpha$-summability of $\{Y_k\}$ to the Gaussian distribution $G(x)$. From this it is proved: if $0<\alpha<\frac{1}{2}$ or if $\sum_{k=0}^{\infty}E(Y^2)<\infty$, there can be no $C_\alpha$-summability of $\{Y_k\}$ to $G(x)$; if $\frac{1}{2}\leqq\alpha<\beta$ and $1\leqq\beta$, $C_\alpha\subseteq C_\beta$; if $\frac{1}{2}\leqq\alpha<\beta$, there always exists $\{Y_k\}$ summable-$C_\alpha$ to $G(x)$ but not summable-$C_\beta$ to $G(x)$, a perhaps surprising result. A close relation is shown between $C_\alpha$-summability of $\{Y_k\}$ to $G(x)$ and summability of $\{E(Y_k^2)\}$ to $+\infty$ by ordinary Riesz means of order $2\alpha-2$. Nörlund summability to 0 of symmetric variables $\{Y_k\}$ is also treated. (Received March 31, 1941.)

### 271. J. A. Greenwood: *A theorem on probability assignments to events.*

Consider the mutually exclusive, exhaustive set of variates of a variable $x$ with a probability distribution function $\phi(x)$ defined over $[a, b]$, and to each value of $x$ an associated number $P(x)$, $0 \leqq P(x) \leqq 1$. $P(x)$ may be called a probability assignment to the event $x$. Definition: $P(x)$ may be said to be a "sufficient" assignment if, for all $x$, $P(x) \geqq LS\int_{D_x} d\phi(t)$ where $D_x = D_x(t \in [a, b]; P(t) \leqq P(x))$. In order to ensure the existence of this integral and the succeeding one which defines $E[P(x)]$, it is assumed that $D_x$ is a Borel set. The following theorem is then provéd: A necessary condition that a probability assignment $P(x)$ to each of the mutually exclusive, exhaustive values of a stochastic variable $x$ be "sufficient" is that $E[P(X)] = LS\int_a^b P(x) d\phi(x) \geqq \frac{1}{2}$. (Received April 1, 1941.)

## 272. P. C. Hammer: *On fitting linear functions when all variables are subject to error.*

Among the many recent papers on fitting lines or planes in case all variates are subject to error is one by C. F. Roos which points out the lack of invariance of methods developed prior to that time. Among others the methods of Adcock, Pearson, and a generalization of Pearson's method by E. C. Rhodes seem to lack invariance. Assuming a general normal distribution of error and proceeding along classical lines this paper develops an implicit method of fitting linear functions in $n$ variables which includes that of Pearson as a special case and which is invariant under coordinate transformation. A few geometrical interpretations are made and the cases not leading to a unique solution itemized. A comparison of this method with that of Roos is then made. (Received February 10, 1941.)

## 273. C. T. Hsu: *Two samples from normal bivariate populations.*

Two samples, each being of two variates $(x_1, x_2)$ and $(x_1', x_2')$, of size $n$ and $n'$ respectively, are supposed to be drawn at random from two independent normal bivariate populations, with the following distributions: (1) $(2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2})^{-1} \exp\{-1/(2(1-\rho^2))((x_1-\xi_1)/\sigma_1)^2 - 2\rho((x_1-\xi_1)/\sigma_1)((x_2-\xi_2)/\sigma_2) + ((x_2-\xi_2)/\sigma_2)^2]\}$, (2) $(2\pi\sigma_1'\sigma_2'(1-\rho'^2)^{1/2})^{-1} \exp\{-1/(2(1-\rho'^2))[((x_1'-\xi_1')/\sigma_1')^2 - 2\rho'((x_1'-\xi_1')/\sigma_2') \cdot ((x_2'-\xi_2')/\sigma_2') + (x_2'-\xi_2')/\sigma_2')^2]\}$, where $\xi_1, \xi_2, \sigma_1, \sigma_2, \rho; \xi_1', \xi_2', \sigma_1', \sigma_2', \rho'$ are the unknown parameters of the populations. Two hypotheses are considered concerning the comparison of correlation coefficients, namely, $H_1$: Assuming $\sigma_1 = \sigma_2$ and $\sigma_1' = \sigma_2'$; to test $\rho = \rho'$. $H_2$: Assuming $\sigma_1 = \sigma_2$, $\xi_1 = \xi_2$ and $\sigma_1' = \sigma_2'$, $\xi_1' = \xi_2'$; to test $\rho = \rho'$. Appropriate test criteria are derived for each hypothesis. The distribution of certain of the statistics are obtained in the special case where $n = n'$. Incidentally the distribution of the $z$ of S. S. Wilks (Biometrika, vol. 24 (1932), p. 471) for $p = 2$ and any values of $a_1$ and $a_2$ is studied. (Received April 1, 1941.)

## 274. Henry Scheffé: *Note on the reduction of $\chi^2$ for fit of a frequency distribution.*

Pearson reduced $\chi^2$ for the fit of an observed to a theoretical frequency distribution by the use of determinant theory and trigonometric identities (Philosophical Magazine, (5), vol. 50 (1900), pp. 160–163). The object of this note is to simplify the reduction by the use of matrix theory. (Received February 15, 1941.)

## 275. Abraham Wald: *On testing statistical hypotheses concerning several unknown parameters.*

Let $f(x, \theta_1, \cdots, \theta_k)$ be the probability density function of a random variable $x$ involving $k$ unknown parameters. For testing the simple hypothesis $\theta = \theta_1', \cdots, \theta_k = \theta_k'$, called hypothesis $H$, by means of $n$ independent observations $x_1, \cdots, x_n$ on $x$ we choose a critical region $W_n$ of the $n$-dimensional sample space and reject the hypothesis $H$ if $E=(x_1, \cdots, x_n)$ falls inside $W_n$. Denote by $P(W_n|\theta_1, \cdots, \theta_k)$ the probability that $E$ will fall in $W_n$ under the assumption that $\theta_1, \cdots, \theta_k$ are the true values of the parameters and denote by $P_n(\theta_1, \cdots, \theta_k, \alpha)$ the least upper bound of $P(Z_n|\theta_1, \cdots, \theta_k)$ with respect to all regions $Z_n$ for which $P(Z_n|\theta_1', \cdots, \theta_k')=\alpha$. A critical region $W_n$ is called a most stringent test of the hypothesis $H$ on the level of significance $\alpha$ if $P(W_n|\theta_1', \cdots, \theta_k')=\alpha$ and l.u.b. $\{P_n(\theta_1, \cdots, \theta_k, \alpha)-P(W_n|\theta_1, \cdots, \theta_k)\}$ $\leq$l.u.b. $\{P_n(\theta_1, \cdots, \theta_k, \alpha)-P(Z_n|\theta_1, \cdots, \theta_k)\}$ (l.u.b. with respect to $\theta_1, \cdots, \theta_k$) for any region $Z_n$ for which $P(Z_n|\theta_1', \cdots, \theta_k')=\alpha$. It is shown that the test of $H$ based on the so-called likelihood ratio introduced by Neyman and Pearson is a most stringent test in the limit if $n\to\infty$. The foregoing definitions and results are extended also to testing composite hypotheses. (Received February 3, 1941.)

## TOPOLOGY

### 276. G. E. Albert: *On separation spaces.* Preliminary report.

A. D. Wallace has introduced separation spaces (abstract 46-7-368), adopting as a primitive concept a binary relation $X|Y$, between pairs of non-vacuous subsets $X$ and $Y$ of an abstract set $S$. Subject to certain axioms, $X|Y$ can be used to define a topology in $S$ which makes the resulting space completely equivalent to a $T_1$-topological space. A prominent part is played by the axiom: $X|Y$ implies $Y|X$. In the present paper separation spaces are studied in which this property of symmetry is discarded. It is shown that, subject to proper alterations of Wallace's axioms, separation spaces can be used to characterize $T_0$-topological spaces. The theory of asymmetrical separation is found to be particularly convenient for the definition of a topology in upper semi-continuous collections of type 2 (see R. L. Moore, Rice Institute Pamphlets, vol. 23, no. 1). Other applications of a more general nature are also indicated. (Received March 10, 1941.)

### 277. G. E. Albert and J. W. T. Youngs: *The structure of locally connected topological spaces.*

The present paper is a continuation of earlier work (abstract 46-3-138). Cyclic elements are defined and studied in the class $L$ of spaces which satisfy the postulates: the space and the empty set are open; the intersection (sum) of finitely (arbitrarily) many open sets is open; the components of an open set are open. The hyperspace $\mathcal{X}$ of all cyclic elements of any space $X\in L$ is topologized in such a way that: (1) $\mathcal{X}$ is a space $L$, (2) $\mathcal{X}$ is a strongly continuous image of $X$, and (3) $\mathcal{X}$ is acyclic. A subclass $H$ of $L$ is called hereditary if $X\in H$ implies that the hyperspace of $X$ is in $H$ and every true cyclic element of $X$ is a space in $H$. The class $L$ is a hereditary class which contains the class $P$ of all Peano spaces; however, it deviates widely from $P$. Hereditary subclasses of $L$ are proposed which approximate more closely the class $P$. For example: one such class is composed of all spaces $L$ which satisfy the $T_0$-separation axiom and which are strongly continuous images of the unit line interval. (Received March 10, 1941.)

### 278. Samuel Eilenberg: *Banach space methods in topology.* I.

For a given topological space $X$ the Banach space $\mathfrak{X}$ of all continuous bounded real functions $\mathfrak{x}$ on $X$ with norm l.u.b. $|\mathfrak{x}(x)|$ is considered. Banach has proved that two compact metric spaces, $X_1$ and $X_2$, are homeomorphic if and only if $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are isometric. Stone (Transactions of this Society, vol. 41 (1937), p. 469) has generalized this theorem to normal bicompact spaces $X$. The author obtains the same result for normal compact spaces satisfying the first countability axiom. The method consists in an analysis of the convex subsets of the surface of the unit sphere in $\mathfrak{X}$. A complete reconstruction of $X$, under either Stone's or the author's hypotheses, may then be obtained. (Received March 12, 1941.)

### 279. Samuel Eilenberg: *Banach space methods in topology.* II.

A Banach space $B$ is a direct product, $B_1 \times B_2$, of two of its closed linear subspaces $B_1$ and $B_2$, if every element $b \in B$ has a unique decomposition $b = b_1 + b_2$, $b_i \in B_i$, and if $\|b\| = \max(\|b_1\|, \|b_2\|)$. Given a topological space $X$ decomposed into two disjoint closed sets $X_1$ and $X_2$, the Banach space $\mathfrak{X}$ (see abstract 47-5-278) is a direct product $\mathfrak{X}_1 \times \mathfrak{X}_2$ if $\mathfrak{X}_i$ is that subset of $\mathfrak{X}$ that consists of all functions $\mathfrak{x}$ which vanish outside $\mathfrak{X}_i$. If $X$ is normal, compact, and satisfies the first countability axiom, then the converse can be proved, that is, the above direct product decomposition is the most general one. In particular $X$ is connected if and only if $\mathfrak{X}$ is indecomposable. (Received March 12, 1941.)

### 280. Samuel Eilenberg: *Irreducible transformations onto manifolds.*

A continuous mapping $f(X) = M$ of a metric compact space onto an $n$-dimensional manifold is called irreducible (equal to strongly irreducible in the terminology of G. T. Whyburn, American Journal of Mathematics, vol. 61 (1939), p. 820) if $M - f(A) \neq 0$ for every closed proper subset $A$ of $X$. A point $y \in M$ is said to be covered essentially by $f$ if there is a neighborhood $U$ of $y$ such that there is no mapping $g(X) \subset M - y$ such that: $g(x) = f(x)$ if $f(x) \in M - U$, $g(x) \in U$ if $f(x) \in U$. The following is proved: if $f$ is irreducible and $y \in M$ is covered essentially, then $f^{-1}(y)$ is a continuum. In particular if $f$ is irreducible and essential in the sense of Hopf, then $f$ is monotone. (Received March 12, 1941.)

### 281. Harlan C. Miller: *Concerning compact unicoherent continua.*

In this paper it is proved that in order that the compact hereditarily decomposable continuum $M$ be hereditarily unicoherent it is necessary and sufficient that $M$ contain no continuum $N$ such that $N$ is a simple closed curve with respect to the elements of an upper semicontinuous collection of mutually exclusive continua filling up $N$. The author has previously proved that in order that the compact continuum $M$ be atriodic and hereditarily unicoherent it is necessary and sufficient that of any three points of $M$ there is one which weakly separates the other two from each other in $M$. It is now shown that (1) if $M$ is a compact hereditarily decomposable continuum and there exists a positive integer $k$ such that of any $k$ points of $M$ there is one which weakly separates two of the others from each other in $M$, then $M$ is hereditarily unicoherent, and that the stipulation that $M$ be hereditarily decomposable can be omitted only if $k < 5$, and (2) if $M$ is a compact hereditarily decomposable continuum such that every subcontinuum of $M$ is irreducible about some (closed) proper subset having only countably many components, then $M$ is hereditarily unicoherent. (Received March 13, 1941.)

### 282. W. T. Puckett: *Concerning a transformation in the plane.* Preliminary report.

The single-valued continuous transformation $T(P) = P'$ will be called *quasi 0-regu-*

*lar* provided that (1) $T(P-Q)$ is interior for some discrete set $Q$ and (2) for each $p\in P-Q$ and neighborhood $U(p)$ there exists a neighborhood $V(p)$ such that $(V(p)-Q)\cdot T^{-1}(x')$ lies in a single component of $(U(p)-Q)\cdot T^{-1}(x')$ for any $x'\in P'$. Under the hypotheses that $P$ is the plane and each $T^{-1}(x')$ has only nondegenerate components it is shown that (1) every point of $P'$ is a local separating point of order 2 and (2) any $T^{-1}(x')$ may be characterized as a discrete collection of open arcs, a finite number of which may have a point $q$ of $Q$ as a common end point. Moreover, the behavior of the transformation near such a point $q$ is exhibited. This theory will be applied to a topological characterization of potential functions. (Received March 10, 1941.)

### 283. G. E. Schweigert: *Border transformations.* II.

If $A$ is compact metric and if the light factor of the monotone light factorization of a continuous transformation, $T(A)=B$, is interior, then $T$ is said to be quasi-monotone. This new definition (A. D. Wallace, Duke Mathematical Journal, vol. 7 (1940)) gives a desirable property which, an example shows, does not hold under the old except for special spaces. Quasi-monotone is characterized by the statement: a sequence converging to $y\in B$ implies that the limit superior of the counterimages of the sequence intersects each component of $T^{-1}(y)$. This permits comparison with the border transformation where the limit of the counterimages is the border of $T^{-1}(y)$. Also, if $I(M)$ denotes the interior of $M$, the necessary condition $T^{-1}(I(Y))+\sum I(T^{-1}(y))=I(T^{-1}(Y))$, where the summation is over the border of $Y$, is shown to be sufficient that $T$ be a border transformation. Two other necessary conditions are shown to be sufficient. (Received March 13, 1941.)

### 284. M. E. Shanks: *The space of all metrics on a compact space.*

If $X$ is a compact metrizable space, the space $M(X)$ of all metrics on $X$, which preserve the topology, and their limits is a complete, separable positively-linear normed space. $M(X)$ is unique, that is, $M(X)$ is congruent to $M(Y)$ if and only if $X$ is homeomorphic to $Y$. $M(X)$ contains in a sense all continuous transformations of $X$. If $M^*(X)$ denotes the linear extension of $M(X)$, then $M^*(X)$ is a universal metric space if $X$ has the power of the continuum. In fact it is possible to define a whole class of universal metric spaces by taking different classes of metrices on $X$. The methods are largely those of Banach. (Received March 14, 1941.)

### 285. N. E. Steenrod and A. W. Tucker: *Real n-quadrics as sphere-bundles.*

By a suitable choice of homogeneous coordinates in real projective $(n+1)$-space the equation of a real $n$-quadric $Q$ can be written $x_0^2+x_1^2+\cdots+x_k^2=x_{k+1}^2+\cdots+x_{n+1}^2$ $(k\leqq n-k)$. The mapping $(x_0:x_1:\cdots:x_{n+1})\rightarrow(x_0:x_1:\cdots:x_k:0:\cdots:0)$ sections the quadric $Q$ into a continuous collection of $(n-k)$-spheres, one for each point of projective $k$-space. This collection is a sphere-bundle in the sense of Whitney (Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 148–153). Using results of Radon on linear sets of orthogonal matrices (Abhandlungen des mathematischen Seminars der Hamburgischen Universität, vol. 1 (1921), pp. 1–14) it can be shown that $Q$ is topologically equivalent to the product $P\times S$ of projective $k$-space $P$ and an $(n-k)$-sphere $S$ provided $n-k$ is odd and $k$ is small enough (if $\lambda$ denotes the exponent of the highest power of 2 which divides $n-k+1$ then, according as $\lambda\equiv 0$, 1, 2, 3 mod 4,

$k<2\lambda+1$, $2\lambda$, $2\lambda$, $2\lambda+2$, respectively). For $n-k$ odd and $k$ larger it seems probable that $Q$ and $P\times S$ are not topologically equivalent, although they have isomorphic homology groups. For $n-k$ even and $k>0$ the quadric $Q$ and the product $P\times S$ differ in orientability but have isomorphic homology groups mod 2. (Received March 14, 1941.)

### 286. A. W. Tucker: *Simplicial bands.*

Consider a simplicial complex formed by a circular sequence of $k$ $n$-simplexes, each of which has an $n-1$-face in common with the next ($k\geqq 2n+1$, necessarily). This *simplicial band* has as many $p$-simplexes as $n-p$-simplexes and its boundary is a regular $n-1$-polyhedron. If $n$ and $k$ are both even or if $n$ is odd, it is the topological product of a circle and a closed $n-1$-cell. If $n$ is even and $k$ odd, it is an $n$-dimensional analogue of a Möbius band, obtained from an $n$-cylinder by a sense-reversing identification of the base and cap. If $n$ or $k$ is odd, it is topologically equivalent to each of the following: (1) the symmetric product of $n$ circles; (2) the closed region of real projective $n$-space which represents the subset of real equations of degree less than or equal to $n$ with real roots, when coefficients are used as homogeneous coordinates; and (3) the quadric-bounded projective region $x_0^2+x_1^2\geqq x_2^2+\cdots+x_n^2$. (Received March 8, 1941.)

### 287. A. D. Wallace: *A fixed-point theorem for trees.*

Let $T$ be a compact Hausdorff space which is locally connected and which is acyclic in the sense that every finite open covering admits a finite open refinement whose nerve is a tree. Let $q$ be a function which assigns to each point $t$ in $T$ a continuum $qt$ in $T$, and which is continuous in the sense that if $t$ and a neighborhood $U$ of $qt$ are given then there is a neighborhood $V$ of $t$ such that if $t'$ is in $V$ then $qt'$ is in $U$. In this note it is shown that *there is a* $t_0$ *in* $T$ *such that* $t_0$ *is in* $qt_0$. This is a generalization of the Scherrer fixed-point theorem for acyclic continuous curves and is analogous to a recent result of Kakutani. As a corollary to our result we deduce a known result on the nonexistence of free monotone transformations into trees and a new theorem on the existence of coincidences of monotone transformations into trees. (Received March 7, 1941.)

### 288. P. A. White: *A decomposition of true cyclic elements by means of continua.*

In this paper a decomposition of a cyclicly connected locally connected continuum is defined by dividing the continuum into certain sets which have the property that no two points in them can be separated by a continuum. These sets are divided into two types according as they do or do not contain points of Menger order two. It is shown that each set of the first type is contained in a simple closed curve and that they are at most countable in number, while each set of the second type is at most countable. Other results are given concerning the components of these sets and of their complements. (Received March 31, 1941.)

### 289. G. W. Whitehead: *Homotopy properties of rotation groups.*

This paper continues the investigation of homotopy properties of the rotation groups $R_n$ of $n$-spheres begun by Hurewicz and Steenrod (Proceedings of the National

Academy of Sciences, vol. 27 (1941), pp. 61–64). The homotopy groups $\Pi_i(R_n)$ are computed for $i=3, 4, 5$ and all $n$, with the following results: $\Pi_3(R_n)$ is the infinite cyclic group ($n=2$, $n\geqq 4$); $\Pi_3(R_3)$ is the direct sum of two such groups; $\Pi_4(R_2)=\Pi_4(R_4)$ is the cyclic group of period 2, $\Pi_4(R_3)$ is the direct sum of two such groups, $\Pi_4(R_n)=0$ ($n\geqq 5$); $\Pi_5(R_n)=0$ ($n\neq 5$), and $\Pi_5(R_5)$ is the infinite cyclic group. It is proved that if $n>1$ and $n\not\equiv 3 \pmod 4$, there exists no map $f$ of $S^n$ into $R_n$ such that the projection $\Pi f$ into $S^n$ has odd degree. Thus for these values of $n$ there do not exist $n$ independent fields of tangent vectors on $S^n$, and the tangent sphere-bundle is not simple. In fact any two vector fields are somewhere dependent. (Received March 13, 1941.)

### 290. G. T. Whyburn: *Almost periodicity.*

Let $f(X)=X$ be a homeomorphism where $X$ is a metric space. It is shown that $f$ is pointwise periodic if and only if for any subset $Y$ of $X$, $f(Y) \subset Y$ implies $f(Y)=Y$ and that, for compact sets $X$, $f$ is pointwise almost periodic (see Ayres, Fundamenta Mathematicae, vol. 33, reprint) if and only if $f(K)\subset K$ implies $f(K)=K$ for every compact subset $K$ of $X$. A lemma is then proven to the effect that if $X$ is a continuum and $X=H+K$ is any division of $X$ into continua $H$ and $K$ such that $H\cdot K$ is a single point $p$ and $H\cdot f(H)\neq 0\neq K\cdot f(K)$, then pointwise almost periodicity of $f$ at $p$ implies $f(p)=p$. On the basis of this characterization and lemma, a considerably simplified development (using nowhere the notion of linear order) is given of the results of Ayres on pointwise almost periodic homeomorphisms—at the same time extending them to semi-locally-connected continua. (Received March 12, 1941.)

### 291. G. T. Whyburn: *Orbit decompositions.*

Let $f(X) \subset X$ be continuous where $X$ is metric. For any $x\in X$, the set $A_x$ of all $y\in X$ such that $f^m(x)=f^n(y)$ for positive integers $m$ and $n$ will be called the *orbit* of $x$. A subset $A$ of $X$ is *completely invariant* provided $f(A)=A$ and $f^{-1}(A)=A$. The set $A_x$ then turns out to be the smallest completely invariant subset of $X$ containing $x$. The decomposition of $X$ into the (disjoint) sets $[A_x]$ is called the *orbit decomposition*. It may be obtained also by means of the equivalence relation: $x\equiv y$ provided $f^m(x)=f^n(y)$ for some $m, n>0$. Assuming $X$ compact and the orbits closed, it is found that the orbit decomposition is upper semi-continuous provided $f$ has *equi-continuous powers*, that is, for any $x\in X$ and $\epsilon>0$ a neighborhood $U$ of $x$ exists so that $\rho[f^n(x), f^n(y)]<\epsilon$ for all $y\in U$ and all $n>0$. If $f$ is pointwise periodic, the converse holds and further, in this case, the condition is equivalent to *regular almost periodicity* of $f$, that is, for any $\epsilon>0$ an $n$ exists so that $\rho[x, f^{mn}(x)]<\epsilon$ for all $x\in X$ and all $m>0$. Also, interiority of $f$ implies pointwise periodicity, and the orbit decomposition is actually continuous. (Received March 12, 1941.)

### 292. G. T. Whyburn: *Regular almost periodicity.*

A homeomorphism $f(X)=X$ is regularly almost periodic if for each $\epsilon>0$ an integer $n$ exists such that $\rho[x, f^{mn}(x)]<\epsilon$ for all $x\in X$ and all integers $m>0$. Regular almost periodicity is cyclicly extensible in semi-locally-connected continua $X$. If each cyclic element of such an $X$ is locally euclidean, any pointwise periodic mapping on the non-end-points of $X$ is regularly almost periodic on $X$. A simplified treatment of the results of Hall and Schweigert (Duke Mathematical Journal, vol. 4 (1938), p. 719; this Bulletin, vol. 46 (1940), p. 963) is given making no use of transfinite ordinals or induction. It is shown that any pointwise periodic mapping $f(X) \subset X$, $X$ compact, is monotoni-

# NEW PUBLICATIONS

ALLEN, E. S. Six-Place Tables. 6th edition. New York and London, McGraw-Hill, 1941. 23+181 pp. $1.50.

BUCHLER, J. The Philosophy of Peirce. Selected Writings. (International Library of Psychology, Philosophy and Scientific Method.) London, Kegan Paul, 1940. 16+386 pp. 16s. 6d.

CHAMBERS, E. G. Statistical Calculation for Beginners. Cambridge, University Press, 1940. 8+110 pp. 7s. 6d.

CHURCHILL, R. V. Fourier Series and Boundary Value Problems. New York and London, McGraw-Hill, 1941. 9+206 pp. $2.50.

CONKWRIGHT, N. B. Introduction to the Theory of Equations. Boston, Ginn, 1941. 8+214 pp. $2.00.

DIAMOND, A. H. See ZANT, J. H.

DICKASON, A. The Geometry of Sheet Metal Work. London, Pitman, 1940. 335 pp. 12s. 6d.

DÖRRIE, H. Determinanten. Munich and Berlin, Oldenbourg, 1940. 216 pp. RM 11.

FERRAR, W. L. Algebra: a Text-Book of Determinants, Matrices and Quadratic Forms. Oxford, Clarendon Press; London, Oxford University Press, 1941. 7+202 pp. 12s. 6d.

FRANK, N. H. Introduction to Electricity and Optics. New York and London, McGraw-Hill, 1940. 12+398 pp. 24s. 6d.

HARKIN, D. Fundamental Mathematics. New York, Prentice-Hall, 1941. 15+434 pp. $3.00.

HELMER, O. See TARSKI, A.

HODGE, W. V. D. The Theory and Applications of Harmonic Integrals. Cambridge, University Press; New York, Macmillan, 1941. 9+281 pp. $4.50.

LUTEYN, P. See PETTIT, H. P.

MILES, H. J. A First Year of College Mathematics. New York, Wiley, 1941.

MILLER, F. H. Partial Differential Equations. New York, Wiley, 1941. 259 pp. $3.00.

MILLER, M. See NIKLAS, H.

NIKLAS, H., and MILLER M. Korrelationsrechnung und ihre Anwendung auf Statistik, Versuchswesen, Vererbungslehre, Wissenschaft und Technik. Leipzig, Lorentz, 1940. 70 pp. RM 5.40.

PEACOCK, G. A Treatise on Algebra. Vol. 1. Arithmetical Algebra. Reprinted from the 1842 edition. 16+399 pp. Vol. 2. On Symbolical Algebra, and Its Applications to the Geometry of Position. Reprinted from the 1845 edition. 10+455 pp. New York, Scripta Mathematica, 1940. $6.50.

PETTIT, H. P., and LUTEYN, P. College Algebra. 2d edition. New York, Wiley, 1941.

PLUMMER, H. C. Four Figure Tables with Mathematical Formulae. London, Macmillan, 1941. 4+90 pp. 3s. 6d.

SEITS, F. The Modern Theory of Solids. (International Series in Physics.) New York and London, McGraw-Hill, 1940. 15+698 pp. 49s.

SIMON, L. E. An Engineers' Manual of Statistical Methods. New York, Wiley, 1941.

SOHON, F. W. The Stereographic Projection. New York, Chemical Publishing Co., 1941. 9+210 pp. $4.00.

TARSKI, A. Introduction to Logic and to the Methodology of Deductive Sciences. Translated by O. Helmer. Enlarged and revised edition. New York, Oxford University Press, 1941. 18+239 pp. $2.75.

Temperature, Its Measurement and Control in Science and Industry. New York, American Institute of Physics, 1939, 13+1362 pp. $11.00.

Wolfle, D. Factor Analysis to 1940. (Psychometric Monographs, no. 3.) University of Chicago Press, 1940. 7+69 pp. $1.25.

Zant, J. H., and Diamond, A. H. Elementary Mathematical Concepts from the Historical and Logical Point of View. Minneapolis, Burgess, 1941. 125 pp. $1.50.

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# SOME RECENT DEVELOPMENTS IN THE THEORY OF CONTINUED FRACTIONS[1]

H. S. WALL

The principal rôle of the continued fraction in analysis has perhaps been that of an intermediary between more familiar and easily handled things, such as between the power series and the integral. This may partly explain the fact that there are certain questions about continued fractions which have remained relatively unexplored. To illustrate what I mean, if one's principal attention were focused upon power series, and continued fractions were used only incidentally, it is unlikely that one would imagine that the convergence region for the continued fraction is, as it now appears, more properly a *parabolic* region than a circular region.

I wish to speak today about some results which have been obtained during the last few years, by a group of men with whom I have been associated. Our investigations have been centered mainly upon the continued fraction itself. In certain instances it has been possible to apply our results to problems not directly connected with continued fractions. Thus, during the course of this lecture I shall have occasion to speak of the problem of moments, of Hausdorff summability, and of certain classes of analytic functions.

1. **Some definitions and formulas.** Before discussing some of the problems with which we have been concerned, I shall put down some necessary definitions and formulas. The continued fractions considered are chiefly of the form

$$\frac{1}{1}+\frac{a_2}{1}+\frac{a_3}{1}+\frac{a_4}{1}\cdots \tag{1.1}$$

in which $a_2, a_3, a_4, \cdots$ are complex numbers. Apart from unessential initial irregularities, any continued fraction in which the partial denominators are different from zero can be thrown into this form. The $n$th *approximant* of (1.1) is the ordinary fraction, $A_n/B_n$, obtained by stopping with the $n$th partial quotient. The *numerators* and *denominators* may be computed by means of the recursion formulas

$$\begin{aligned} &A_0 = 0, \quad B_0 = 1, \quad A_1 = 1, \quad B_1 = 1,\\ &A_n = A_{n-1} + a_nA_{n-2}, \quad B_n = B_{n-1} + a_nB_{n-2}, \quad n = 2, 3, 4, \cdots . \end{aligned} \tag{1.2}$$

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[1] An address delivered before the Detroit meeting of the Society, November 23, 1940, by invitation of the Program Committee.

An important consequence of the recursion formulas is the determinant formula

$$(1.3)\qquad A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1}a_2a_3\cdots a_n.$$

To assign a meaning to the continued fraction, we suppose, first, that $B_n \neq 0$ from and after some $n$. If, then, the limit $\lim_{n\to\infty} (A_n/B_n)$ exists and is finite, the continued fraction is said to converge, and the value of this limit is the *value* of the continued fraction. The convergence problem is the problem of determining conditions upon $a_2, a_3, a_4, \cdots$ which are sufficient for the convergence of the continued fraction. For instance, one may determine a "convergence region" in which $a_2, a_3, a_4, \cdots$ may vary independently and the continued fraction remains convergent. Some idea of the nature of this problem may be had from the fact that it is possible to choose $a_2, a_3, a_4, \cdots$ in such a way that these numbers form an everywhere dense set in the complex plane, and the continued fraction is convergent.[2] Worpitzky[3] showed that (1.1) converges if $|a_n| \leqq 1/4$. This is the best circular convergence region with center at the origin, inasmuch as (1.1) diverges if $a_n = -\frac{1}{4} - c$, $c > 0$. Szász[4] found that if $a$ is not a real number $\leqq -\frac{1}{4}$, then there exists a positive number $r$ such that (1.1) converges if $|a_n - a| \leqq r$. We[5] found recently that if $R(a) > -1/4$ $r$ may be taken equal to $\frac{1}{2}(|1+2a| - 2|a|)$. Again, if $|a_{2n+1}| \leqq 1/4$, $|a_{2n}| \geqq 25/4$, (1.1) is convergent.[6]

There is one important *necessary* condition for convergence due to von Koch,[7] which applies whenever the partial numerators are different from zero. Write the continued fraction in the form

$$(1.4)\qquad \frac{1}{1} + \frac{1}{b_2} + \frac{1}{b_3} + \frac{1}{b_4} + \cdots$$

where $a_n = 1/b_{n-1}b_n$, $(n = 2, 3, 4, \cdots, b_1 = 1)$. Then, if the series $\sum |b_n|$ *converges*, the sequences of even and odd approximants have distinct limits (one may be infinite), and the continued fraction diverges by oscillation.

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[2] Walter Leighton and H. S. Wall, *On the transformation and convergence of continued fractions* (to be referred to as T), American Journal of Mathematics, vol. 58 (1936), pp. 267–281, p. 269.

[3] Worpitzky, Jahresbericht, Friedrichs-Gymnasium und Realschule, Berlin, 1865, pp. 3–39. Independent proofs of this theorem were given later by Pringsheim and by Van Vleck.

[4] O. Perron, *Die Lehre von den Kettenbrüchen* (to be referred to as "Perron"), 2d edition, Leipzig and Berlin, 1929, p. 282.

[5] W. T. Scott and H. S. Wall, *A convergence theorem for continued fractions* (to be referred to as CT), Transactions of this Society, vol. 47 (1940), pp. 155–172, p. 171.

[6] T, p. 278.

[7] Perron, pp. 235–236.

This natural breaking up of the sequence of approximants into two sequences makes it often of importance to consider the continued fraction

$$(1.5)\qquad \frac{1}{1+a_2} - \frac{a_2a_3}{1+a_3+a_4} - \frac{a_4a_5}{1+a_5+a_6} - \cdots,$$

having $A_{2n}/B_{2n}$ as its $n$th approximant; and the continued fraction

$$(1.6)\qquad 1 - \frac{a_2}{1+a_2+a_3} - \frac{a_3a_4}{1+a_4+a_5} - \frac{a_5a_6}{1+a_6+a_7} - \cdots,$$

having $A_{2n+1}/B_{2n+1}$ as its $n$th approximant.[8] We shall refer to these as the *even part* and the *odd part*, respectively, of (1.1).

2. **Some ideas on convergence.** A natural procedure to follow in attacking the convergence problem is to turn to the infinite series

$$\frac{A_1}{B_1} + \left(\frac{A_2}{B_2} - \frac{A_1}{B_1}\right) + \left(\frac{A_3}{B_3} - \frac{A_2}{B_2}\right) + \cdots,$$

which, by (1.3), may be written in the form

$$(2.1)\qquad 1 - \frac{a_2}{B_1B_2} + \frac{a_2a_3}{B_2B_3} - \frac{a_2a_3a_4}{B_3B_4} + \cdots.$$

The starting point in our investigation is the idea of forming a majorant for the series (2.1) by requiring that the test-ratio

$$\rho_n = \frac{(-1)^n a_2a_3\cdots a_{n+1}}{B_nB_{n+1}} : \frac{(-1)^{n-1}a_2a_3\cdots a_n}{B_{n-1}B_n} = \frac{-a_{n+1}B_{n-1}}{B_{n+1}}$$

remain numerically less than or equal to some number $r_n$:

$$(2.2)\qquad |\rho_n| \leqq r_n, \qquad n = 1, 2, 3, \cdots.$$

If this holds, then it is necessarily true that $B_n \neq 0$, $(n=1, 2, 3, \cdots)$. Hence the continued fraction converges if some $a_n$ vanishes, or, in any case, if the majorant series $1+\sum r_1r_2\cdots r_n$ converges. Moreover, the sum of this majorant series is an upper bound for the absolute value of the continued fraction. If the $a_n$'s are variables, and the $r_n$'s are independent of the $a_n$'s when the latter lie in a certain domain, then the continued fraction converges uniformly over this domain if the majorant series converges.

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[8] Perron, p. 201.

Now, in order to translate the condition (2.2) into a condition upon $a_2, a_3, a_4, \cdots$ we have the simple relation

$$(2.3) \qquad \rho_n = \frac{-a_{n+1}(1+\rho_{n-1})}{1+a_{n+1}(1+\rho_{n-1})}, \qquad n = 1, 2, 3, \cdots, \rho_0 = 0.$$

Using this, the desired translations may be made in a great many different ways. We have found[9] the following to be especially useful:

$$(2.4) \qquad r_n|1+a_n+a_{n+1}| \geqq r_n r_{n-2}|a_n| + |a_{n+1}|,$$

$(n=1, 2, 3, \cdots)$, where we agree to put $r_{-1}=r_0=a_1=0$. If, for example, we take $r_n=n/(n+2)$, so that the majorant series $1+\sum r_1 r_2 \cdots r_n=2$, we find at once the theorem of Worpitsky mentioned in §1, namely: (1.1) converges (uniformly) for $|a_n| \leqq 1/4$, $(n=2, 3, 4, \cdots)$. The upper bound 2 which we find in this case is the least upper bound inasmuch as the value of (1.1) is 2 when $a_n=-1/4$.

Again, by making an appropriate choice of the $r_n$'s, one can derive the theorem that the continued fraction

$$(2.5) \qquad \frac{g_1}{1+}\;\frac{(1-g_1)g_2x_2}{1}\;+\;\frac{(1-g_2)g_3x_3}{1}\;+\;\cdots$$

in which $0<g_1<1$, $0\leqq g_n<1$, $(n=2, 3, 4, \cdots)$, converges uniformly for $|x_n| \leqq 1$, $(n=2, 3, 4, \cdots)$, and its absolute value in this domain does not exceed

$$(2.6) \qquad 1 - \frac{1}{1+\displaystyle\sum_{n=1}^{\infty} \frac{g_1g_2\cdots g_n}{(1-g_1)(1-g_2)\cdots(1-g_n)}}.$$

This value is actually attained by the continued fraction if $x_n=-1$. We thus improve in important respects the well known *Pringsheim criteria*. An immediate consequence is the theorem that

$$(2.7) \qquad \frac{1}{1+}\;\frac{g_1x_1}{1\;+}\;\frac{(1-g_1)g_2x_2}{1}\;+\;\frac{(1-g_2)g_3x_3}{1}\;+\;\cdots$$

converges uniformly for $|x_n| \leqq 1$, $(n=1, 2, 3, \cdots)$, if the series in (2.6) converges; and for $|x_1| \leqq h<1$, $|x_n| \leqq 1$, $(n=2, 3, 4, \cdots)$, if this same series diverges or converges. Part of this theorem was proved by E. B. Van Vleck,[10] who seems to have been the first to recognize the importance of continued fractions of this form.

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[9] CT, p. 155 ff.

The relationship of the inequalities (2.4) to continued fractions of the form (2.7) is indicated by the following theorem.

THEOREM 2.1. *If the inequalities* (2.4) *hold with actual inequality for* $n=1, 2,$ *and if* $a_n \neq 0$, $(n=2, 3, 4, \cdots)$, *then the even and odd parts of* (1.1) *can, except for initial irregularities, be written in the form* (2.7). *In fact, the even part can be put into the form*

$$(2.8)\qquad \frac{1}{1+a_2}\left\{\frac{1}{1}\,+\,\frac{g_1x_1}{1}\,+\,\frac{(1-g_1)g_2x_2}{1}\,+\,\frac{(1-g_2)g_3x_3}{1}\,+\,\cdots\right\}$$

*where* $|x_1|<1$, $|x_n|\leqq 1$, $(n=2, 3, 4, \cdots)$, $0<g_n<1$, $(n=1, 2, 3, \cdots)$; *and the odd part can be put into the form*

$$(2.9)\qquad 1-\frac{a_2}{1+a_2+a_3}\left\{\frac{1}{1}\,+\,\frac{h_1y_1}{1}\,+\,\frac{(1-h_1)h_2y_2}{1}\,+\,\frac{(1-h_2)h_3y_3}{1}\,+\,\cdots\right\},$$

*where* $|y_1|<1$, $|y_n|\leqq 1$, $(n=2, 3, 4, \cdots)$, $0<h_n<1$, $(n=1, 2, 3, \cdots)$.

Put

$$g_n=\frac{r_{2n+1}|1+a_{2n+1}+a_{2n+2}|-|a_{2n+2}|}{r_{2n+1}|1+a_{2n+1}+a_{2n+2}|},$$

$$h_n=\frac{r_{2n+2}|1+a_{2n+2}+a_{2n+3}|-|a_{2n+3}|}{r_{2n+2}|1+a_{2n+2}+a_{2n+3}|}.$$

It follows from the inequalities (2.4) that

$$g_1\geqq\frac{r_1|a_3|}{|1+a_3+a_4|},\qquad h_1\geqq\frac{r_2|a_4|}{|1+a_4+a_5|},$$

$$(1-g_{n-1})g_n\geqq\left|\frac{a_{2n}a_{2n+1}}{(1+a_{2n-1}+a_{2n})(1+a_{2n+1}+a_{2n+2})}\right|,$$

$$(1-h_{n-1})h_n\geqq\left|\frac{a_{2n+1}a_{2n+2}}{(1+a_{2n}+a_{2n+1})(1+a_{2n+2}+a_{2n+3})}\right|.$$

If then we put

$$\frac{-a_3a_4}{(1+a_2+a_3)(1+a_4+a_5)}=h_1y_1,\qquad\frac{-a_2a_3}{(1+a_2)(1+a_3+a_4)}=g_1x_1,$$

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[10] E. B. Van Vleck, Transactions of this Society, vol. 2 (1901), pp. 476–483.

$$\frac{-a_{2n+1}a_{2n+2}}{(1+a_{2n}+a_{2n+1})(1+a_{2n+2}+a_{2n+3})} = (1-h_{n-1})h_n y_n,$$

$$\frac{-a_{2n}a_{2n+1}}{(1+a_{2n-1}+a_{2n})(1+a_{2n+1}+a_{2n+2})} = (1-g_{n-1})g_n x_n,$$

we see on referring to (1.5), (1.6) that the theorem is true.

Now, both the continued fractions (2.8), (2.9) are convergent. Hence, to prove (1.1) convergent it is required to show that the values of (2.8) and (2.9) are equal. For that purpose we find that[11]

$$\left|\frac{A_n}{B_n} - \frac{A_{n+1}}{B_{n+1}}\right| = \frac{1}{|Q_n Q_{n+1}|}$$

where

$$r_1 r_3 \cdots r_{2n+1} |Q_{2n+2}| \geqq e_1\left(1 + \sum_{k=1}^{n} r_1 r_3^2 \cdots r_{2k-1}^2 r_{2k+1} |b_{2k+2}|\right),$$

$$r_2 r_4 \cdots r_{2n+2} |Q_{2n+3}| \geqq e_2\left(1 + \sum_{k=1}^{n+1} r_2 r_4^2 \cdots r_{2k-2}^2 r_{2k} |b_{2k+1}|\right),$$

$e_1$, $e_2$ being positive constants, $b_1 = 1$, $a_n = 1/b_{n-1}b_n$ (cf. (1.4)). It is now easy to see that (1.1) converges, under certain conditions, even if the majorant series $1 + \sum r_1 r_2 \cdots r_n$ diverges. It suffices, for example, to have simply $\liminf (r_1 r_2 \cdots r_n) = 0$ provided actual inequality holds for $n = 1, 2$ in (2.4). Other interesting cases are cited in the following theorem.

THEOREM 2.2. *If* (2.4) *holds with actual inequality for* $n = 1, 2$, *then* (1.1) *converges if any one of the following conditions holds*:

(a) $|a_n| < M$, $n = 2, 3, 4, \cdots$, (*$M$ a positive constant*);

(b) $\liminf |a_n| = 0$;

(c) $\liminf (r_1 r_2 \cdots r_n) < \infty$, $\sum (1/|a_n|)$ *diverges*;

(d) $r_1 r_3 \cdots r_{2n-1} < M$, $r_2 r_4 \cdots r_{2n} < M$ (*$M$ a positive constant*), $\sum |b_n|$ *diverges*; *or*

(e) $\liminf (r_1 r_2 \cdots r_n) < \infty$, $r_1 r_3 \cdots r_{2n-1}$ *and* $r_2 r_4 \cdots r_{2n}$ *bounded away from zero*, $\sum |b_n|$ *diverges*.

3. **The parabola theorem.** As noted in §1, the domain $|a_n| \leqq 1/4$ is the best possible circular convergence region with center at the origin. Perhaps the most interesting result of our investigations of the convergence problem is the theorem which follows.[12]

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[11] CT, p. 164.

[12] CT, p. 166.

THEOREM 3.1. *A set of points $W$ which is symmetrical with respect to the real axis is a convergence region*[13] *for the continued fraction* (1.1) *if and only if $W$ is bounded, and is contained in or upon the parabola*

$$|z| - R(z) = \tfrac{1}{2}. \tag{3.1}$$

The necessity of the conditions follows from the fact that if $z$ lies outside the parabola (3.1), then

$$\frac{1}{1} + \frac{z}{1} + \frac{\bar{z}}{1} + \frac{z}{1} + \frac{\bar{z}}{1} + \cdots \tag{3.2}$$

diverges; and from the fact that a convergence region must be bounded, for otherwise $a_2, a_3, a_4, \cdots$ could be chosen in the region in such a way that the series $\sum |b_n|$ of (1.4) would be convergent. The sufficiency follows immediately from Theorem 2.2 (a), since it is easy to verify that the inequalities (2.4) hold with $r_n = 1$, and with actual inequality for $n = 1, 2$, when the $a_n$'s lie in or upon the parabola (3.1).

From Theorem 2.2 (d) we have further that if the $a_n$'s lie in or upon the parabola[14] (3.1), then (1.1) converges if (and only if) the series $\sum |b_n|$ diverges.

No less important and interesting than the convergence region problem is the companion value region problem:[15] if $U$ is a convergence region, to determine a value region $V$ in which the value of (1.1) must lie when $a_2, a_3, a_4, \cdots$ lie in $U$. If it is known that

(a) $u$ in $U$ implies that $1/(1+u)$ is in $V$;

(b) $u$ in $U$, $v$ in $V$ implies that $1/(1+uv)$ is in $V$,

then it is plain that the approximants $A_n/B_n$, $(n = 2, 3, 4, \cdots)$, all lie in $V$ whenever $a_2, a_3, a_4, \cdots$ lie in $U$. By employing this interpretation of the continued fraction as a succession of linear transformations we have succeeded in determining the exact region $V$ in which the approximants of (1.1) must lie when $a_2, a_3, a_4, \cdots$ lie in the parabola (3.1). We find that $V$ is the region

$$|z - 1| \leq 1, \qquad z \neq 0. \tag{3.3}$$

Every value $z$ satisfying (3.3) is assumed by some continued fraction of the form (1.1) having its partial numerators in the parabola (3.1); and no value $z$ outside this region is so assumed.

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[13] I.e. the continued fraction converges when $a_2, a_3, a_4, \cdots$ have arbitrary values in $W$.

[14] Not necessarily a bounded portion of the parabola.

[15] W. T. Scott and H. S. Wall, *Value regions for continued fractions* (to be referred to as VR), to be published in this Bulletin.

We have also investigated the value region problem corresponding to a circular convergence region with center at an arbitrary point $a$ not real and $\leqq -1/4$.

It is interesting to consider the following "restricted" value region problem. Suppose that $a_2, a_3, \cdots, a_n$ are fixed in $U$. To determine a value region $V$ in which the approximants must lie when $a_{n+1}, a_{n+2}, a_{n+3}, \cdots$ vary independently in $U$. We have considered this problem for the case that $U$ is the region bounded by the parabola (3.1). To illustrate the sort of result obtained, let $a_2=\frac{1}{2}$, $a_3=\frac{1}{6}$, $a_4=\frac{1}{3}$, so that the continued fraction starts out like the simple continued fraction for log 2. Then if (1.1) converges, and $a_5, a_6, a_7, \cdots$ lie in the parabola (3.1), the value of the continued fraction must lie in a circle with center[16] $z=.69375$ ($\log 2=.69315\cdots$) and radius .00625.

It is interesting to apply these considerations to special continued fractions. For instance, one may show in this way that the hypergeometric function $F(\alpha, 1, \gamma, x)$, ($\alpha, \gamma$, real, $0<\alpha<\gamma$), satisfies the inequality[17]

$$\left| F(\alpha, 1, \gamma+1, x) - \frac{\gamma^2}{\gamma^2-\alpha^2} \right| \leqq \frac{\alpha\gamma}{\gamma^2-\alpha^2}, \qquad |x| \leqq 1.$$

## 4. The moment problem[18] for the interval (0, 1).

We shall return now to the continued fraction (2.7), in which we shall put $x_n=x$. We shall also multiply by a positive constant factor $c_0$. We then have:

$$\frac{c_0}{1+}\,\frac{g_1x}{1+}\,\frac{(1-g_1)g_2x}{1\quad+}\,\frac{(1-g_2)g_3x}{1\quad+}\cdots, \tag{4.1}$$

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[16] Formulas for the center and radius in terms of $a_2, a_3, a_4$ are given in VR.

[17] H. S. Wall, *A class of functions bounded in the unit circle*, Duke Mathematical Journal, vol. 7 (1940), pp. 146–153.

[18] By the "moment problem" is ordinarily understood the problem of determining a monotone nondecreasing function $\phi(u)$ satisfying the equations $c_n=\int_a^b u^n d\phi(u)$, ($n=0, 1, 2, \cdots$), when $\{c_n\}$ is a given sequence. This problem was solved by Stieltjes for the case $a=0$, $b=+\infty$ in 1894 (*Oeuvres*, vol. 2, pp. 402–566). In 1903 E. B. Van Vleck (*On an extension of the 1894 memoir of Stieltjes*, Transactions of this Society, vol. 4 (1903), pp. 297–332) investigated the problem for the case $a=-\infty$, $b=+\infty$. He did not obtain a complete solution. H. Hamburger (Mathematische Annalen, vol. 81 (1920), pp. 235–319, and vol. 82 (1921), pp. 120–164, 168–187) obtained a complete solution in the general case. At about the same time, E. Hellinger (Mathematische Annalen, vol. 86 (1922), pp. 18–29) gave a complete solution in a twelve-page article, by resolving the problem into a question of solving a system of infinitely many linear equations. Another solution was given by T. Carleman, *Sur les équations intégrales singulières à noyau symétrique*, Uppsala, 1923, pp. 189–220.

with $c_0>0$, $0<g_n<1$, $(n=1, 2, 3, \cdots)$. This is a *Stieltjes* type continued fraction inasmuch as the constant factors in the partial numerators are all positive. Since this continued fraction converges uniformly for $|x| \leqq c$ for every positive $c<1$, it follows that the corresponding Stieltjes integral has the form

$$\int_0^1 \frac{d\phi(u)}{1+xu}, \tag{4.2}$$

in which $\phi(u)$ is bounded and monotone nondecreasing and has infinitely many points of increase, $0 \leqq u \leqq 1$. Hence, if $P(x)=\sum c_n(-x)^n$ is the corresponding Stieltjes series, it follows that the sequence $\{c_n\}$ is a *totally monotone sequence*, i.e.,

$$\Delta^m c_n = \int_0^1 (1-u)^m u^n d\phi(u) \geqq 0, \qquad m, n = 0, 1, 2, \cdots .$$

Conversely, let $\{c_n\}$ be a totally monotone sequence. Then, as Hausdorff showed, there exists a function $\phi(u)$ such that

$$c_n = \int_0^1 u^n d\phi(u), \qquad n = 0, 1, 2, \cdots ,$$

where $\phi(u)$ is bounded and monotone nondecreasing. If $\phi(u)$ has infinitely many points of increase ("there is an infinite distribution of mass"), the series $P(x)=\sum c_n(-x)^n$ has a corresponding Stieltjes continued fraction of the form

$$\frac{c_0}{1} + \frac{a_2 x}{1} + \frac{a_3 x}{1} + \cdots, \tag{4.3}$$

in which $a_2, a_3, a_4, \cdots$ are real and positive. Then one may show[19] that (4.3) must be of the form (4.1). Thus, in order for the real sequence $\{c_n\}$ to be totally monotone and correspond to an infinite distribution of mass it is necessary and sufficient that the corresponding Stieltjes continued fraction be of the form (4.1).

As for totally monotone sequences corresponding to a finite distribution of mass, they are completely characterized by having a corresponding continued fraction of the form (4.1) which terminates. In this case the last $g_n$ may be 0 or 1.

The Stieltjes moment problem for the interval (0, 1), for which we have completely characterized the corresponding continued fraction, is of particular importance on account of its relation to Hausdorff methods of summation. I shall discuss some of our work on this theory in a later section.

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[19] H. S. Wall, *Continued fractions and totally monotone sequences* (to be referred to as TM), Transactions of this Society, vol. 48 (1940), pp. 165–184.

5. **A class of real functions bounded in the unit circle.** Let $f(x)$ denote the moment generating function represented by (4.1). Then $f(x)$ has an integral representation of the form (4.2), and a power series expansion $f(x)=\sum c_n(-x)^n$ in which the sequence of coefficients, $\{c_n\}$, is totally monotone. We shall include in our discussion the case where (4.1) terminates, in which event the last $g_n$ may be 0 or 1.

Let $M(f)=\text{l.u.b.}_{|x|<1}|f(x)|$. Then $M(f)\leqq 1$ if and only if the continued fraction can be put into the form[20]

$$\frac{g_1}{1}+\frac{(1-g_1)g_2x}{1}+\frac{(1-g_2)g_3x}{1}+\cdots \tag{5.1}$$

where $0\leqq g_n\leqq 1$, $(n=1, 2, 3, \cdots)$, and we agree that in case equality holds here for some $n$, then the continued fraction shall terminate with the first identically vanishing partial quotient. Again, $M(f)\leqq 1$ if and only if $\sum c_n\leqq 1$; or, if and only if

$$f(x)=\int_0^1\frac{(1-u)d\phi(u)}{1+xu},$$

where $\phi(u)$ is monotone and $0\leqq\phi(1)-\phi(0)\leqq 1$.

With each function $e(x)$, analytic for $|x|<1$, and with $M(e)\leqq 1$, Schur[21] associated a sequence of numbers $\{\gamma_n\}$ in the following way. Put $e(x)\equiv e_0(x)$,

$$e_{n+1}(x)=\frac{1}{x}\,\frac{\gamma_n-e_n(x)}{1-\bar{\gamma}_ne_n(x)},\quad \gamma_n=e_n(0),\qquad n=0, 1, 2, \cdots. \tag{5.2}$$

He showed that either (a) $|\gamma_n|<1$, $n=0, 1, 2, \cdots$; or else (b) $|\gamma_n|<1$, $n=0, 1, 2, \cdots, N-1$, $|\gamma_N|=1$, $\gamma_n=0$, $n>N$. Conversely, if $\{\gamma_n\}$ is any sequence satisfying (a) or (b) then there is uniquely determined a function $e(x)$, analytic for $|x|<1$ and with $M(e)\leqq 1$, such that (5.2) holds.

We shall confine our attention to the class $E$ of *real* functions $e(x)$, analytic for $|x|<1$ and with $M(e)\leqq 1$ In this case the sequence $\{\gamma_n\}$ associated with $e(x)$ in accordance with the result of Schur satisfies the condition (a) $-1<\gamma_n<+1$, $n=0, 1, 2, \cdots$; or else the condition (b) $-1<\gamma_n<+1$, $n=0, 1, 2, \cdots, N-1$, $|\gamma_N|=1$, $\gamma_n=0$, $n>N$. The subclass consisting of all moment generating functions of $E$ will be denoted by $F$. If $e(x)$ is in $F$, then all the functions $e_1(x), e_2(x), e_3(x), \cdots$ given by (5.2) are also in $F$.

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[20] TM, p. 179. Naturally the $g_n$'s are not the same in (5.1), (5.2).

[21] I. Schur, *Ueber Potenzreihen, die im Innern des Einheitskreises beschränkt sind*, Journal für die reine und angewandte Mathematik, vol. 147 (1916), pp. 205–232, and vol. 148 (1917), pp. 122–145.

We shall now prove the theorem which follows.

THEOREM 5.1. *There is a one-to-one correspondence between the functions of $E$ and of $F$ such that if $e(x)$ of $E$ corresponds to $f(z)$ of $F$, then for $|x|<1$ we have*

$$(5.3)\qquad \frac{(1-x)}{2}\,\frac{1-e(x)}{1+xe(x)} = f(z), \qquad z = 4x/(1-x)^2.$$

This may be formulated in terms of monotone functions: to every function $e(x)$ of $E$ there corresponds a monotone function $\phi(u)$ such that $0\leqq\phi(1)-\phi(0)\leqq 1$ and

$$(5.4)\qquad \frac{1-x}{x}\,\frac{1-e(x)}{1+xe(x)} = \int_0^1 \frac{(1-u)d\phi(u)}{1+zu}, \qquad z = 4x/(1-x)^2, \qquad |x|<1.$$

Conversely, if $\phi(u)$ is any monotone function such that $0\leqq\phi(1)-\phi(0)\leqq 1$, then there exists a function $e(x)$ of $E$ such that (5.4) holds.

Let $e_0(x)\equiv e(x), e_1(x), e_2(x), \cdots$ be determined by (5.2), and put

$$p_n(x) = \frac{1-e_n(x)}{1+xe_n(x)}, \qquad n = 0, 1, 2, \cdots.$$

Then from (5.2) we have the relation

$$p_n(x) = \frac{1-\gamma_n}{1-x+(1+\gamma_n)xp_{n+1}(x)}, \qquad n = 0, 1, 2, \cdots,$$

and consequently there is the *formal* continued fraction expansion

$$(5.5)\qquad \tfrac{1}{2}(1-x)\frac{1-e(x)}{1+xe(x)} = \frac{g_1}{1+}\,\frac{(1-g_1)g_2z}{1}\,\frac{}{+}\,\frac{(1-g_2)g_3z}{1}\,\frac{}{+}\cdots,$$

where $z=4x/(1-x)^2$, $g_n=\frac{1}{2}(1-\gamma_{n-1})$, $(n=1, 2, 3, \cdots)$. If some $\gamma_n$ is $+1$ or $-1$, this continued fraction terminates and (5.5) is then an identity. If, on the other hand, $-1<\gamma_{n-1}<+1$, then $0<g_n<1$, and the continued fraction converges uniformly for $|z|\leqq 1$, or for $x$ in a sufficiently small neighborhood of the origin. One may easily show that the power series expansion in ascending powers of $x$ of the function on the left agrees term-by-term with the power series expansion in ascending powers of $x$ of the $n$th approximant of the continued fraction, for more and more terms as $n$ is increased. From these facts one may conclude[22] that (5.5) is a true equality for $x$ in a sufficiently small neighborhood of the origin. Now,[23] the function $z=4x/(1-x)^2$ maps the interior of the circle $|x|=1$ in a one-to-one manner upon the $z$-plane exterior to the cut along the real axis from $-1$ to $-\infty$. The continued fraction represents an analytic function of $z$ in this cut plane, and therefore represents an analytic function of $x$ for $|x|<1$. Thus (5.5) is a true equality for $|x|<1$.

As previously pointed out, this continued fraction represents a function $f(z)$ of the class $F$.

Conversely, starting with the function $f(z)$ of $F$, so that the numbers $g_n$ may be found, we then determine the sequence $\{\gamma_n\}$ and hence the function $e(x)$ of $E$, by putting $\gamma_{n-1}=1-2g_n$, $(n=1, 2, 3, \cdots)$. If $0<g_n<1$, then $-1<\gamma_n<+1$, and the $g_n$'s and $\gamma_n$'s are uniquely determined. In case the continued fraction for $f(z)$ terminates, we may always assume that the last $g_n$ is either 0 or 1, and take subsequent $g_n$'s all equal to $\frac{1}{2}$. In this case we determine $e(x)$ as before by taking $\gamma_{n-1}=1-2g_n$. The function $e(x)$ clearly satisfies (5.5), and the theorem is proved.

Denote by $q(x)$ the function in the left member of (5.5). Then we find that

$$\frac{1-q(x)}{1+zq(x)}=\tfrac{1}{2}(1-x)\frac{1+e(x)}{1-xe(x)}\cdot$$

Now if we replace $e(x)$ by $-e(x)$ in (5.5), it is clear that the effect upon the continued fraction on the right is to replace $\gamma_{n-1}$ by $-\gamma_{n-1}$, which is the same as replacing $g_n$ by $1-g_n$. Hence we have the theorem[24] that if

$$p(x)=\frac{g_1}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{(1-g_1)g_2x}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{(1-g_2)g_3x}{1}\,\genfrac{}{}{0pt}{}{}{+}\cdots,$$

then the continued fraction for $[1-p(x)]/[1+xp(x)]$ is obtained from that for $p(x)$ by replacing $g_n$ by $1-g_n$, $(n=1, 2, 3, \cdots)$. It follows that if $p(x)$ is in the class $F$, so is $[1-p(x)]/[1+xp(x)]$. Therefore we have, in consequence of Theorem 5.1, the following theorem:

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[22] Cf. the argument in Perron, p. 343.

[23] Cf. E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionentheorie*, Berlin, 1929, p. 112.

[24] This theorem also follows from Theorem 2.1, p. 166, of TM. This was used in the paper referred to in Footnotes 14, 24 and also in the paper by H. S. Wall: *A continued fraction related to some partition formulas of Euler*, American Mathematical Monthly, vol. 48 (1941), pp. 102–108.

THEOREM 5.2. *To each function $p(x)$ of the class $F$ there corresponds uniquely a function $f(z)$ of $F$ such that*

$$(5.6) \qquad (1-x)p(x) = 2f(z), \qquad z = 4x/(1-x)^2, \ |x| < 1.$$

From Theorem 5.1 we obtain the following result:

THEOREM 5.3. *Let $e(x)$ be a function of $E$ such that $-1<\gamma_n<+1$, $(n=0, 1, 2, \cdots)$. Let the series*

$$S = 1 + \sum_{n=0}^{\infty} \frac{(1-\gamma_0)(1-\gamma_1)\cdots(1-\gamma_n)}{(1+\gamma_0)(1+\gamma_1)\cdots(1+\gamma_n)}$$

*be convergent. Then $M(e)=1$.*

It is interesting to compare this with the result of Schur which states that if the series $\sum|\gamma_n|$ converges, then $M(e)<1$.

To prove the theorem, let us recall that the series $S$ is the series appearing in (2.6) with $g_n=\frac{1}{2}(1-\gamma_{n-1})$. Hence, when this series converges we conclude by Theorem 5.1 that

$$|q(x)| = \left| \frac{1-x}{2} \, \frac{1-e(x)}{1+xe(x)} \right| \leqq (1-1/S) < 1$$

if $|4x/(1-x)^2|<1$. Let $x$ be real, and let $x\to -1$, $|x|<1$. Then $|4x/(1-x)^2|<1$ and $4x/(1-x)^2\to -1$, so that $q(x)\to 1-(1/S)$. It follows that $e(x)\to 1$, so that $M(e)=1$.

Inasmuch as $M[e(x)]=M[-e(x)]=M[e(-x)]=M[-e(-x)]$, it follows that the convergence of one of the series obtained from $S$ by replacing $\gamma_n$ by $-\gamma_n$, $\gamma_{2n+1}$ by $-\gamma_{2n+1}$, or $\gamma_{2n}$ by $-\gamma_{2n}$, $(n=0, 1, 2, \cdots)$, is sufficient in order that $M(e)=1$.

By a result of Stieltjes, the function $f(z)$ represented by the continued fraction on the right of (5.5) is a meromorphic function of $z$ if and only if $\lim_{n\to\infty}(1-g_n)g_{n+1}=0$, that is:

$$\lim_{n=\infty} (1+\gamma_{n-1})(1-\gamma_n) = 0.$$

In this case we may write

$$q(x) = \frac{(1-x)[1-e(x)]}{2[1+xe(x)]} = (1-x)^2 \sum_{n=1}^{\infty} \frac{M_n}{(x-r_n)(x-\bar{r}_n)}$$

where $M_n > 0$, $(n = 1, 2, 3, \cdots)$, $\sum M_n$ converges, and $r_n = 1 - 2u_n + 2i[u_n(1-u_n)]^{1/2}$, $1 > u_1 > u_2 > \cdots > 0$, $\lim u_n = 0$. Thus the singularities of $q(x)$ consist of simple poles $r_n$, $\bar{r}_n$ lying upon the circle $|x| = 1$, together with the point $x = 1$ which is the unique limit point of these poles. The function $q(x)$ satisfies the relation $q(1/x) = q(x)$. The following theorem is now evident.

THEOREM 5.4. *If $e(x)$ is a function of $E$ such that $-1 < \gamma_n < +1$, $\lim_{n=\infty} (1+\gamma_{n-1})(1-\gamma_n) = 0$, then $e(x)$ has an essential singularity at $x = 1$, and no other singularities in the extended plane except poles. The function*

$$q(x) = \frac{1-x}{2} \frac{1-e(x)}{1+xe(x)}$$

*has as its singularities an infinite sequence of simple poles lying on the circle $|x| = 1$ with the single limit point $x = 1$, and no others; and $q(1/x) = q(x)$.*

As an example, if

$$e(x) = \frac{1}{x} \frac{\dfrac{2x^{1/2}}{1+x} - \tanh \dfrac{2x^{1/2}}{1-x}}{\dfrac{2x^{1/2}}{1+x} + \tanh \dfrac{2x^{1/2}}{1-x}},$$

then $q(x) = 3^{-1}/1 + 3^{-1}5^{-1}z/1 + 5^{-1}7^{-1}z/1 + \cdots$, $z = 4x/(1-x)^2$, as one may verify by means of the continued fraction of Lambert for $\tanh z$. This function $e(x)$ satisfies the conditions of Theorem 5.4. When $z = -1$ in the continued fraction for $q(x)$, we find that its value is $1 - (\tan 1)^{-1} < 1$, and consequently the series $S$ of Theorem 5.3 converges. From that theorem it then follows that $M(e) = 1$. This can be verified directly by letting $x$ approach $-1$ in the above expression for $e(x)$.

Theorem 5.1 admits of the following interpretation. In (5.3) put $[1-e(x)]/[1+xe(x)] = c_0 - c_1x + c_2x^2 - \cdots$, and $f(z) = C_0 - C_1z + C_2z^2 - \cdots$. Considered as an identity in $x$, (5.3) yields the equations[25].

$$c_n = 2^{2n+3}C_{2n,0}C_n - 2^{2n+1}C_{2n-1,1}C_{n-1} + \cdots + (-1)^n 2^3 C_{n,n}C_0, \tag{5.7}$$

which constitute a linear transformation of the sequence $\{C_n\}$ into the sequence $\{c_n\}$. The inverse transformation is found to be

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[25] W. T. Scott and H. S. Wall, *Linear manifolds of Hausdorff means* (to be referred to as LM). To appear in the Transactions of this Society as Part I of a paper: *The transformation of series and sequences.*

$$(5.8)\qquad \begin{aligned} C_n = {} & \frac{1}{(n+1)2^{2n+1}} C_{2n,n}c_0 + \frac{3}{(n+2)2^{2n+1}} C_{2n,n-1}c_1 + \cdots \\ & + \frac{2n+1}{(2n+1)2^{2n+1}} C_{2n,0}c_n. \end{aligned}$$

The transformation (5.8) has the property that it carries any sequence $\{c_n\}$ generated by a function $[1-e(x)]/[1+xe(x)]$ where $e(x)$ is in $E$, into a totally monotone sequence $\{C_n\}$ with sum $\sum C_n \leqq 1$; and (5.7) has the property that it carries any totally monotone sequence $\{C_n\}$ with sum $\leqq 1$ into a sequence $\{c_n\}$ which is generated by a function of the form $[1-e(x)]/[1+xe(x)]$ where $e(x)$ is in $E$. This leads us to the topic to be discussed in the next section.

6. **Hausdorff means.** The familiar $(C, 1)$ transform of a sequence $\{s_n\}$, namely: $t_m = (s_0 + s_1 + \cdots + s_m)/(m+1)$, may be written in the form

$$t_m = \sum_{n=0}^{m} C_{m,n} \int_0^1 (1-u)^{m-n} u^n s_n du.$$

One may generalize this by replacing $du$ by $d\phi(u)$ where $\phi(u)$ is any function of bounded variation on the interval $0 \leqq u \leqq 1$, which is continuous at $u=0$, and which satisfies the conditions $\phi(0)=0$, $\phi(1)=1$. The resulting transform of the sequence $\{s_n\}$ is the Hausdorff mean.[26] This defines a regular method of summation, i.e., $s_n \to s$ implies $t_m \to s$, which is denoted by the symbol $[H, \phi(u)]$.

If we put $c_n = \int_0^1 u^n d\phi(u)$, $(n = 0, 1, 2, \cdots)$, the Hausdorff mean can be written in the form

$$t_m = \sum_{n=0}^{m} C_{m,n} \Delta^{m-n} c_n \cdot s_n.$$

The function $\phi(u)$, subject to the conditions imposed above, is called a *regular mass function*; and the sequence $\{c_n\}$ is called a *regular moment sequence*. Included among the Hausdorff means are the Hölder means $(H, \alpha)$, the Cesàro means $(C, \alpha)$, and many others.

Let $[H, \phi_a(u)]$ and $[H, \phi_b(u)]$ be two Hausdorff methods of summation, and let $\{a_n\}$ and $\{b_n\}$ be the corresponding regular moment sequences. Then[27] $[H, \phi_a(u)] \supset [H, \phi_b(u)]$, when $b_n \neq 0$, $(n = 0, 1, 2, \cdots)$, if and only if there exists a regular moment sequence $\{c_n\}$ such that

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[26] F. Hausdorff, *Summationsmethoden und Momentfolgen*, I and II, Mathematische Zeitschrift, vol. 9 (1921), pp. 74–109, 280–299. For an elementary account, see H. L. Garabedian, *Hausdorff matrices*, American Mathematical Monthly, vol. 46 (1939), pp. 390–410.

$$a_n = b_n c_n, \qquad n = 0, 1, 2, \cdots . \tag{6.1}$$

This is the basic theorem of the Hausdorff theory. When (6.1) holds it is convenient to say that $\{a_n\}$ is *divisible* by $\{b_n\}$. The condition that $\{a_n\}$ be divisible by $\{b_n\}$ may be formulated in a number of different ways.[28] One of the most natural ways is in terms of the moment generating functions

$$f_a(x) = \sum a_n(-x)^n, \qquad f_b(x) = \sum b_n(-x)^n.$$

The regular moment sequence $\{a_n\}$ is divisible by the regular moment sequence $\{b_n\}$ if and only if there exists a regular mass function $\phi_c(u)$ such that

$$f_a(x) = \int_0^1 f_b(ux)d\phi_c(u)$$

for all $x$ not real and $\leqq -1$. By means of the Stieltjes inversion formula one may show that this holds if and only if

$$\phi_a(u) = \phi_c(u) + \int_u^1 \phi_b(u/v)d\phi_c(v)$$

for all except at most a countable set of values of $u$.

If $\phi(u)$ is monotone nondecreasing, then $\{c_n\}$ is a totally monotone sequence, and hence the moment generating function $f_c(x)$ has a corresponding continued fraction of the form (4.1). It is easy to obtain conditions on (4.1) which are necessary and sufficient for $\{c_n\}$ to be a regular moment sequence.[29] An interesting example is afforded by the continued fraction of Gauss for the hypergeometric function $F(\alpha, 1, \gamma, x)$, $\alpha, \gamma$ real, $0 < \alpha < \gamma$. We have investigated in some detail the "hypergeometric summability" defined in this way.

It is interesting that in some cases one may operate directly with the continued fraction (4.1) to prove inclusion relationships between Hausdorff methods.[30]

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[27] This means that every sequence summable $[H, \phi_b(u)]$ is summable $[H, \phi_a(u)]$, and is read "$[H, \phi_a(u)]$ includes $[H, \phi_b(u)]$."

[28] H. L. Garabedian, Einar Hille, and H. S. Wall, *Formulations of the Hausdorff inclusion problem*, Duke Mathematical Journal, vol. 8 (1941), pp. 193–213.

[29] H. L. Garabedian and H. S. Wall, *Hausdorff methods of summation and continued fractions*, Transactions of this Society, vol. 48 (1940), pp. 185–207, p. 188.

[30] H. L. Garabedian and H. S. Wall, *Continued fractions and Hausdorff methods of summation*, Northwestern University Studies, in press.

Beginning with the interpretation of Theorem 5.1 given at the end of §5, we have developed a theory of linear manifolds of Hausdorff means.[31] Returning to (5.8), let us put

$$\beta_n(u) = \frac{1}{2^{2n+1}}\left\{\frac{1}{n+1}C_{2n,n} + \frac{3}{n+2}C_{2n,n-1}u + \cdots + \frac{2n+1}{2n+1}C_{2n,0}u^n\right\}.$$

Then if $c_n = \int_0^1 u^n d\phi(u)$, we may write the transformation (5.8) as

$$C_n = \int_0^1 \beta_n(u)d\phi(u), \qquad n = 0, 1, 2, \cdots. \tag{6.2}$$

From the discussion in §5 it follows that if $\{c_n\}$ is totally monotone and[32] $\sum c_n \leqq 1$, the sequence $\{C_n\}$ is of like character. This is, however, only part of the story. We find that if $\phi(u)$ is any function of bounded variation on the interval $0 \leqq u \leqq 1$, such that $\phi(1) - \phi(0) = 1$, then the sequence $\{C_n\}$ is a regular moment sequence.

Let $M(\beta_n(u)) = M(\beta)$ denote the set of all moment sequences $\{C_n\}$ obtained from (6.2) by letting $\phi(u)$ run through the class of all functions of bounded variation on the interval $0 \leqq u \leqq 1$. It is observed that if $\{C_n'\}$ and $\{C_n''\}$ are any two sequences in $M(\beta)$ then the sequences $\{C_n' + C_n''\}$, and $\{KC_n'\}$, ($K$ a constant), are in $M(\beta)$. Thus $M(\beta)$ is a *linear manifold* of moment sequences.

Any suitably chosen sequence of functions $\{\beta_n(u)\}$ determines in this same way a linear manifold of moment sequences. We have called $\{\beta_n(u)\}$ the *basis* of the manifold. In laying down the outline for a general theory of these manifolds, we have obtained conditions on a sequence of functions $\{\beta_n(u)\}$ in order that it form the basis of a manifold; have obtained conditions under which the sequences of the manifold are all essentially regular; and conditions under which the Hausdorff methods defined by the sequences of a manifold all include a given Hausdorff method.

As an example, let $\beta_n(u) = (u+1)/(u+n+1)$. For every fixed $u$ in the interval $0 \leqq u \leqq 1$, the sequence $\{\beta_n(u)\}$ is totally monotone. From this one may conclude that $\{\beta_n(u)\}$ is a basis for a manifold $M(\beta)$. Further, it can be shown that every sequence $\{C_n\}$ in $M(\beta)$ for which $C_0 \neq 0$ is essentially regular; and that every Hausdorff method of summation defined by the sequences of $M(\beta)$ includes $(C, 1)$ but not all of them include $(C, 1+t)$, $t > 0$.

**7. Continued fraction expansions for arbitrary power series.** The continued fraction expansions for power series which we have used up to this point all have partial quotients of the form $a_nx/1$. This is true only when the power series is of a special kind. Any power series $P(x)=1+c_1x+c_2x^2+\cdots$ has a corresponding continued fraction in which the partial quotients are of the form $a_nx^{\alpha_n}/1$, $\alpha_n$ a positive integer.[33] The continued fraction has many of the properties of semi-normal continued fractions. For example, it terminates if and only if $P(x)$ represents a rational function of $x$; and if the continued fraction converges uniformly in the vicinity of the origin, the power series also converges in the neighborhood of the origin and the power series is equal to its continued fraction.

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[31] LM.

[32] If $f(x)$ is in $F$ and $f(x) = \sum c_n(-x)^n$, then $M(f) = \sum c_n$. Cf. TM, p. 181.

We have investigated these "corresponding type" continued fractions with a view toward obtaining formulas for the $a_n$'s and $\alpha_n$'s in terms of the power series.[34]

We found first of all that there is a fairly practical step-by-step process for expanding $P(x)$ into a continued fraction. This is based upon the observation that if we have carried out the expansion to the point where we have

$$1+\frac{a_1x^{\alpha_1}}{1}+\frac{a_2x^{\alpha_2}}{1}+\cdots+\frac{a_nx^{\alpha_n}}{1},$$

and we denote this fraction by $A_n(x)/B_n(x)$, then the formal power series for

$$\frac{A_n(x)}{B_n(x)}-P(x) \tag{7.1}$$

must begin with the term $(-1)^{n+1}a_1a_2\cdots a_{n+1}x^{\alpha_1+\alpha_2+\cdots+\alpha_{n+1}}$. Hence, knowing $a_1, a_2, \cdots, a_n, \alpha_1, \alpha_2, \cdots, \alpha_n$ we may determine $a_{n+1}$ and $\alpha_{n+1}$ as soon as we know the term of lowest degree in the power series for (7.1). This process requires expansions in powers of $x$ for only a *finite* number of terms at each stage.

We have found formulas for the elements of the continued fraction in terms of the power series in the case of a large class of continued fractions which we have termed *absolutely regular*, namely those for which $\alpha_1=1$ and

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[33] Walter Leighton and W. T. Scott, *A general continued fraction expansion*, Bulletin of this Society, vol. 45 (1939), pp. 596–605.

[34] W. T. Scott and H. S. Wall, *Continued fraction expansions for arbitrary power series*, Annals of Mathematics, (2), vol. 41 (1940), pp. 328–349.

$$\begin{aligned} \alpha_2 + \alpha_4 + \cdots + \alpha_{2n} &\geqq \alpha_1 + \alpha_3 + \cdots + \alpha_{2n-1}, \\ \alpha_3 + \alpha_5 + \cdots + \alpha_{2n+1} &\geqq \alpha_2 + \alpha_4 + \cdots + \alpha_{2n}, \end{aligned} \qquad n = 1, 2, 3, \cdots .$$

In case the $a_n$'s are real and positive, these conditions are equivalent to the condition that all the approximants of the continued fraction be Padé approximants for the corresponding power series. The class of absolutely regular continued fractions contains the class of semi-normal continued fractions, and the formulas found are natural extensions of the well known formulas for the semi-normal case.

A remarkable situation arises when each exponent in the power series $P(x)$ is at least twice the preceding. In this case the $a_n$'s in the continued fraction depend only upon the coefficients in the power series, while the $\alpha_n$'s in the continued fraction depend only upon the exponents in the power series. For this reason, certain operations upon the power series and continued fraction, such as differentiation, integration, forming reciprocals, and a sort of Hadamard composition, may be readily performed.

There is considerable evidence to support the conjecture that corresponding type continued fractions represent functions having circles as natural boundaries, provided the exponents $\alpha_n$ increase sufficiently rapidly. We have found that this is the case when the $a_n$'s are suitably restricted and the $\alpha_n$'s form a geometric progression.

NORTHWESTERN UNIVERSITY

# ON BIORTHOGONAL MATRICES[1]

Y. K. WONG

**Introduction.** Consider the basis consisting of a number system $\mathfrak{A}$ of type $D$, two general ranges $\mathfrak{P}^1$, $\mathfrak{P}^2$, and two positive hermitian matrices $\epsilon^1$, $\epsilon^2$. We introduce two binary relations for pairs of non-modular matrices. The matrices $\kappa^{12}$, $\phi^{21}$ are said to be contraceding as to $\epsilon^1\epsilon^2_\kappa$, $\epsilon^2$ in case $\kappa^{12}$, $\phi^{21}$ are by columns of $\mathfrak{M}(\epsilon^1)$, $\mathfrak{M}(\epsilon^2)$ respectively and such that $J^{2\kappa}\kappa^{12}\mu^2 = J^2\phi^{*12}\mu^2$ for every $\mu^2$ in the set $\mathfrak{M}(\epsilon^2_\kappa \cap \epsilon^2)$. It is evident that when $\kappa^{12}$ is of type $\mathfrak{M}(\epsilon^1)\overline{\mathfrak{M}}(\epsilon^2)$, then the contracedence property implies that $J^2\kappa^{12}\phi^{21} = \epsilon^1_\kappa$ but not conversely. The main results are stated in Theorems 2 and 3. We next consider $\epsilon^1_0$, $\epsilon^1_1$ both idempotent as to $\epsilon^1$. Suppose that $\kappa^{12}$ is by columns of $\mathfrak{M}(\epsilon^1_0)$ and $\phi^{21}$ is by rows-conjugate of $\mathfrak{M}(\epsilon^1_1)$. Take any pair of vectors $\mu^1$, $\nu^1$ modular as to $\epsilon^1_0$, $\epsilon^1_1$ respectively such that $J^1\kappa^{*21}\mu^1$, $J^1\phi^{21}\nu^1$ are in $\mathfrak{M}(\epsilon^2)$. If $J^1\bar{\mu}^1\nu^1$ is equal to the inner product $J^2(J^1\bar{\mu}^1\kappa^{12}, J^1\phi^{21}\nu^1)$, then $\kappa^{12}$, $\phi^{21}$ are said to be biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^1_1\epsilon^2$. When $\epsilon^1_1 = \epsilon^1_0$, then $\kappa^{12}$, $\phi^{21}$ are said to be biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^2$ in case they are biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^1_0\epsilon^2$. With proper restrictions imposed upon $\kappa^{12}$, $\phi^{21}$, we obtain the contracedence property. In a later paper, we shall establish the relations of biorthogonality and a certain mode of interchange of integration processes.

1. **Preliminary results.** Consider the basis $\mathfrak{A}$, $\mathfrak{P}^1$, $\mathfrak{P}^2$, $\epsilon^1$, and $\kappa^{12}$ which is by columns of $\mathfrak{M}(\epsilon^1)$. E. H. Moore's generalized Fourier processes give $\epsilon^2_\kappa \equiv J^1\kappa^{*21}\kappa^{12}$ and $\epsilon^1_\kappa \equiv J^{2\kappa}\kappa^{12}\kappa^{*21}$. The spaces $\mathfrak{M}(\epsilon^1_\kappa)$ and $\mathfrak{M}(\epsilon^2_\kappa)$ are in one-to-one correspondence (denoted by $\leftrightarrow$) via the transformations $J^1\kappa^{*21}$ and $J^{2\kappa}\kappa^{12}$, and the correspondences are orthogonal in the sense that the moduli of the corresponding vectors are preserved.[2]

(A)[3] Suppose that $\mathfrak{M}_1(\epsilon^1_\kappa) \leftrightarrow \mathfrak{M}_1(\epsilon^2_\kappa)$ and $\mathfrak{M}_2(\epsilon^1_\kappa) \leftrightarrow \mathfrak{M}_2(\epsilon^2_\kappa)$ via the transformations $J^1\kappa^{*21}$, $J^{2\kappa}\kappa^{12}$. Then $\mathfrak{M}_1(\epsilon^1_\kappa)$ is a subset of $\mathfrak{M}_2(\epsilon^1_\kappa)$ if and only if $\mathfrak{M}_1(\epsilon^2_\kappa)$ is a subset of $\mathfrak{M}_2(\epsilon^2_\kappa)$; $\mathfrak{M}_1(\epsilon^1_\kappa)$ is linearly $J^1$-closed if and only if $\mathfrak{M}_1(\epsilon^2_\kappa)$ is linearly $J^{2\kappa}$-closed; and $\mathfrak{M}_1(\epsilon^1_\kappa)$ is everywhere dense in $\mathfrak{M}_2(\epsilon^1_\kappa)$ if and only if $\mathfrak{M}_1(\epsilon^2_\kappa)$ is everywhere dense in $\mathfrak{M}_2(\epsilon^2_\kappa)$.

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[1] Presented to the Society, June 20, 1940.

[2] For a concise outline of Moore's generalized Fourier theory and its related topics, *see* Moore, *General Analysis*, I, pp. 19–26. For an important classical instance, *see* E. Schmidt, *Über die Auflösung linearer Gleichungen mit unendlichvielen Unbekannten*, Rendiconti del Circolo Matematico di Palermo, vol. 25 (1908), pp. 56–77.

[3] For the demonstrations of the following results, *see* the author's forthcoming paper *On non-modular matrices*.

Let $\epsilon^2$ be a positive hermitian matrix. (B) The class of all vectors $\mu^1$ modular as to $\epsilon^1$ such that $J^1\kappa^{*21}\mu^1$ is in $\mathfrak{M}(\epsilon^2)$ will be denoted by $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)$. The intersection of $\mathfrak{M}(\epsilon^2)$ and $\mathfrak{M}(\epsilon^2_\kappa)$ will be denoted by $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)$. (C) The set $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)$ is identical with the class of vectors $J^1\kappa^{*21}\mu^1$ for all $\mu^1$ in $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)$. Moreover, (D) the sets $\mathfrak{M}(\epsilon^1_\kappa\kappa^*\epsilon^2)$ and $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)$ are in one-to-one (orthogonal) correspondence via the transformations $J^1\kappa^{*21}$ and $J^{2\kappa}\kappa^{12}$. (E) The set $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)$ is the linear extension of the sum of $\mathfrak{M}(\epsilon^1_\kappa\kappa^*\epsilon^2)$ and the orthogonal complement within $\mathfrak{M}(\epsilon^1)$ of $\mathfrak{M}(\epsilon^1_\kappa)$. (F) The class $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)$ is everywhere dense in $\mathfrak{M}(\epsilon^1)$ if and only if $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)$ is everywhere dense in $\mathfrak{M}(\epsilon^2_\kappa)$.

2. **Contraceding pairs of matrices.** The basis of the paper consists of a number system $\mathfrak{A}$ of type $D$, two general ranges $\mathfrak{P}^1$, $\mathfrak{P}^2$, and two positive hermitian matrices $\epsilon^1$, $\epsilon^2$.

LEMMA 1. *Suppose that $\kappa^{12}$ is by columns of $\mathfrak{M}(\epsilon^1)$. Let $\mathfrak{M}_0(\epsilon^1)$ be a subset of $\mathfrak{M}(\epsilon^1)$; then $\mathfrak{M}(\epsilon^1_\kappa\kappa^*\epsilon^2)$ is contained in (contains) $\mathfrak{M}_0(\epsilon^1)$ only if $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)$ is contained in (contains) the class of vectors $J^1\kappa^{*21}\mu^1$ for all $\mu^1$ in $\mathfrak{M}_0(\epsilon^1)$. The converses are valid provided that $\mathfrak{M}_0(\epsilon^1)\subset\mathfrak{M}(\epsilon^1_\kappa)$.*

PROOF. The lemma follows from (D) and (A) in §1.

If $\phi^{21}$ is by columns of $\mathfrak{M}(\epsilon^2)$, we introduce the notations due to E. H. Moore:

$$\epsilon^1_\phi \equiv J^2\phi^{*12}\phi^{21}, \qquad \epsilon^2_\phi \equiv J^{1\phi}\phi^{21}\phi^{*12}.$$

LEMMA 2. *Suppose that $\kappa^{12}$, $\phi^{21}$ are by columns of $\mathfrak{M}(\epsilon^1)$, $\mathfrak{M}(\epsilon^2)$ respectively. Then $\mathfrak{M}(\epsilon^1_\kappa\kappa^*\epsilon^2)$ is contained in (contains) $\mathfrak{M}(\epsilon^1_\kappa\cap\epsilon^1_\phi)$ if and only if $\mathfrak{M}(\epsilon^2_\kappa\cap\epsilon^2)$ is contained in (contains) the class of vectors $J^1\kappa^{*21}\mu^1$ for all $\mu^1$ in $\mathfrak{M}(\epsilon^1_\kappa\cap\epsilon^1_\phi)$.*

PROOF. This lemma is a special instance of Lemma 1.

DEFINITION 1. *The matrices $\kappa^{12}$, $\phi^{21}$ are said to be contraceding as to $\epsilon^1\epsilon^2_\kappa\epsilon^2$ in case $\kappa^{12}$, $\phi^{21}$ are by columns of $\mathfrak{M}(\epsilon^1)$, $\mathfrak{M}(\epsilon^2)$ respectively and such that $J^{2\kappa}\kappa^{12}\mu^2 = J^2\phi^{*12}\mu^2$ for every $\mu^2$ in the set $\mathfrak{M}(\epsilon^2_\kappa\cap\epsilon^2)$.*

THEOREM 1. *Suppose that $\kappa^{12}$, $\phi^{21}$ are contraceding as to $\epsilon^1\epsilon^2_\kappa\epsilon^2$. Then*
(1) $\mathfrak{M}(\epsilon^1_\kappa\kappa^*\epsilon^2) = [J^2\phi^{*12}\mu^2 \mid \mu^2 \text{ in } \mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)] \subset \mathfrak{M}(\epsilon^1_\kappa\cap\epsilon^1_\phi)$;
(2) $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa) \subset \mathfrak{M}(\epsilon^2\phi\epsilon^1_\kappa)$;
(3) $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2) \subset \mathfrak{M}(\epsilon^1\cap\epsilon^1_\phi)$ *if and only if every $\mu^1$ for which $J^1\epsilon^1_\kappa\mu^1 = 0^1$ is in $\mathfrak{M}(\epsilon^1_\phi)$.*

PROOF. Part (1) is a consequence of the fact that

$$\mathfrak{M}(\epsilon^1_\kappa\kappa^*\epsilon^2) = [J^{2\kappa}\kappa^{12}\mu^2 \mid \mu^2 \text{ in } \mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)].$$

By hypothesis, we may replace $J^{2\kappa}\kappa^{12}$ by $J^2\phi^{*12}$. Since $\phi^{*12}$ is modular as to $\epsilon_\phi^1\epsilon^2$ and $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ is a subset of $\mathfrak{M}(\epsilon_\kappa^1)$, every vector of the form $J^2\phi^{*12}\mu^2$ must belong to $\mathfrak{M}(\epsilon_\kappa^1\cap\epsilon_\phi^1)$. This proves (1). Part (2) is obvious from (1). Part (3) follows from (1) and (E).

THEOREM 2. *Suppose that* $\phi^{21}$, $\kappa^{12}$ *are contraceding as to* $\epsilon^2\epsilon_\phi^1\epsilon^2$. *Then*

(1) $O^1\mathfrak{M}(\epsilon_\kappa^1)\subset\mathfrak{M}(\epsilon_\phi^1)$ *if and only if* $\kappa^{12}$ *is complete by columns of* $\mathfrak{M}(\epsilon^1)$;

(2) *the following four assertions are equivalent*: (i) $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)\subset\mathfrak{M}(\epsilon_\phi^1)$; (ii) $\kappa^{12}$ *is complete by columns of* $\mathfrak{M}(\epsilon^1)$ *and* $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)\subset\mathfrak{M}(\epsilon_\phi^2\phi^*\epsilon^1)$; (iii) $\kappa^{12}$ *is complete by columns of* $\mathfrak{M}(\epsilon^1)$ *and* $\kappa^{12}$, $\phi^{21}$ *are contraceding as to* $\epsilon^1\epsilon_\kappa^2\epsilon^2$; (iv) $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)\subset[J^2\phi^{*12}\mu^2 \mid \mu^2$ *in* $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)]$.

*If one of the four conditions in* (2) *is valid, then* $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)=\mathfrak{M}(\epsilon^1\cap\epsilon_\phi^1)$ *and* $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)=\mathfrak{M}(\epsilon_\phi^2\phi^*\epsilon^1)$.

PROOF. If $\kappa^{12}$ is complete by columns of $\mathfrak{M}(\epsilon^1)$, then $O^1\mathfrak{M}(\epsilon_\kappa^1)=[0^1]$, which is, of course, contained in $\mathfrak{M}(\epsilon_\phi^1)$. Conversely, every $\mu^1$ satisfying $J^1\bar{\mu}^1\kappa^{12}=0^2$ is in the orthogonal complement within $\mathfrak{M}(\epsilon^1)$ of $\mathfrak{M}(\epsilon_\kappa^1)$, and hence is in $\mathfrak{M}(\epsilon_\phi^1)$. Thus $J^{1\phi}\phi^{21}\mu^1=0^2$. Since $\phi^{21}$ is complete by rows-conjugate of $\mathfrak{M}(\epsilon_\phi^1)$, we have $\mu^1=0^1$, proving that $\kappa^{12}$ is complete by columns of $\mathfrak{M}(\epsilon^1)$.

To prove (i) implies (ii), we observe by (E) that $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)$ is the linear extension of the sum of $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ and $O^1\mathfrak{M}(\epsilon_\kappa^1)$. If (i) is valid, then by part (1) just proved, $\kappa^{12}$ is complete by columns of $\mathfrak{M}(\epsilon^1)$. By Lemma 2, the condition that $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)\subset\mathfrak{M}(\epsilon_\phi^1)$ is equivalent to

$$\text{(a)} \qquad \mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)\subset[J^1\kappa^{*21}\mu^1 \mid \mu^1 \text{ in } \mathfrak{M}(\epsilon^1\cap\epsilon_\phi^1)].$$

By hypothesis, we may replace $J^1\kappa^{*21}$ by $J^{1\phi}\phi^{21}$. Since, by (D),

$$\text{(b)} \qquad [J^{1\phi}\phi^{21}\mu^1 \mid \mu^1 \text{ in } \mathfrak{M}(\epsilon^1\cap\epsilon_\phi^1)]=\mathfrak{M}(\epsilon_\phi^2\phi^*\epsilon^1),$$

we have (i)→(ii). To prove (ii)→(iii), consider any vector $\mu^2$ in $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)$. If (ii) is valid, then, by (D) (cf. (b) above), there exists a vector $\mu^1$ in $\mathfrak{M}(\epsilon^1\cap\epsilon_\phi^1)$ such that

$$\text{(c)} \qquad \mu^2=J^{1\phi}\phi^{21}\mu^1,$$

whence

$$J^2\phi^{*12}\mu^2=J^2\phi^{*12}J^{1\phi}\phi^{21}\mu^1=J^{1\phi}\epsilon_\phi^1\mu^1=\mu^1.$$

Now by hypothesis and (c), we have $\mu^2=J^1\kappa^{*21}\mu^1$. As $\epsilon_\kappa^1=\epsilon^1$, it follows that

$$J^{2\kappa}\kappa^{12}\mu^2=J^{2\kappa}\kappa^{12}J^1\kappa^{*21}\mu^1=J^1\epsilon_\kappa^1\mu^1=\mu^1,$$

proving the condition (iii). By Theorem 1, we have (iii)→(iv). Since $\phi^{*12}$ is modular as to $\epsilon_\phi^1\epsilon^2$, we secure (iv)→(i).

The final statement follows by Theorem 1.

THEOREM 3. *The matrices* $\phi^{21}$, $\kappa^{12}$ *are contraceding as to* $\epsilon^2\epsilon_\phi^1\epsilon^1$, $\phi^{21}$ *is complete by columns of* $\mathfrak{M}(\epsilon^2)$, *and* $\mathfrak{M}(\epsilon^1\kappa^*\epsilon^2)\subset\mathfrak{M}(\epsilon_\phi^1)$ *if and only if* $\kappa^{12}$, $\phi^{21}$ *are contraceding as to* $\epsilon^1\epsilon_\kappa^2\epsilon^2$, $\kappa^{12}$ *is complete by columns of* $\mathfrak{M}(\epsilon^1)$, *and* $\mathfrak{M}(\epsilon^2\phi^*\epsilon^1)\subset\mathfrak{M}(\epsilon_\kappa^2)$.

PROOF. Apply Theorems 1 and 2.

3. **Biorthogonal matrices.** By considering $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ and $\mathfrak{M}(\epsilon_\kappa^2\cap\epsilon^2)$ as subsets of $\mathfrak{M}(\epsilon_\kappa^1)$ and $\mathfrak{M}(\epsilon_\kappa^2)$ respectively, we have established the one-to-one correspondence between those two subsets. The correspondences are orthogonal. But the Fourier coefficient function of every vector in $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ is also modular as to $\epsilon^2$. This property gives rise to another direction of studying the correspondences between the aforementioned subsets.

DEFINITION 2. *Suppose that* $\epsilon_0^1$, $\epsilon_1^1$ *are idempotent*[4] *as to* $\epsilon^1$, $\kappa^{12}$ *is by columns of* $\mathfrak{M}(\epsilon_0^1)$, *and* $\phi^{21}$ *is by rows-conjugate of* $\mathfrak{M}(\epsilon_1^1)$. *Then* $\kappa^{12}$, $\phi^{21}$ *are said to be biorthogonal as to* $\epsilon^1\epsilon_0^1\epsilon_1^1\epsilon^2$ *in case*

$$J^1\bar{\mu}^1\nu^1 = J^2(J^1\bar{\mu}^1\kappa^{12}, J^1\phi^{21}\nu^1)$$

*for every pair of vectors* $\mu^1$, $\nu^1$ *in* $\mathfrak{M}(\epsilon_0^1\kappa^*\epsilon^2)$, $\mathfrak{M}(\epsilon_1^1\phi\epsilon^2)$ *respectively.*

It is obvious that $\kappa^{12}$, $\phi^{21}$ are biorthogonal as to $\epsilon^1\epsilon_0^1\epsilon_1^1\epsilon^2$ if and only if $\phi^{*12}$, $\kappa^{*21}$ are biorthogonal as to $\epsilon^1\epsilon_1^1\epsilon_0^1\epsilon^2$.

DEFINITION 3. *Suppose that* $\epsilon_0^1$ *is idempotent as to* $\epsilon^1$ *and* $\kappa^{12}$, $\phi^{*12}$ *are by columns of* $\mathfrak{M}(\epsilon_0^1)$. *Then* $\kappa^{12}$, $\phi^{21}$ *are said to be biorthogonal as to* $\epsilon^1\epsilon_0^1\epsilon^2$ *if they are biorthogonal as to* $\epsilon^1\epsilon_0^1\epsilon_0^1\epsilon^2$.

THEOREM 4. *Suppose that* $\epsilon_0^1$, $\epsilon_1^1$ *are idempotent as to* $\epsilon^1$, *and* $\kappa^{12}$, $\phi^{*12}$ *are by columns of* $\mathfrak{M}(\epsilon_0^1)$, $\mathfrak{M}(\epsilon_1^1)$ *respectively. Then*

(i) $\kappa^{12}$, $\phi^{21}$ *are biorthogonal as to* $\epsilon^1\epsilon_\kappa^1\epsilon_1^1\epsilon^2$ *if* $\kappa^{12}$, $\phi^{21}$ *are biorthogonal as to* $\epsilon^1\epsilon_0^1\epsilon_1^1\epsilon^2$, *and only if* $\kappa^{12}$, $\phi^{21}$ *are biorthogonal as to* $\epsilon^1\epsilon_\kappa^1\epsilon_{\phi*}^1\epsilon^2$;

(ii) $\kappa^{12}$, $\phi^{21}$ *are biorthogonal as to* $\epsilon^1\epsilon_\kappa^1\epsilon_1^1\epsilon^2$ *if and only if*

$$J^2(J^1\bar{\mu}^1\phi^{*12}, \mu^2) = J^{2\kappa}(J^1\bar{\mu}^1\kappa^{12}, \mu^2) \tag{4.1}$$

*holds for every* $\mu^1$ *in* $\mathfrak{M}(\epsilon_1^1\phi\epsilon^2)$ *and every* $\mu^2$ *in* $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)$.

(iii) $\kappa^{12}$, $\phi^{21}$ *are biorthogonal as to* $\epsilon^1\epsilon_\kappa^1\epsilon_{\phi*}\epsilon^2$ *if and only if*

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[4] $\epsilon_0^1$ is idempotent as to $\epsilon^1$ in case $\epsilon_0^1$ is by columns of $\mathfrak{M}(\epsilon^1)$ and $J^1\epsilon_0^{1*}\epsilon_0^1 = \epsilon_0$. See G.A., I, pp. 23–24.

(4.2) $$J^2\xi^2\eta^2 = J^1(J^{2\phi^*}\bar{\xi}^2\phi^{21}, J^{2\kappa}\kappa^{12}\eta^2)$$

*holds for every* $\xi^2$ *in* $\mathfrak{M}(\epsilon^2\cap\epsilon^2_{\phi^*})$ *and every* $\eta^2$ *in* $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)$.

PROOF. Since $\mathfrak{M}(\epsilon^1_\kappa)$ and $\mathfrak{M}(\epsilon^1_{\phi^*})$ are subsets of $\mathfrak{M}(\epsilon^1_0)$ and $\mathfrak{M}(\epsilon^1_1)$ respectively, we have the conclusion (i).

To prove the necessity in (ii), consider any $\mu^2$ in the set $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)$ and any vector $\mu^1$ in $\mathfrak{M}(\epsilon^1_1\phi\epsilon^2)$. Theorem (D) shows the existence of a vector $\nu^2$ in $\mathfrak{M}(\epsilon^1_\kappa\kappa^*\epsilon^2)$ such that

(a) $$\nu^1 = J^{2\kappa}\kappa^{12}\mu^2, \qquad \mu^2 = J^1\kappa^{*21}\nu^1.$$

By using the fact that $\kappa^{12}$ is modular as to $\epsilon^1\epsilon^2_\kappa$, we have

(b) $$J^{2\kappa}(J^1\bar{\mu}^1\kappa^{12}, \mu^2) = J^1\bar{\mu}^1\nu^1.$$

Equations (a) and (b) show that the condition is necessary. The sufficiency can be proved similarly.

Now observe that $\kappa^{12}$, $\phi^{21}$ are biorthogonal as to $\epsilon^1\epsilon^1_\kappa\epsilon^1_{\phi^*}\epsilon^2$ if and only if

(c) $$J^2(J^1\bar{\mu}^1\phi^{*12}, \eta^2) = J^1(\bar{\mu}^1, J^{2\kappa}\kappa^{12}\eta^2)$$

for every $\mu^1$ in $\mathfrak{M}(\epsilon^1_{\phi^*}\phi\epsilon^2)$ and every $\eta^2$ in $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)$. The latter condition is obviously satisfied if and only if (4.2) holds, since $\mathfrak{M}(\epsilon^1_{\phi^*}\phi\epsilon^2)$ and $\mathfrak{M}(\epsilon^2_{\phi^*}\cap\epsilon^2)$ are in one-to-one correspondence via the transformations $J^1\phi^{21}$ and $J^{2\phi^*}\phi^{*12}$ by (D).

THEOREM 5. *Suppose that* $\epsilon^1_0$, $\epsilon^1_1$, $\kappa^{12}$, $\phi^{21}$ *have the same properties as in the hypothesis of Theorem* 4, *and* $\mathfrak{M}(\epsilon^1_\kappa)$ *contains either* $\mathfrak{M}(\epsilon^1_1)$ *or* $\mathfrak{M}(\epsilon^1_0)$. *Then* $\kappa^{12}$, $\phi^{21}$ *are biorthogonal as to* $\epsilon^1\epsilon^1_0\epsilon^1_1\epsilon^2$ *if and only if* $\kappa^{12}$, $\phi^{21}$ *are biorthogonal as to* $\epsilon^1\epsilon^1_\kappa\epsilon^1_1\epsilon^2$.

PROOF. The condition is necessary by Theorem 4. If $\mathfrak{M}(\epsilon^1_\kappa)$ contains $\mathfrak{M}(\epsilon^1_0)$, the condition is obviously sufficient. If $\mathfrak{M}(\epsilon^1_\kappa)$ contains $\mathfrak{M}(\epsilon^1_1)$, the sufficiency is proved by the following argument: Let $\mu^1$, $\nu^1$ be vectors in $\mathfrak{M}(\epsilon^1_1\phi\epsilon^2)$, $\mathfrak{M}(\epsilon^1_0\kappa^*\epsilon^2)$ respectively. Since $\mu^1$ is modular as to $\epsilon^1_\kappa$ by hypothesis, we have

$$J^1\bar{\mu}^1\nu^1 = J^1(J^1\bar{\mu}^1\epsilon^1_\kappa, \nu^1) = J^1(\bar{\mu}^1, J^1\epsilon^1_\kappa\nu^1).$$

Also

$$J^1\kappa^{*21}\nu^1 = J^1J^1\kappa^{*21}\epsilon^1_\kappa\nu^1 = J^1(\kappa^{*21}, J^1\epsilon^1_\kappa\nu^1).$$

The vector $J^1\epsilon^1_\kappa\nu^1$ is evidently in the set $\mathfrak{M}(\epsilon^1_\kappa\kappa^*\epsilon^2)$.

COROLLARY 6. *Suppose that* $\kappa^{12}$, $\phi^{*12}$ *are by columns of* $\mathfrak{M}(\epsilon^1)$, $\mathfrak{M}(\epsilon^1_\kappa)$

*respectively. Then* $\kappa^{12}$, $\phi^{21}$ *are biorthogonal as to* $\epsilon^1\epsilon_0^1\epsilon_\kappa^1\epsilon^2$ *if and only if* $\kappa^{12}$, $\phi^{21}$ *are biorthogonal as to* $\epsilon^1\epsilon_\kappa^1\epsilon^2$.

THEOREM 7. *Suppose that* $\epsilon_0^1$, $\epsilon_1^1$, $\kappa^{12}$, $\phi^{21}$ *have the same properties as in the hypothesis of Theorem* 4, *and* $\mathfrak{M}(\epsilon_\kappa^2\cap\epsilon^2)$ *is everywhere dense in* $\mathfrak{M}(\epsilon_\kappa^2)$. *Then* $\kappa^{12}$, $\phi^{21}$ *are biorthogonal as to* $\epsilon^1\epsilon_0^1\epsilon_1^1\epsilon^2$ *and the orthogonal complement within* $\mathfrak{M}(\epsilon_1^1)$ *of* $\mathfrak{M}(\epsilon_{\phi*}^1)$ *is a subset of* $\mathfrak{M}(\epsilon_0^1)$ *if and only if* $\kappa^{12}$, $\phi^{21}$ *are biorthogonal as to* $\epsilon^1\epsilon_0^1\epsilon_{\phi*}^1\epsilon^2$ *and* $\phi^{21}$ *is complete by rows-conjugate of* $\mathfrak{M}(\epsilon_1^1)$.

PROOF. The condition is obviously sufficient. For the necessity, we need to show only that $\phi^{21}$ is complete by rows-conjugate of $\mathfrak{M}(\epsilon_1^1)$. Consider any $\xi^1$ modular as to $\epsilon_1^1$ such that $J^{11}\phi^{21}\xi^1=J^1\phi^{21}\xi^1=0^2$. Then $\xi^1$ belongs to $\mathfrak{M}(\epsilon_1^1\phi\epsilon^2)$. Hence

$$J^1\bar{\mu}^1\xi^1 = J^2(J^1\bar{\mu}^1\kappa^{12}, J^1\phi^{21}\xi^1) = 0$$

for every $\mu^1$ belonging to $\mathfrak{M}(\epsilon_0^1\kappa^*\epsilon^2)$. Now if $\mathfrak{M}(\epsilon_\kappa^2\cap\epsilon^2)$ is everywhere dense in $\mathfrak{M}(\epsilon_\kappa^2)$, then $\mathfrak{M}(\epsilon_0^1\kappa^*\epsilon^2)$ is everywhere dense in $\mathfrak{M}(\epsilon_0^1)$, or, $O^1\mathfrak{M}(\epsilon_0^1\kappa^*\epsilon^2)=O^1\mathfrak{M}(\epsilon_0^1)$. Since $\xi^1$ is in $\mathfrak{M}(\epsilon_0^1)$ and also in $O^1\mathfrak{M}(\epsilon_0^1)$, it follows that $\xi^1=0^1$.

THEOREM 8. *Assume that* $\kappa^{12}$ *is of type* $\mathfrak{M}(\epsilon^1)\overline{\mathfrak{M}}(\epsilon^2)$, *and* $\phi^{21}$ *is by rows-conjugate of* $\mathfrak{M}(\epsilon^1)$ *such that* $\kappa^{12}$, $\phi^{21}$ *are biorthogonal as to* $\epsilon^1\epsilon_\kappa^1\epsilon_{\phi*}^1\epsilon^2$. *Then*

(1) $\phi^{*12}$, $\kappa^{*21}$ *are contraceding as to* $\epsilon^1\epsilon_{\phi*}^2\epsilon^2$;
(2) $\mathfrak{M}(\epsilon^2\cap\epsilon_{\phi*}^2)\subset\mathfrak{M}(\epsilon^2\kappa\epsilon_\kappa^1)$;
(3) $\mu^1=J^2\kappa^{12}J^1\phi^{21}\mu^1$ *for every* $\mu^1$ *in* $\mathfrak{M}(\epsilon_{\phi*}^1\phi\epsilon^2)$;
(4) $\mathfrak{M}(\epsilon^2\cap\epsilon_{\phi*}^2)\subset[J^1\phi^{21}\mu^1\,|\,\mu^1$ *in* $\mathfrak{M}(\epsilon_\kappa^1\cap\epsilon_{\kappa*}^1)]$;
(5) $\mathfrak{M}(\epsilon_\kappa^1\phi\epsilon^2)\supset\mathfrak{M}(\epsilon_\kappa^1\cap\epsilon_{\kappa*}^1)$ *if and only if* $\mathfrak{M}(\epsilon^2\kappa\epsilon_\kappa^1)$ *contains* $[J^1\phi^{21}\mu^1\,|\,\mu^1$ *in* $\mathfrak{M}(\epsilon_\kappa^1\cap\epsilon_{\kappa*}^1)]$.

PROOF. Since $\kappa^{12}$ is by rows-conjugate of $\mathfrak{M}(\epsilon^2\cap\epsilon_\kappa^2)$, we make use of (4.2) in Theorem 4 and secure

$$J^2\kappa^{12}\mu^2 = J^1\epsilon_\kappa^1 J^{2\phi^*}\phi^{*12}\mu^2 = J^{2\phi^*}\phi^{*12}\mu^2$$

for every $\mu^2$ belonging to $\mathfrak{M}(\epsilon^2\cap\epsilon_{\phi*}^2)$. This proves (1). Part (2) is an immediate consequence of (1) just proved. For the demonstration of (3), we note that $\epsilon_\kappa^1$ is by columns of $\mathfrak{M}(\epsilon_\kappa^1\kappa^*\epsilon^2)$ and hence by Definition 2, we have

$$J^1\epsilon_\kappa^1\eta^1 = J^2(J^1\epsilon_\kappa^1\kappa^{12}, J^1\phi^{21}\eta^1) = J^2\kappa^{12}J^1\phi^{21}\eta^1$$

for every $\eta^1$ in $\mathfrak{M}(\epsilon_{\phi*}^1\phi\epsilon^2)$. Since $\mathfrak{M}(\epsilon_{\phi*}^1)$ is a subset of $\mathfrak{M}(\epsilon_\kappa^1)$, we have $J^1\epsilon_\kappa^1\eta^1=\eta^1$, and hence, part (3). To prove (4), consider any $\mu^2$ in

$\mathfrak{M}(\epsilon^2\cap\epsilon^2_{\phi^*})$. Let $\mu^1$ be a vector in $\mathfrak{M}(\epsilon^1_\kappa\cap\epsilon^1_{\kappa^*})$ for which $J^{2\phi^*}\phi^{*12}\bar{\mu}^2=\mu^1$. Then by (1)

$$J^1\phi^{21}\mu^1 = J^1\phi^{21}J^{2\phi^*}\phi^{*12}\mu^2 = J^{2\phi^*}\epsilon^2_{\phi^*}\mu^2 = \mu^2,$$

whence (4) follows. In part (5), if $\mathfrak{M}(\epsilon^1_\kappa\phi\epsilon^2)\supset\mathfrak{M}(\epsilon^1_\kappa\cap\epsilon^1_{\kappa^*})$, then

$$[J^1\phi^{21}\mu^1 \mid \mu^1 \text{ in } \mathfrak{M}(\epsilon^1_\kappa\phi\epsilon^2)] \supset [J^1\phi^{21}\mu^1 \mid \mu^1 \text{ in } \mathfrak{M}(\epsilon^1_\kappa\cap\epsilon^1_{\kappa^*})].$$

By (C), the left-hand side is identical with $\mathfrak{M}(\epsilon^2\cap\epsilon^2_{\phi^*})$, which by part (2) is a subset of $\mathfrak{M}(\epsilon^2\kappa\epsilon^1_\kappa)$. The converse is obvious.

THEOREM 9. *Let $\epsilon^1_0$ be idempotent as to $\epsilon^1$, and $\kappa^{12}$ be of type $\mathfrak{M}(\epsilon^1_0)\overline{\mathfrak{M}}(\epsilon^2)$. Suppose that $\phi^{21}$ is by rows-conjugate of $\mathfrak{M}(\epsilon^1_0)$ and $\kappa^{12}$, $\phi^{21}$ are biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^2$. Then we have the following conclusions:*

(1) *$\phi^{21}$ is complete by rows-conjugate of $\mathfrak{M}(\epsilon^1_0)$;*

(2) *$\mu^1=J^2\kappa^{12}J^1\phi^{21}\mu^1$ for every $\mu^1$ in $\mathfrak{M}(\epsilon^1_0\phi\epsilon^2)$;*

(3) *$\mathfrak{M}(\epsilon^1_0\phi\epsilon^2)=[J^2\kappa^{12}\mu^2 \mid \mu^2$ in $\mathfrak{M}(\epsilon^2\cap\epsilon^2_{\phi^*})]\subset\mathfrak{M}(\epsilon^1_0\cap\epsilon^1_{\kappa^*})$;*

(4) *$\mathfrak{M}(\epsilon^2\cap\epsilon^2_{\phi^*})\subset\mathfrak{M}(\epsilon^2\kappa\epsilon^1_0)$;*

(5) *if $\phi^{21}$ is by columns of $\mathfrak{M}(\epsilon^2)$, then $\epsilon^1_0=J^2\kappa^{12}\phi^{21}$ and $\kappa^{12}$ is complete by columns of $\mathfrak{M}(\epsilon^1_0)$;*

(6) *$\mathfrak{M}(\epsilon^1_0\phi\epsilon^2)\supset\mathfrak{M}(\epsilon^1_0\cap\epsilon^1_{\kappa^*})$ if and only if every vector in $\mathfrak{M}(\epsilon^1_0\cap\epsilon^1_{\kappa^*})$ is expressible in the form $J^2\kappa^{12}\mu^2$, where $\mu^2$ is a vector in $\mathfrak{M}(\epsilon^2\cap\epsilon^2_{\phi^*})$.*

PROOF. Part (1) follows from Theorem 7, for $\mathfrak{M}(\epsilon^2\cap\epsilon^2_\kappa)$ is everywhere dense in $\mathfrak{M}(\epsilon^2_\kappa)$ when $\kappa^{12}$ is of type $\mathfrak{M}(\epsilon^1_0)\overline{\mathfrak{M}}(\epsilon^2)$. Part (2) is proved in the same way as part (3) of Theorem 8 with the replacement of $\epsilon^1_\kappa$ by $\epsilon^1_0$. By Theorem (C), part (3) is a consequence of (2). Part (4) follows from (3). For part (5), we note that $\epsilon^1_0$ is by columns of $\mathfrak{M}(\epsilon^1_0\phi\epsilon^2)$; hence the first conclusion follows from (2) whereas the second follows from (1) and the fact that $\phi^{*12}$, $\kappa^{*21}$ are biorthogonal as to $\epsilon^1\epsilon^1_0\epsilon^2$. Part (6) is a consequence of (3).

THEOREM 10. *Suppose that $\kappa^{12}$ is of type $\mathfrak{M}(\epsilon^1)\overline{\mathfrak{M}}(\epsilon^2)$, $\phi^{21}$ is by rows-conjugate of $\mathfrak{M}(\epsilon^1_\kappa)$, and $\kappa^{12}$, $\phi^{21}$ are biorthogonal as to $\epsilon^1\epsilon^1_\kappa\epsilon^2$. Then the following assertions are equivalent:*

(1) *$\mathfrak{M}(\epsilon^2\cap\epsilon^2_{\phi^*})\supset\mathfrak{M}(\epsilon^2\kappa\epsilon^1_\kappa)$;*

(2) *$\kappa^{12}$ is complete by rows-conjugate of $\mathfrak{M}(\epsilon^2)$ and $\mathfrak{M}(\epsilon^1_\kappa\phi\epsilon^2)$ contains $\mathfrak{M}(\epsilon^1_\kappa\cap\epsilon^1_{\kappa^*})$;*

(3) *$\kappa^{12}$ is complete by rows-conjugate of $\mathfrak{M}(\epsilon^2)$ and $\mathfrak{M}(\epsilon^2\kappa\epsilon^1_\kappa)$ contains $[J^1\phi^{21}\mu^1 \mid \mu^1$ in $\mathfrak{M}(\epsilon^1_\kappa\cap\epsilon^1_{\kappa^*})]$;*

(4) *$\kappa^{*21}$, $\phi^{*12}$ are contraceding as to $\epsilon^2\epsilon^1_{\kappa^*}\epsilon^1$, and $\kappa^{12}$ is complete by rows-conjugate of $\mathfrak{M}(\epsilon^2)$;*

(5) $[J^1\phi^{21}\mu^1 | \mu^1 \text{ in } \mathfrak{M}(\epsilon^1_\kappa \cap \epsilon^1_{\kappa^*})] \supset \mathfrak{M}(\epsilon^2 \kappa \epsilon^1_\kappa)$.
*If one of the preceding five conditions is valid, then* $\epsilon^2 = \epsilon^2_{\kappa^*} = J^1\phi^{21}\kappa^{12}$.

Proof. The equivalence of the second and third conditions follows from (5) of Theorem 8. Conditions (1), (2), (4), and (5) are equivalent because of Theorem 2, where $\kappa$, $\phi$, $\epsilon^1$, $\epsilon^2$, $\epsilon^1_\phi$ are replaced by $\kappa^*$, $\phi^*$, $\epsilon^2$, $\epsilon^1_\kappa$, $\epsilon^2_{\phi^*}$ respectively. The relation $\epsilon^2 = J^1\phi^{21}\kappa^{12}$ follows from (4), since $\kappa^{12}$ is by columns of $\mathfrak{M}(\epsilon^1_\kappa \cap \epsilon^1_{\kappa^*})$.

The University of Chicago

# ON SPHERICAL CYCLES[1]

SAMUEL EILENBERG

Given a metric separable space $\Upsilon$, we consider the homology group $B^n(\Upsilon)$ obtained using $n$-dimensional singular cycles in $\Upsilon$ with integer coefficients. Every continuous mapping $f \in \Upsilon^{S^n}$ of the oriented $n$-dimensional sphere $S^n$ into $Y$ defines uniquely an element $h(f)$ of $B^n(\Upsilon)$. Clearly if $f_0, f_1 \in \Upsilon^{S^n}$ are two homotopic mappings, then $h(f_0) = h(f_1)$.

The homology classes $h(f)$ will be called *spherical homology classes*. A cycle will be called *spherical* if its homology class is spherical.[2]

THEOREM 1. *If $\Upsilon$ is arcwise connected, the spherical homology classes form a subgroup of $B^n(\Upsilon)$.*

Let $p \in S^n$, $q \in \Upsilon$, and let $S^n = S^n_+ + S^n_-$ be a decomposition of $S^n$ into two hemispheres such that $p \in S^n_+ \cdot S^n_-$. Consider $f_0, f_1 \in \Upsilon^{S^n}$. It is well known that, replacing if necessary $f_0$ and $f_1$ by homotopic mappings, we may assume that $f_0(S^n_+) = q$ and that $f_1(S^n_-) = q$. Defining $f = f_0$ on $S^n_-$ and $f = f_1$ on $S^n_+$ we clearly have

$$f \in \Upsilon^{S^n}, \qquad h(f) = h(f_0) + h(f_1)..$$

The homology class $h(f_0) + h(f_1)$ is therefore spherical.

Let $M^r$ be an $r$-dimensional (finite or infinite) manifold[3] and $P^{r-n-1}$ $(n>0)$ an at most $(r-n-1)$-dimensional subpolyhedron of $M^r$.

THEOREM 2. *Every $n$-dimensional cycle $\gamma^n$ in $M^r - P^{r-n-1}$ such that $\gamma^n \sim 0$ in $M^r$ is spherical (with respect to $M^r - P^{r-n-1}$).*

Let $a^{r-n-1}$ be an $(r-n-1)$-dimensional simplex of $M^r$ and $b^{n+1}$ the $(n+1)$-cell dual to it. The boundary $\partial b^{n+1}$ is contained in $M^r - P^{r-n-1}$ and is a spherical cycle. Since $M^r - P^{r-n-1}$ is connected, the spherical homology classes of $B^n(M^r - P^{r-n-1})$ form a group. It follows that each cycle of the form

$$(*) \qquad \partial\left(\sum_i \alpha_i b_i^{n+1}\right)$$

is a spherical cycle with respect to $M^r - P^{r-n-1}$. The cycle $\gamma^n$ is homologous in $M^r - P^{r-n-1}$ to a cycle of the form (*). Therefore $\gamma^n$ is spherical.

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[1] Presented to the Society, April 13, 1940.

[2] Spherical cycles were considered by W. Hurewicz, Proceedings, Akademie van Wetenschappen te Amsterdam, vol. 38 (1935), pp. 521–528.

[3] See K. Reidemeister, *Topologie der Polyëder*, Leipzig, 1938, p. 151.

THEOREM 3. *Let $\gamma^n$ be a spherical cycle in $M^r$ and let $r > 2n$. Then there is a simplicial homeomorphism*[4] $g \in M^{rS^n}$ *such that* $\gamma^n \in h(g)$.

This is an immediate consequence of Theorem 5 below. Using Theorem 2 we obtain the following:

THEOREM 4. *Given an n-cycle* $\gamma^n \subset M^r - P^{r-n-1}$ $(r > 2n)$ *such that* $\gamma^n \sim 0$ *in* $M^r$, *there is a cycle* $\gamma_1^n \subset M^r - P^{r-n-1}$ *which is a simplicial and homeomorphic image of* $S^n$ *such that* $\gamma^n \sim \gamma_1^n$ *in* $M^r - P^{r-n-1}$.

THEOREM 5. *Let $Q^n$ be a finite n-dimensional polyhedron and let $r > 2n$. Every continuous mapping $f \in M^{rQ^n}$ can be approached by simplicial homeomorphisms $g \in M^{rQ^n}$.*

We may admit that the mapping $f$ is simplicial. Let $a_1, a_2, \cdots, a_k$ be the vertices of the complex $f(Q^n)$ and let $\sigma_1, \sigma_2, \cdots, \sigma_k$ be the corresponding stars.[5] Let us choose $\delta > 0$ so that $x \in f(Q^n)$ will imply $\rho(x, M^r - \sigma_i) > \delta$ for some $i = 1, 2, \cdots, k$.

Let $\delta > 2\epsilon > 0$. We are going to define a sequence $f = f_0, f_1, \cdots, f_k$ of simplicial maps of $Q^n$ into $M^r$ such that

$$(1) \qquad | f_i(x) - f_{i-1}(x) | < \frac{\epsilon}{k},$$

$$(2) \qquad f_i(x_1) = f_i(x_2) \quad \text{implies} \quad f_{i-1}(x_1) = f_{i-1}(x_2),$$

$$(3) \qquad x_1 \neq x_2 \text{ and } f_i(x_1) = f_i(x_2) = y \text{ imply } \rho(y, M^r - \sigma_i) < \delta \frac{2k - i}{2k}.$$

Suppose that $f_0, f_1, \cdots, f_{i-1}$ are already defined. Let

$$f_i(x) = f_{i-1}(x) \quad \text{if} \quad f_{i-1}(x) \in M^n - \sigma_i,$$

and let $Q_i^n = f_{i-1}^{-1}(\sigma_i)$.

$M^r$ being a manifold, $\sigma_i$ is simplicially homeomorphic with a convex $r$-cell in a euclidean $r$-dimensional space. Since $r > 2n$, then using the very well known[6] procedure of making vertices linearly independent we find a simplicial map $f_i(Q_i^n) \subset \sigma_i$ such that $f_i(x) = f_{i-1}(x)$ if $f_{i-1}(x)$ is on the boundary of $\sigma_i$ and satisfying (1)–(3).

Taking $g = f_k$ it follows from (1) that

$$| g(x) - f(x) | < \epsilon.$$

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[4] With respect to certain simplicial subdivisions of $M^r$ and $S^n$.

[5] $\sigma_i$ consists of all closed simplices of $M^r$ containing $a_i$.

[6] See for instance W. Hurewicz, Sitzungsberichte der Preussischen Akademie der Wissenschaften, vol. 24 (1933), p. 758.

Now if $x_1 \neq x_2$ and $g(x_1) = g(x_2)$, then according to (2) we have

$$f_i(x_1) = f_i(x_2) = y_i \qquad \text{for } i = 0, 1, \cdots, k.$$

Owing to the definition of $\delta$ there is an index $j = 0, 1, \cdots, k$ such that

$$\rho(y_0, M^n - \sigma_j) > \delta.$$

Combining this with (1) we see that

$$\rho(y_i, M^n - \sigma_j) > \delta - \frac{\epsilon i}{k} > \delta - \frac{\delta i}{2k} = \delta \, \frac{2k - i}{2k} \cdot$$

Taking $i = j$ we obtain a contradiction with (3).

University of Michigan

# A NEW CONCEPT IN DISTANCE GEOMETRY WITH APPLICATIONS TO SPHERICAL SUBSETS[1]

LEONARD M. BLUMENTHAL

1. **Introduction.** A semimetric space is a set of abstract elements (*points*) with a structure imposed by attaching to each pair $p$, $q$ of elements a non-negative real number $pq$ (*distance*) such that $pq=qp$ and $pq=0$ if and only if $p=q$. Some of the more important of these spaces (for example, euclidean, hyperbolic, and spherical spaces of $n$-dimensions) have the exceedingly useful property that the existence of a function mapping an arbitrary semimetric space congruently (that is, *with preservation of distances*) upon the space follows from the congruent embedding in the space of each set of $k$ points (not necessarily pairwise distinct) of the semimetric space.[2] Such spaces are said to have *congruence order* $k$ with respect to all semimetric spaces—a term introduced by Menger, who proved that the $n$-dimensional euclidean space $E_n$ has this property with $n+3$ as the smallest value of $k$.[3]

The importance of the possession by a given space of a congruence order is clear, since it evidently reduces the characterization problem of determining necessary and sufficient conditions for the congruent embedding of *any* semimetric space in the given one to the finite problem of finding under what conditions a semimetric set of $k$ points is congruently contained in the space. The desirability of having $k$ as small as possible is obvious.

After showing that $n+3$ is the minimum congruence order of the $n$-dimensional euclidean space $E_n$, Menger proved the more difficult theorem that if a semimetric space $S$ is not congruently contained in the $E_n$, but each set of $n+2$ of its points is congruent with $n+2$ points of the $E_n$, then $S$ consists of exactly $n+3$ points. Hence the congruent embedding in the $E_n$ of each set of $n+2$ points of a semimetric space $S$ insures that $S$ is congruently contained in $E_n$ *whenever $S$ has more than $n+3$ points.* This property was labelled "quasi congruence order

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[1] Presented to the Society, April 12, 1940.

[2] See Chapter III of the author's *Distance Geometries*, University of Missouri Studies, vol. 13, 1938.

[3] The proof is given in Menger, *Untersuchungen über allgemeine Metrik*, Mathematische Annalen, vol. 100 (1928), pp. 75–163. The term "congruence order $n+3$" describing this property was introduced in *Bemerkungen zur zweiten Untersuchung über allgemeine Metrik*, Proceedings of the Royal Academy, Amsterdam, vol. 30 (1927), pp. 1–5.

CUTTA

$n+2$."[4] It is enjoyed by the $n$-dimensional hyperbolic space also, but not by the $n$-dimensional elliptic nor the $n$-dimensional spherical spaces. The notions of congruence and quasi congruence orders have been the principal ones employed in the metric characterizations of spaces.

Recent investigations of characterization problems for spherical *subsets* (that is, subsets of the surface of the sphere with geodesic (shorter arc) metric) have revealed the inadequacy of these two notions to describe the results obtained.[5] It is the purpose of this paper to present a general concept which systematizes all of these results, contains the older notions as two links of a chain of notions which occur in a special case, and which orders between these two links infinitely many other links. This new concept gives rise to a whole field of new problems in distance geometry and, besides, induces a re-examination of older, solved problems with a view towards a "better" solution. The meaning of the latter part of the above remark will become clear later.

2. **$\sigma$-relative congruence indices.** Let $S$ and $\sigma$ be two given semimetric spaces. Then $S$ has *$\sigma$-relative congruence indices* $(n, k)$ *with respect to a given class* $(\Sigma)$ *of semimetric spaces* provided any space $\Sigma$ of $(\Sigma)$ with more than $n+k$ pairwise distinct points is congruently contained in $S$ whenever each $n$ of its points (not necessarily pairwise distinct) is congruently contained in $\sigma$. It seems appropriate to call $\sigma$ a catalytic space of index $(n, k)$ of $S$ with respect to the class $(\Sigma)$, since it facilitates the desired reaction of congruent embedding of $\Sigma$ in $S$ but plays in general no part in the result itself, for $\Sigma$ is not necessarily embeddable in $\sigma$. If $\sigma=S$ the indices $(n, k)$ are called *congruence indices* of $S$ with respect to $(\Sigma)$. It is this case with which the present paper is principally concerned.

REMARK 1. *A semimetric space $S$ has congruence indices* $(n, 0)$ *with respect to* $(\Sigma)$ *if and only if $S$ has congruence order $n$ with respect to* $(\Sigma)$.

For suppose $S$ has congruence indices $(n, 0)$ with respect to $(\Sigma)$ and let $\Sigma$ be any space of $(\Sigma)$ with each $n$ of its points congruently contained in $S$. If $\Sigma$ contains more than $n$ pairwise distinct points, then $\Sigma$ is congruently contained in $S$ by hypothesis and definition. On the other hand, if $\Sigma$ consists of $n$ or fewer pairwise distinct points, the congruence of $\Sigma$ with a subset of $S$ is a trivial consequence of the congruent embedding in $S$ of each $n$ points of $\Sigma$. Hence $S$ has congruence order $n$ with respect to $(\Sigma)$.

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[4] The proof appears in *Untersuchungen über allgemeine Metrik*, loc. cit., but the term "quasi congruence order" is first defined in Menger, *New foundations of euclidean geometry*, American Journal of Mathematics, vol. 53 (1931), pp. 721–745.

[5] C. V. Robinson, *Contributions to Distance Geometry*, University of Missouri thesis.

Finally, if $S$ has congruence order $n$ with respect to $(\Sigma)$, then congruence of any space $\Sigma$ of $(\Sigma)$ follows from the congruent embedding in $S$ of each $n$ points of $\Sigma$, and the situation described by $S$ having congruence indices $(n, 0)$ with respect to $(\Sigma)$ evidently exists.

REMARK 2. *A semimetric space $S$ has congruence indices $(n, 1)$ with respect to $(\Sigma)$ if and only if $S$ has the quasi congruence order $n$ with respect to $(\Sigma)$.*

The proof is similar to that of Remark 1.

Thus the notions of congruence and quasi congruence orders correspond to particular $\sigma$-relative congruence indices in the special case $\sigma = S$.

Let $(n, k)\rightarrow(n', k')$ symbolize the statement, "if $S$ has congruence indices $(n, k)$ with respect to $(\Sigma)$, then $S$ has congruence indices $(n', k')$ with respect to $(\Sigma)$."

REMARK 3. *If $n \leqq n'$ and $n+k \leqq n'+k'$, then*

$$(n, k)\rightarrow(n', k').$$

The proof follows immediately from the definition of congruence indices.

Some special cases of Remark 3 are of interest. Thus, $(n, 1)\rightarrow(n+1, 0)$; that is, quasi congruence order $n$ implies congruence order $n+1$. Also $(n, 1)\rightarrow(n, 2)\rightarrow(n+1, 1)$. This relation, which interpolates a property between quasi congruence order $n$ and quasi congruence order $n+1$ illustrates one of the advantages of the concept of congruence indices. Previous to its introduction, if a space $S$ were found not to have quasi congruence order $n$, the next step was to examine it for a higher quasi congruence order. Consideration was thus shifted from the congruent embedding in $S$ of sets of $n$ points to the more difficult problems concerning embedding in $S$ of sets of $n+1$ or more points. The possession by $S$ of congruence indices $(n, 2)$ implies quasi congruence order $n+1$, but, in addition, characterizes $S$ in terms of the congruent embedding in $S$ of each $n$ points of a semimetric space rather than each $n+1$ points.

In the light of the foregoing observations, any one of the congruence symbols $(n, k)$, $(k=0, 1, 2, \cdots)$, is to be preferred to the symbol $(n+1, 0)$, though only for $k=0, 1$ is the implication $(n, k)\rightarrow(n+1, 0)$ valid. Similarly, the symbols $(n, 1)$, $(n, 2)$, $(n, 3)$, $\cdots$ are each preferable to $(n+1, 1)$, though only the first two imply the latter. Thus, there is interpolated between quasi congruence orders $n$ and $n+1$ an infinity of symbols, each of which is to be preferred to quasi congruence order $n+1$. It is clear that the indices $(n, k)$ characterize a space only with respect to all semimetric spaces *containing more than* $n+k$ *points*, but this slight restriction of the class of comparison spaces is more than offset by the simplicity gained from keeping the number of points to be embedded as small as possible. In addition to the possibility of bettering old results (e.g., the possession of indices $(n, 2)$ where previously only $(n+1, 1)$ and not $(n, 1)$ had been proved), new problems arise (e.g., the possibility of indices $(n, 3)$ when neither $(n, 1)$ nor $(n+1, 1)$ exists for the space).

The above considerations suggest the following preferential (lexicographical) ordering of congruence indices. If $(n, k)$ and $(n', k')$ are two congruence symbols of $S$, that one is preferred for which (1) the first index is the smaller or (2) if $n=n'$, then the preferred symbol has the smaller second index. That symbol $(n, k)$ of $S$ is *best* provided no symbol of $S$ is preferred to it. It is emphasized that this ordering is *not* that of logical implication. With respect to the latter criterion congruence symbols form only a partially ordered set (for example, the symbols $(n, 3)$ and $(n+1, 1)$ are not logically comparable).

Since $(n, k) \rightarrow (n+k, 0)$, $S$ has no congruence indices with respect to $(\Sigma)$ if $S$ has no congruence order with respect to $(\Sigma)$. Conversely, if $S$ has congruence order $m$ with respect to $(\Sigma)$, then by Remark 1, $S$ has congruence indices with respect to $(\Sigma)$. Thus, $S$ possesses congruence indices if and only if $S$ has a congruence order.

## 3. Lemmas concerning pseudo-spherical sets.

The $n$-dimensional spherical space $S_{n,r}$ of radius $r$ is the "surface" of a sphere of radius $r$ in a euclidean space of $n+1$ dimensions, with geodesic (shorter arc) metric. The $S_{n,r}$ has indices $(n+3, 0)$ but not $(n+2, 1)$, and there exist semimetric spaces of arbitrary power exceeding $n+2$ which are not contained congruently in $S_{n,r}$ though every set of $n+2$ points is congruently embeddable in $S_{n,r}$. Such spaces are called pseudo-$S_{n,r}$ sets. From the characterization of these spaces (given elsewhere) we take the following results:[6]

PROPERTY 1. *Let $P$ be a pseudo-$S_{n,r}$ set of more than $n+3$ pairwise distinct points, no two of which have distance $\pi r$. If $p$, $q$ are any two distinct points of $P$, then*

$$\cos (pq/r) = \pm 1/(n+1).$$

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[6] L. M. Blumenthal and G. R. Thurman, *The characterization of pseudo-spherical sets*, American Journal of Mathematics, vol. 62 (1940), pp. 835–854.

PROPERTY 2. *Let $P$ be a pseudo-$S_{n,r}$ set of $m$ pairwise distinct points*

$$p_1, p_2, \cdots, p_m, \qquad m > n + 3$$

*no two of which have distance $\pi r$, and consider the determinant*

$$\Delta_m(p_1, p_2, \cdots, p_m) = \left| \cos (p_ip_j/r) \right|, \qquad i, j = 1, 2, \cdots, m,$$

*of the set. Apart from the elements of the first and jth rows, ($j=2, 3, \cdots, m$), either each element in the first column of $\Delta_m$ is equal to the corresponding element in the jth column, or each element in the first column is the negative of the corresponding element in the jth column. If the first column of $\Delta_m$ contains at least one positive element different from the element* 1, *then apart from the elements in the first and second rows, each element in the second column is the negative of the corresponding element in the first column. Due to the symmetry of $\Delta_m$ and the first part of this property, the sign of every element of $\Delta_m$ is then determined.*[7]

PROPERTY 3. *If $P$ is a pseudo-$S_{n,r}$ set of more than $n+3$ pairwise distinct points, no two of which have distance $\pi r$, then every set of $m$ points of $P$, ($m \geqq n+3$), is a pseudo-$S_{n,r}$ set.*

LEMMA 1. *Apart from the labelling, there is a unique pseudo-$S_{n,r}$ set of $2n+2$ pairwise distinct points, ($n>1$), free from diametral point-pairs (that is, pairs of points with distance $\pi r$), which does not contain an equilateral set of $n+2$ points (that is, a set of $n+2$ points with all distances equal).*[8]

PROOF. Let $P$ be any pseudo-$S_{n,r}$ set of $2n+2$ points, no two of which have distance $\pi r$. Since $n>1$, $2n+2>n+3$, and the determinant $\Delta_{2n+2}$ of $P$ has Properties 1, 2 listed above. To prove the lemma it suffices to show that apart from a (symmetric) shifting of rows and columns, there is a single distribution of the signs in $\Delta_{2n+2}$ which will not yield a principal minor of order $n+2$ with all elements outside the principal diagonal negative. (Since each set of $n+2$ points of $P$ is congruently contained in $S_{n,r}$, an $(n+2)$-nd order principal minor of $\Delta_{2n+2}$ with all positive elements is impossible. Thus an equilateral $(n+2)$-tuple has each distance equal to $r \cdot \cos^{-1} -1/(n+1)$.)

Applying Property 2, a brief consideration of $\Delta_{2n+2}$ shows that the first column of the determinant must contain *exactly* $n+1$ positive elements (apart from 1) and an appropriate (symmetric) shifting of rows and columns gives (schematically):

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[7] See L. M. Blumenthal, *Metric methods in determinant theory*, American Journal of Mathematics, vol. 61 (1939), pp. 912–922.

[8] This lemma is not valid for $n=1$.

$$\begin{array}{r} n+2\left\{\vphantom{\begin{matrix}1\\+\\+\\ \cdot\\+\end{matrix}}\right. \\ n\left\{\vphantom{\begin{matrix}-\\-\\ \cdot\\-\end{matrix}}\right. \end{array}\left|\begin{matrix} 1 & + & + & \cdot & + & - & - & \cdot & - \\ + & 1 & - & \cdot & - & + & + & \cdot & + \\ + & - & 1 & \cdot & - & + & + & \cdot & + \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ + & - & - & \cdot & 1 & + & + & \cdot & + \\ - & + & + & \cdot & + & 1 & - & \cdot & - \\ - & + & + & \cdot & + & - & 1 & \cdot & - \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ - & + & + & \cdot & + & - & - & \cdot & 1 \end{matrix}\right|.$$

COROLLARY. *Every pseudo-*$S_{n,r}$ *set of* $2n+2$ *pairwise distinct points,* $(n>1)$, *no two with distance* $\pi r$, *contains an equilateral* $(n+1)$*-tuple with each distance equal to* $r\cdot\cos^{-1}-1/(n+1)$.

For the only such set without an equilateral $(n+2)$-tuple is, apart from labelling, the set exhibited in the determinant of Lemma 1. Deleting the 2nd, 3rd, $\cdots$, $(n+2)$-nd rows and columns of this determinant, one obtains the determinant of a set of the kind specified in the corollary. Another such set is given by deleting the 1st, $(n+3)$-rd, $\cdots$, $(2n+2)$-nd rows and columns.

LEMMA 2. *Each pseudo-*$S_{n,r}$ *set* $P$ *of more than* $2n+2$ *pairwise distinct points,* $(n>1)$, *no two of which have distance* $\pi r$, *contains an equilateral* $(n+2)$*-tuple.*[9]

PROOF. Since $n>1$, $2n+2>n+3$, and hence $P$ contains more than $n+3$ pairwise distinct points. By Property 3, each set $p_1, p_2, \cdots, p_{2n+3}$ of $2n+3$ points of $P$ is a pseudo-$S_{n,r}$ set. It suffices to show that these points contain an equilateral $(n+2)$-tuple. Again by Property 3, the points $p_1, p_2, \cdots, p_{2n+2}$ form a pseudo-$S_{n,r}$ set of more than $n+3$ points, and according to Lemma 1 either these points contain an equilateral $(n+2)$-tuple or they constitute a set of the kind exhibited in the proof of that lemma.

In the latter case, consider the determinant $\Delta_{2n+3}$ of the points $p_1, p_2, \cdots, p_{2n+3}$. The principal minor of order $2n+2$ in the upper left-hand corner of $\Delta_{2n+3}$ is the determinant of Lemma 1. If the element in the last row and first column of $\Delta_{2n+3}$ is positive, then the principal minor of order $n+2$ obtained by deleting the 1st, $(n+4)$-th, $\cdots$, $(2n+3)$-rd rows and columns of $\Delta_{2n+3}$ has all its elements outside the principal diagonal negative, and hence the corresponding $(n+2)$-tuple is equilateral. If, on the other hand, the element in the last row and first column of $\Delta_{2n+3}$ is negative, deleting the 2nd, 3rd, $\cdots$, $(n+2)$-nd rows and columns yields the determinant of an equilateral $(n+2)$-tuple.

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[9] We suppose $n>1$ since Lemma 1 is applied. It is easily shown that Lemma 2 is valid for $n=1$.

4. **Applications to spherical subsets.** The subsets of the $S_{n,r}$ to which the foregoing is applied in this paper are spherical caps $\kappa_{n,\rho}$ of spherical radius $\rho$, $0<\rho\leqq\pi r/2$ (that is, the locus of points of $S_{n,r}$ whose distance from a fixed point of $S_{n,r}$ is less than or equal to $\rho$).

THEOREM 1. *Let* $\sigma$ *denote the spherical cap* (*n-dimensional hemisphere*) $\kappa_{n,\pi r/2}$, $n>1$, *with base circle removed. Then* $S_{n,r}$ *has* $\sigma$-*relative congruence indices* $(n+2, n)$ *with respect to all semimetric spaces* $(\Sigma)$.[10]

PROOF. We must show that each semimetric space $\Sigma$ containing more than $2n+2$ pairwise distinct points and having each $n+2$ of them congruent with $n+2$ points of $\sigma$ is congruently contained in $S_{n,r}$. Suppose this is not the case. Then $\Sigma$ is a pseudo-$S_{n,r}$ set of more than $2n+2$ points, no two of which have distance $\pi r$ (since each $n+2$ points of $\Sigma$ are congruently contained in $\sigma$) and hence, by Lemma 2, $\Sigma$ contains an equilateral set of $n+2$ points with each of the $\frac{1}{2}(n+1)(n+2)$ mutual distances equal to $r\cdot\cos^{-1}-1/(n+1)$. But this is not possible, for such an $(n+2)$-tuple is clearly not contained in $\sigma$.[11] This contradiction yields the theorem. Obvious examples show that $\Sigma$ is not necessarily contained in $\sigma$.

Thus the open $n$-dimensional hemisphere is a catalytic set of index $(n+2, n)$ of the containing sphere.

THEOREM 2. *Let* $\kappa_{n,\rho}$, $n>1$, *be an n-dimensional spherical cap of radius* $\rho$, $(\rho<\pi r/2)$. *Then* $\kappa_{n,\rho}$ *has congruence indices* $(n+2, n)$ *with respect to all semimetric spaces* $(\Sigma)$.

PROOF. Let $\Sigma$ be any semimetric space containing more than $2n+2$ points, each $n+2$ of which are congruently contained in $\kappa_{n,\rho}$. Then it follows from Theorem 1 that $\Sigma$ is congruently contained in $S_{n,r}$. But it has been shown that *with respect to subsets of* $S_{n,r}$, the cap $\kappa_{n,\rho}$, $(\rho < \pi r/2)$, has congruence indices $(n+2, 0)$.[12] Hence $\Sigma$ is congruently contained in $\kappa_{n,\rho}$, and the theorem is proved.

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[10] Since any arc $\kappa_{1,\rho}$, $\rho\leqq\pi r/2$, of a circle $S_{1,r}$ is evidently congruent with a straight line segment (which has indices (3, 1) with respect to $(\Sigma)$), Theorems 1, 2 of this section hold for $n=1$.

[11] In the contrary case there exist $n+2$ positive numbers $b_1, b_2, \cdots, b_{n+2}$ such that the $(n+3)$rd order determinant obtained by bordering the determinant of the equilateral set by these numbers (with 1 in the intersection of the bordering row and column) is zero. This yields quite easily $\sum_{i=1}^{n+2}b_i=0$, which is impossible.

REMARK 1. Since $(n+2, n) \rightarrow (2n+1, 1)$, the cap $\kappa_{n,\rho}$, $(\rho < \pi r/2)$, has quasi congruence order $2n+1$ with respect to semimetric spaces. As $n$ increases, however, the indices on the left-hand side of the above implication tend to place conditions on only *half* the number of points that must be isometrically embedded if the characterization is given in terms of the indices on the right-hand side of the implication.

THEOREM 3. *The cap* $\kappa_{n,\rho}$, $(\rho < r \cdot \cos^{-1} 1/(n+1))$, $n > 1$, *has congruence indices* $(n+2, n-1)$ *with respect to all semimetric spaces* $(\Sigma)$.[13]

PROOF. If $\Sigma$ is any semimetric space of more than $2n+1$ pairwise distinct points, each $n+2$ of which are embeddable in $\kappa_{n,\rho}$, $(\rho < r \cdot \cos^{-1} 1/(n+1))$, then $\Sigma$ is congruently contained in $S_{n,r}$, for an assumption to the contrary implies that $\Sigma$ is a pseudo-$S_{n,r}$ set of at least $2n+2$ pairwise distinct points (no two with a distance $\pi r$). Then by the corollary to Lemma 1, $\Sigma$ contains an equilateral $(n+1)$-tuple with each distance equal to $r \cdot \cos^{-1} - 1/(n+1)$. Such an $(n+1)$-tuple is not embeddable in any cap of radius $\rho < r \cdot \cos^{-1} 1/(n+1)$. Since, now, $\Sigma$ is congruently contained in $S_{n,r}$ and each cap $\kappa_{n,\rho}$ of radius $\rho < \pi r/2$ has congruence indices $(n+2, 0)$ with respect to subsets of $S_{n,r}$, it follows that $\Sigma$ is congruently contained in $\kappa_{n,\rho}$, $(\rho < r \cdot \cos^{-1} 1/(n+1))$.

REMARK 1. For $n=2$, the cap $\kappa_{2,\rho}$, $(\rho < r \cdot \cos^{-1} 1/3)$, has indices $(4,1)$; that is, quasi congruence order 4 with respect to all semimetric spaces. Thus spherical caps with radii less than $r \cdot \cos^{-1} 1/3$ behave like the *euclidean plane* in this important respect. The indices $(4,1)$ cannot be improved no matter how small $\rho$ is taken.

REMARK 2. For $n=3$, the indices $(5,2)$ of the cap $\kappa_{3,\rho}$, $\rho < r \cdot \cos^{-1} 1/4$, given by Theorem 3, can be bettered. A brief consideration shows that every pseudo-$S_{3,r}$ set of *more than six points* (no two with distance $\pi r$) contains at least one equilateral set of four points with mutual distances equal to $r \cdot \cos^{-1} - 1/4$. A proof quite similar to that of Theorem 3 demonstrates that $\kappa_{3,\rho}$, $\rho < r \cdot \cos^{-1} 1/4$, has the indices $(5,1)$ and hence behaves like the euclidean three-dimensional spaces in this respect.

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[12] C. V. Robinson, loc. cit.

[13] This theorem is not valid for $n=1$.

REMARK 3. If $n=4$, the indices (6,3) given by Theorem 3 for the cap $\kappa_{4,\rho}$, ($\rho<r\cdot\cos^{-1} 1/5$), cannot be reduced to (6,1) by a simple argument analogous to that presented in the preceding remark.[14] This arises from the fact that there exists a pseudo-$S_{4,r}$ set of eight points (no two diametral) that does not contain an equilateral quintuple with mutual distances equal to $r\cdot\cos^{-1}-1/5$. Such a pseudo-$S_{4,r}$ set is represented schematically by

$$\begin{vmatrix} 1 & + & + & + & + & - & - & - \\ + & 1 & - & - & - & + & + & + \\ + & - & 1 & - & - & + & + & + \\ + & - & - & 1 & - & + & + & + \\ + & - & - & - & 1 & + & + & + \\ - & + & + & + & + & 1 & - & - \\ - & + & + & + & + & - & 1 & - \\ - & + & + & + & + & - & - & 1 \end{vmatrix}.$$

There is no fifth-order principal minor of this determinant with all elements outside the principal diagonal negative.

The problems of (1) finding the best indices for caps $\kappa_{n,\rho}$, $\rho<r\cdot\cos^{-1} 1/(n+1)$, $(n>1)$, and (2) ascertaining for each $n$ the $n$-dimensional cap of maximum spherical radius $\rho$ that, *like the n-dimensional euclidean space*, has congruence indices $(n+2, 1)$, will be investigated in a later paper.

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[14] It is easily shown that the indices (6,3) can be reduced to (6,2). Whether the cap $\kappa_{4,\rho}$, ($\rho<r\cos^{-1}\frac{1}{5}$), does behave like the four-dimensional euclidean space with regard to the symbol (6,1) is not known.

# THE FINITE DIFFERENCES OF POLYGENIC FUNCTIONS[1]

RUFUS P. ISAACS

By a polygenic function $f(z)$ we shall mean a function analytic in $x$ and $y$ separately, but whose real and imaginary parts are not required to satisfy the Cauchy-Riemann equations. At any point $z$ the derivative of such a function will depend on $\theta$, the angle at which the incremented point (used in defining the derivative) approaches $z$. The set of these numbers, for a fixed $z$, but for different $\theta$, form a circle. The equation for the derivative was given by Riemann in his classic dissertation (1851), but Kasner was the first to point out that it was a circle and make a detailed study of its geometry.[2] Hedrick called it the Kasner circle.

In this paper we shall be concerned with the finite difference quotients of polygenic functions. We shall show how a surface can be constructed for each point $z$ representing the difference quotient, and the derivative circle is a cross section of this surface.

**The conjugate form.** Regard

$$z = x + iy, \qquad \bar{z} = x - iy$$

as a linear substitution, and perform its inverse

$$x = \frac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z - \bar{z})$$

on $f(z)$. The resulting $F(z, \bar{z})$ will be called the conjugate form of $f$. Let $D_zF$ and $D_{\bar{z}}F$ be the partial derivatives[3] of $F(z, \bar{z})$, regarding $z$ and $\bar{z}$ as independent variables. That is,

$$\begin{aligned} D_zF &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \tfrac{1}{2}(D_x - D_y)f, \\ D_{\bar{z}}F &= \tfrac{1}{2}(D_x + D_y)f. \end{aligned} \tag{1}$$

**The operator $E^{\omega}$.** Let $\omega = \rho e^{i\theta}$. We define

$$E^{\omega}f(z) = f(z + \omega).$$

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[1] Presented to the Society, February 25, 1939, under the title *A geometric interpretation of the difference quotient of polygenic functions.*

[2] *General theory of polygenic or non-monogenic functions*; *The derivative congruence of circles*, Proceedings of the National Academy of Sciences, vol. 13 (1928), pp. 75–82. *A new theory of polygenic functions*, Science, vol. 66 (1927). Also, *The Geometry of Polygenic Functions*, Kasner and DeCicco—a book in the course of preparation.

[3] In Kasner's notation, these are $\mathfrak{M}(f)$ and $\mathfrak{P}(f)$.

More precisely, this means

$$E^{\omega}f(x, y) = f(x + \rho \cos \theta,\ y + \rho \sin \theta).$$

$E$ may also have a "partial" meaning:

$$E_x^a f(x, y) = f(x + a, y).$$

And we see the equivalence of the two operators:

$$E^{\omega} = E_x^{\rho \cos \theta} \cdot E_y^{\rho \sin \theta}. \tag{2}$$

Taylor's expansion may be written in the form

$$E_x^a f(x, y) = \left(1 + aD_x + \frac{1}{2!} a^2 D_x^2 + \cdots\right) f(x, y) = \exp (aD_x) f. \tag{3}$$

Now, by combining (2) and (3) we obtain the operational equivalence

$$E^{\omega} = \exp (\rho \cos \theta\ D_x + \rho \sin \theta\ D_y).$$

$(\cos \theta\ D_x + \sin \theta\ D_y) f$ is nothing more than the directional derivative of $f$, which we designate by $\mathcal{D}f$. Substituting from (1) for $D_x$ and $D_y$, we see that

$$\mathcal{D} = e^{i\theta} D_z + e^{-i\theta} D_{\bar{z}}, \tag{4}$$

$$E^{\omega} = \exp (\rho \mathcal{D}). \tag{5}$$

**The differential quotient.** We define

$${}_{\omega}\Delta f(z) = \frac{f(z + \omega) - f(z)}{\omega},$$

whence, operationally,

$${}_{\omega}\Delta = \frac{1}{\omega} [E^{\omega} - 1].$$

Expanding by means of (4) and (5)

$$\begin{aligned} {}_{\omega}\Delta f(z) &= \frac{1}{\rho e^{i\theta}} \left[\rho \mathcal{D} + \frac{\rho^2 \mathcal{D}^2}{2!} + \cdots\right] f(z) \\ &= \left[D_z + e^{-2i\theta} D_{\bar{z}} + \frac{\rho}{2!} (e^{i\theta} D_z^2 + 2e^{-i\theta} D_z D_{\bar{z}} \right. \\ &\qquad \left. + e^{-3i\theta} D_{\bar{z}}^2) + \cdots\right] f(z). \end{aligned} \tag{6}$$

To obtain the derivative we let $\rho\rightarrow 0$. The resulting expression

$$[D_z + e^{-2i\theta}D_{\bar{z}}]f(z),$$

which Kasner calls $\gamma$, is immediately seen to be the points of a circle, when $z$ is fixed and $\theta$ varies.

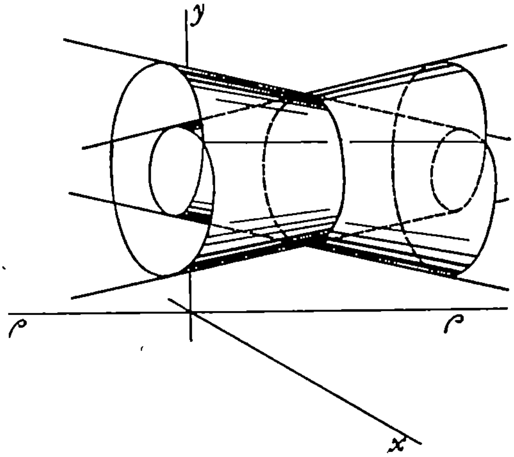

FIG. 1

For a geometric interpretation of (6), let us add a third coordinate $\rho$ to the $x$, $y$ plane. In this 3-space, (6) represents a surface for each fixed value of $z$. Such a surface we will call a Kasnercoid or K-coid of the function.

**Example.** $f(z)=x^2+y^2+i(x+y)$. Its conjugate form is $z\bar{z}+\frac{1}{2}(i+1)z+\frac{1}{2}(i-1)\bar{z}$. Using (6) we obtain for the difference quotient:

$$_{\omega}\Delta f = \bar{z} + \tfrac{1}{2}(i+1) + e^{-2i\theta}[z + \tfrac{1}{2}(i-1)] + \rho e^{-i\theta}.$$

The K-coid of a typical point is sketched in Figure 1.

This of course is a very simple case. In regard to the nature of K-coids in general, we state a theorem.

A curve will be called a doubler of a circle when it has the following properties:

1. In traversing the doubler once, we pass around the center of the circle twice.
2. No circle intersects the doubler in more than six points.
3. Every point on the circle is the midpoint of two points on the doubler.

(See Figure 2.)

THEOREM. *If $\rho^2$ be neglected, then sections of the K-coid sufficiently near the derivative circle are doublers of the derivative circle.*

Compare with the monogenic case:

*For a monogenic function, the derivative circle reduces to a point and the nearby sections are nearly circles.*

The purpose of this paper is to throw some light on the nature of derivatives. We may consider the K-coid compressed along the $\rho$-axis so as to be a two-sheeted (at least for small $\rho$) Riemann surface with the derivative circle acting as "branch circle." Let us consider the familiar expression:

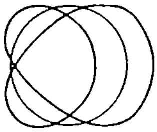

FIG. 2

$$\lim_{\omega\to 0}\frac{f(z+\omega)-f(z)}{\omega},$$

and suppose $z+\omega$, in approaching $z$, travels along some curve. To each point on the curve correspond two points on the K-coid which approach a common point on the derivative circle.

**Connection with the second derivative.** Let us write (6) in the form

$$[e^{-i\theta}\mathcal{D}+(\omega/2!)e^{-2i\theta}\mathcal{D}^2+\cdots]f(z).$$

Is the coefficient of $\omega/2!$ the second derivative of $f(z)$? This depends on what is meant by the second derivative, as there are several alternative methods of defining it.[4] According to Kasner's method,[5] the second derivative depends not only on the slope of the path of the incremented point but also on the curvature of the path. The expression is

$$[e^{-2i\theta}\mathcal{D}^2-2ie^{-3i\theta}\kappa D_z]f(z),$$

where $\kappa$ is the curvature.

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[4] See my unpublished Master's Essay, Columbia University.

[5] *The second derivative of a polygenic function*, Transactions of this Society, vol. 30, no. 4.

The corresponding expression for what I call the type A derivative—based on another, but equally logical definition—is merely the first term of the above expression.

COLUMBIA UNIVERSITY

# ON THE ASYMPTOTIC LINES OF A RULED SURFACE

GUIDO FUBINI

Many mathematicians have studied the surfaces every *asymptotic curve* of which belongs to a linear complex. I will here be content with the results given on pages 112–116 and 266–288 of a treatise[1] written by myself and Professor A. Cech. This treatise gives (p. 113) a very simple proof of the following theorem:

*If every non-rectilinear asymptotic curve of a ruled surface $S$ belongs to a linear complex, all these asymptotic curves are projective to each other.*

We will find all the ruled surfaces, the non-rectilinear asymptotic curves of which *are projective to each other*, and prove conversely that *every one of these asymptotic curves belongs to a linear complex*. If $c$, $c'$ are two of these asymptotic curves and if $A$ is an arbitrary point of $c$, we can find on $c'$ a point $A'$ such that the straight line $AA'$ is a straight generatrix of $S$. The projectivity, which, according to our hypothesis, transforms $c$ into $c'$, will carry $A$ into a point $A_1$ of $c'$. We will prove that *the two points $A'$ and $A_1$ are identical*; but this theorem is not obvious and therefore our demonstration cannot be very simple. The generalization to nonruled surfaces seems to be rather complicated: and we do not occupy ourselves here with such a generalization.

If the point $x=x(u, v)$ generates a ruled surface $S$, for which $u=$const. and $v=$const. are asymptotic curves, we can suppose (loc. cit., p. 182)

$$x = y + uz \tag{1}$$

in which $y$ and $z$ are functions of $v$. More clearly, if $x_1, x_2, x_3, x_4$ are homogeneous projective coordinates of a point of $S$, we can find eight functions $y_i$ and $z_i$ of $v$ such that

$$x_i = y_i(v) + uz_i(v), \qquad i = 1, 2, 3, 4. \tag{1$_{\text{bis}}$}$$

From the general theory of surfaces, it is known (loc. cit., p. 90) that we can find five functions $\theta$, $\beta$, $\gamma$, $p_{11}$, $p_{22}$ of $u$, $v$ such that

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[1] *Geometria Proiettiva Differenziale*, Bologna, Zanichelli.

$$(2)\qquad \begin{cases} x_{uu} = \theta_u x_u + \beta x_v + p_{11} x, \\ x_{vv} = \gamma x_u + \theta_v x_v + p_{22} x, \end{cases}$$

$$x_u = \frac{\partial x}{\partial u},\ \theta_u = \frac{\partial \theta}{\partial u},\ x_{uu} = \frac{\partial^2 x}{\partial u^2},\ \cdots;$$

$$x = x_i;\quad i = 1, 2, 3, 4;\quad x = y + uz.$$

Since now $x_{uu}=0$, the former of these equations becomes

$$0 = \theta_u x_u + \beta x_v + p_{11} x,$$

and therefore (since the points $x$, $x_u$, $x_v$ are independents):

$$\theta_u = \beta = p_{11} = 0.$$

Equation (2) becomes

$$y_{vv} + uz_{vv} = \gamma z + \theta_v(y_v + uz_v) + p_{22}(y + uz).$$

And, by differentiating two times with respect to $u$,

$$0 = \frac{\partial^2 p_{22}}{\partial u^2} y + \frac{\partial^2(\gamma + up_{22})}{\partial u^2} z.$$

Therefore

$$0 = \frac{\partial^2 p_{22}}{\partial u^2} = \frac{\partial^2}{\partial u^2}(\gamma + up_{22})$$

and we can write

$$(3)\qquad p_{22} = A + Bu, \qquad \gamma + up_{22} = C + Du,$$

$$(4)\qquad \gamma = (C + Du) - \mu(A + Bu),$$

in which $A$, $B$, $C$, $D$ are functions only of $v$. We can multiply the $x_i$ or, what is the same, the $y_i$ and the $z_i$ by a factor of proportionality (function *only of* $v$) such that $\theta=$const., and $\theta_v=0$ (or that the determinant of the $y_i$, $z_i$, $y_i' = \partial y_i/\partial v$, $z_i' = \partial z_i/\partial v$ becomes a constant). The second equation of (2) becomes

$$x_{vv} = \gamma z + px, \qquad p = p_{22} = A + Bu,$$

and, by differentiating with respect to $u$,

$$z_{vv} = Dz + By = (D - uB)z + Bx.$$

Therefore

$$\gamma z = x_{vv} - px, \qquad p = A + Bu = p_{22},$$

$$\frac{\partial^2}{\partial v^2}\,\frac{x_{vv} - px}{\gamma} - (D - uB)\,\frac{x_{vv} - px}{\gamma} - Bx = 0;$$

or

$$x'''' - 2\,\frac{\gamma'}{\gamma}\,x''' + \left[2\left(\frac{\gamma'}{\gamma}\right)^2 - \frac{\gamma''}{\gamma} - A - D\right] x'' + 2\,\frac{pq' - p'q}{\gamma}\,x'$$

$$(5) \qquad + \left[2\,\frac{\gamma'}{\gamma}\,\frac{p'q - pq'}{\gamma} + \frac{pq'' - p''q}{\gamma} + AD - BC\right] x = 0,$$

$$p = A + Bu, \qquad q = C + Du, \qquad \gamma = q - up,$$

$$p' = \frac{\partial p}{\partial v}, \qquad \gamma' = \frac{\partial \gamma}{\partial v}, \qquad x' = \frac{\partial x}{\partial v}, \ \cdots .$$

This is the differential equation which defines the asymptotic curves $u = \text{const.}$ If we put $x = X\gamma^{1/2}$, this equation becomes

$$X'''' + lX'' + mX' + nX = 0$$

in which

$$l = 2\,\frac{\gamma''}{\gamma} - \frac{5}{2}\left(\frac{\gamma'}{\gamma}\right)^2 - (A + D), \qquad n = -\frac{35}{16}\left(\frac{\gamma'}{\gamma}\right)^4 + \frac{r}{\gamma^3},$$

$$m = 2\,\frac{\gamma'''}{\gamma} - 7\,\frac{\gamma'\gamma''}{\gamma^2} + 5\left(\frac{\gamma'}{\gamma}\right)^3 + 2\,\frac{pq' - qp'}{\gamma} - (A + D)\,\frac{\gamma'}{\gamma}$$

($r$ is a polynomial of the variable $u$). The projective invariants (or covariants) of the curve defined by this equation are

$$Udv^3, \qquad V_1dv^2, \qquad Wdv^4.$$

We have put

$$U = l' - m = \frac{\epsilon}{\gamma}, \qquad [\epsilon = \{(A' - D')C - (A - D)C'\}$$

$$+ 2(CB' - BC')u + \{(A' - D')B - B'(A - D)\}u^2],$$

$$W = 20l'' - 50m' - 9l^2 + 100n = k\left(\frac{\gamma'}{\gamma}\right)^4 + \frac{R}{\gamma^3},$$

$$k = \text{const.} = 175 \neq 0;\ R \text{ a polynomial of } u,$$

and (if $U \neq 0$)

$$V_1 = 6[\log U]'' - \left(\frac{U'}{U}\right)^2 - \frac{36}{5}\,l, \qquad U' = \frac{\partial U}{\partial v}.$$

If $U=0$, the curve belongs to a linear complex; if $U\neq 0$,

$$\frac{W^3}{U^4}=\frac{(k\gamma'^4+R\gamma)^3}{\gamma^8\epsilon^4}$$

is a projective invariant. If all the asymptotic curves $u=$const. are projective to each other, this ratio must not be dependent upon $u$. And therefore the values of $u$, for which $\gamma=0$, must also satisfy the equation $\gamma'=0$. Therefore

$$\frac{C'}{D'}=\frac{D'-A'}{D-A}=\frac{B'}{B}$$

or

$$\gamma=C+(D-A)u-Bu^2=V(c+bu+au^2)$$

($V$ function of $v$; $a$, $b$, $c=$const.) Therefore $\epsilon=0$, $U=0$ and every asymptotic curve $u=$const. belongs to a linear complex. In this case

$$\frac{\gamma'}{\gamma}=\frac{V'}{V},\qquad \frac{\gamma''}{\gamma}=\frac{V''}{V},\qquad B'=\frac{V'}{V}B,$$

$$\frac{pq'-p'q}{\gamma}=\frac{p(q'-up')-p'(q-up)}{\gamma}=p\frac{\gamma'}{\gamma}-p'$$

$$=(A+Bu)\frac{V'}{V}-\left(A'+Bu\frac{V'}{V}\right)=\frac{AV'-A'V}{V}.$$

And analogously

$$\frac{pq''-p''q}{\gamma}=\frac{AV''-A''V}{V}.$$

Therefore no one of the coefficients of (5) is dependent upon $u$, and consequently we can suppose that the projectivity, which carries an asymptotic curve $u=$const. into another, carries every point of the former into that point of the latter which belongs to the same rectilinear generatrix of the surface (because the corresponding value of $v$ is not changed by this projectivity).

We have in this manner completely demonstrated the stated theorems.

THE INSTITUTE FOR ADVANCED STUDY

# A SEQUENCE OF LIMIT TESTS FOR THE CONVERGENCE OF SERIES[1]

MARGARET MARTIN

In this paper, we shall develop a sequence of limit tests for the convergence and divergence of infinite series of positive terms which is similar in form to the De Morgan and Bertrand sequence but involves the ratio of two successive values of the test ratio rather than the test ratio itself. The proof will be based on the following integral test by R. W. Brink:[2]

"THEOREM VI. *Given the sequence* $\{u_n\}$. *Let* $r_n = u_{n+1}/u_n$ *and* $R_n = r_{n+1}/r_n = u_{n+2}u_n/u_{n+1}^2$. *If* $\lim_{n\to\infty} r_n = 1$, *and if* $R(x)$ *is a function such that* $R(n) = R_n$, *and such that* $R(x) \geqq R(x')$ *when* $x' > x$, *a necessary and sufficient condition for the convergence of the series* $\sum_{n=0}^{\infty} u_n$ *is the convergence of the integral*

$$\int_0^\infty \exp\left\{-\int_0^x \int_x^\infty \log R(x)\,dx\,dx\right\} dx."$$

Since a finite number of terms does not affect convergence or divergence, the conditions of Theorem VI need hold only for $n$ greater than some fixed number $\nu$, in which case zero is to be replaced by $\nu$ as a lower limit of integration.

The foregoing theorem admits a generalization similar to that given by C. T. Rajagopal[3] in the case of another theorem of Brink's.[4] However, Brink's Theorem VI is sufficient for the purposes of the present paper.

LEMMA. *Let* $\{u_n\}$ *and* $\{u_n'\}$ *be sequences of positive terms with ratios* $r_n = u_{n+1}/u_n$, $R_n = r_{n+1}/r_n$, $r_n' = u_{n+1}'/u_n'$, *and* $R_n' = r_{n+1}'/r_n'$, *such that* $\lim_{n\to\infty} r_n = \lim_{n\to\infty} r_n' = 1$.

1. *If the series* $\sum_{n=\nu}^{\infty} u_n'$ *converges and if* $R_n \geqq R_n'$ *for all values of* $n \geqq \nu$, *then the series* $\sum_{n=\nu}^{\infty} u_n$ *converges.*

2. *If the series* $\sum_{n=\nu}^{\infty} u_n'$ *diverges and if* $R_n \leqq R_n'$ *for all values of* $n \geqq \nu$, *then the series* $\sum_{n=\nu}^{\infty} u_n$ *diverges.*

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[1] Presented to the Society, April 27, 1940, under the title *A sequence of tests for the convergence and divergence of infinite series.*

[2] R. W. Brink, *A new sequence of integral tests for the convergence and divergence of infinite series*, Annals of Mathematics, vol. 21 (1919), pp. 39–60.

[3] C. T. Rajagopal, *On an integral test of R. W. Brink for the convergence of series*, this Bulletin, vol. 43 (1937), pp. 405–412.

[4] R. W. Brink, *A new integral test for the convergence and divergence of infinite series*, Transactions of this Society, vol. 19 (1918), pp. 186–204.

PROOF. In case 1, if $n \geqq \nu$,

$$\frac{r_{n+1}}{r_n} \geqq \frac{r'_{n+1}}{r'_n}, \quad \frac{r_{n+2}}{r_{n+1}} \geqq \frac{r'_{n+2}}{r'_{n+1}}, \quad \cdots, \quad \frac{r_{N+1}}{r_N} \geqq \frac{r'_{N+1}}{r'_N}.$$

Multiplying these inequalities, we have $r_{N+1}/r_n \geqq r'_{N+1}/r'_n$, and taking the limit as $N$ becomes infinite, we obtain $r_n \leqq r'_n$, $(n \geqq \nu)$. Hence if $\sum_{n=\nu}^{\infty} u'_n$ converges, $\sum_{n=\nu}^{\infty} u_n$ also converges. A similar proof can be given for the case of divergence.

THEOREM 1. *Let* $\{u_n\}$ *be a sequence of positive terms with ratios* $r_n = u_{n+1}/u_n$, $R_n = r_{n+1}/r_n$, *such that* $\lim_{n\to\infty} r_n = 1$. *If of the limits*

$$\lim_{n\to\infty} n^2 \log R_n = a_0,$$

$$\lim_{n\to\infty} \log n(n^2 \log R_n - 1) = a_1,$$

$$\lim_{n\to\infty} \log\log n\{\log n(n^2 \log R_n - 1) - 1\} = a_2,$$

$$\lim_{n\to\infty} \log\log\log n[\log\log n\{\log n(n^2 \log R_n - 1) - 1\} - 1] = a_3,$$

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

$a_k$ *is the first which is finite and different from* 1, *or the first to be positively or negatively infinite, the series* $\sum_{n=\nu}^{\infty} u_n$ *converges if* $a_k > 1$ *and diverges if* $a_k < 1$.

PROOF. Let $l_1 = \log n$, $l_k = \log l_{k-1}$, $(k > 1)$;

$$L_k(n, \alpha) = \frac{1}{l_1} + \frac{1}{l_1 l_2} + \cdots + \frac{1}{l_1 l_2 \cdots l_{k-1}} + \frac{\alpha}{l_1 l_2 \cdots l_k}, \quad k \geqq 1,$$

$$L_0(n, \alpha) = 0.$$

By hypothesis,

$$\lim_{n\to\infty} l_k[l_{k-1}\{l_{k-2}( \cdots l_2[l_1(n^2 \log R_n - 1) - 1] \cdots ) - 1\} - 1] = a_k. \tag{1}$$

Hence

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{1+L_k(n, 1)} & [l_k\{l_{k-1}(l_{k-2}[ \cdots l_2\{l_1(n^2 \log R_n - 1) - 1\} \cdots ] - 1) - 1\} \\ & - l_k l_{k-1} \cdots l_2 L_1(n, 1) - l_k l_{k-1} \cdots l_3 L_2(n, 1) - \cdots - l_k l_{k-1} L_{k-2}(n, 1) \\ & - l_k L_{k-1}(n, 1)] = a_k, \end{aligned} \tag{2}$$

since $\lim_{n\to\infty} 1/[1 + L_k(n, 1)] = 1$ and $\lim_{n\to\infty} l_k l_{k-1} \cdots l_j L_{j-1}(n, 1) = 0$, $(1 \leqq j \leqq k)$.

(a) If $a_k > 1$, let $\alpha_1$ be a number such that $1 < \alpha_1 < a_k$. Let $N_1$ be chosen sufficiently large so that for $n \geqq N_1$, $l_k(n)$ is defined and positive and

$$\begin{aligned}\frac{1}{1+L_k(n,1)}\big[&l_k\{l_{k-1}(l_{k-2}[\cdots l_2\{l_1(n^2\log R_n-1)-1\}\cdots]-1)-1\}\\ &-l_kl_{k-1}\cdots l_2L_1(n,1)-l_kl_{k-1}\cdots l_3L_2(n,1)-\cdots-l_kl_{k-1}L_{k-2}(n,1)\\ &-l_kL_{k-1}(n,1)\big]>\alpha_1.\end{aligned}\tag{3}$$

It follows that

$$\begin{aligned}\log R_n > \frac{1}{n^2}+\frac{1}{n^2l_1}[1+L_1(n,1)]+\frac{1}{n^2l_1l_2}[1+L_2(n,1)]+\cdots\\ +\frac{1}{n^2l_1l_2\cdots l_{k-1}}[1+L_{k-1}(n,1)]\\ +\frac{\alpha_1}{n^2l_1l_2\cdots l_k}[1+L_k(n,1)].\end{aligned}\tag{4}$$

Let

$$\begin{aligned}M_k(x,\alpha)=\frac{1}{x^2}+\frac{1+l_1}{x^2l_1^2}+\frac{1+l_2+l_2l_1}{x^2l_1^2l_2^2}+\cdots\\ +\frac{1+l_{k-1}+l_{k-1}l_{k-2}+\cdots+l_{k-1}\cdots l_1}{x^2l_1^2l_2^2\cdots l_{k-1}^2}\\ +\alpha\frac{1+l_k+l_kl_{k-1}+\cdots+l_k\cdots l_1}{x^2l_1^2l_2^2\cdots l_k^2},\end{aligned}$$

where now $l_1 = \log x$, and so on. Then (4) can be written

$$\log R_n > M_k(n, \alpha_1).\tag{5}$$

(b) If $a_k < 1$, let $\alpha_2$ be a positive number such that $a_k < \alpha_2 \leqq 1$. Proceeding as in (a), we can show that for $n$ greater than a suitably chosen number $N_2$,

$$\log R_n < M_k(n, \alpha_2).\tag{6}$$

Consider the series

$$\sum_{n=\nu}^{\infty} u'_{n;\alpha} \qquad u'_{n;\alpha} = \exp\left\{-\sum_{j=\nu}^{n-1}\sum_{i=j}^{\infty} M_k(i,\alpha)\right\}, \qquad \nu > N_1, N_2.\tag{7}$$

For this series, $r_n' = \exp\left\{-\sum_{i=n}^{\infty} M_k(i,\alpha)\right\}$, $R_n' = \exp\left\{M_k(n,\alpha)\right\}$. It is easily shown that the conditions of Brink's Theorem VI are satisfied with $R'(x) = \exp\{M_k(x, \alpha)\}$ and $n \geqq \nu$. The test integral then has the form

$$\int_\nu^\infty \exp\left\{-\int_\nu^x \int_x^\infty M_k(x, \alpha)dxdx\right\}dx = K\int_\nu^\infty \frac{1}{xl_1l_2 \cdots l_{k-1}l_k^\alpha}dx,$$

where $K$ is constant. This integral, and hence the series $\sum_{n=\nu}^\infty u'_{n;\alpha}$, converges for $\alpha > 1$ and diverges for $\alpha \leqq 1$. We now apply the lemma to the series $\sum_{n=\nu}^\infty u_n$ and $\sum_{n=\nu}^\infty u'_{n;\alpha}$.

In case (a), we set $\alpha = \alpha_1$ in series (7). $\sum_{n=\nu}^\infty u'_{n;\alpha_1}$ converges since $\alpha_1 > 1$. From (5), we have $R_n > \exp\{M_k(n, \alpha_1)\} = R_n$, $n \geqq \nu$. The conditions of part 1 of the lemma are satisfied, and hence $\sum_{n=\nu}^\infty u_n$ converges.

In case (b), we set $\alpha = \alpha_2$ in series (7). $\sum_{n=\nu}^\infty u'_{n;\alpha_2}$ diverges, and from (6),

$$R_n < \exp\{M_k(n, \alpha_2)\} = R'_n, \qquad n \geqq \nu.$$

Hence $\sum_{n=\nu}^\infty u_n$ diverges by part 2 of the lemma.

The tests of Theorem 1 apply to series for which an explicit expression for $R_n$ is known. The general term of such a series has the form $u_n = \prod_{j=\nu}^{n-1}\prod_{m=j}^\infty \phi(m)$.

*Example* 1. Consider the series $\sum_{n=3}^\infty u_n$, where

$$u_n = \exp\left\{-\sum_{j=2}^{n-1}\sum_{m=j}^\infty \phi(m)\right\}, \qquad \phi(m) = \frac{\alpha + \beta \log(m^2)}{m^2 \log(m^2)}, \qquad \beta > 0.$$

We have $r_n = \exp\{-\sum_{m=n}^\infty \phi(m)\}$; $\lim_{n\to\infty} r_n = 1$ since $\sum_{m=3}^\infty \phi(m)$ converges for all values of $\alpha$ and $\beta$; $R_n = \exp\{\phi(n)\}$; $\log R_n = \phi(n)$. We apply the first test of Theorem 1,

$$\lim_{n\to\infty} n^2 \log R_n = \lim_{n\to\infty} \frac{\alpha + \beta \log(n^2)}{\log(n^2)} = \beta.$$

Thus the series converges for $\beta > 1$ and diverges for $\beta < 1$, regardless of the value of $\alpha$. For the case $\beta = 1$, we apply the second test,

$$\lim_{n\to\infty} \log n(n^2 \log R_n - 1) = \lim_{n\to\infty} \log n\left[\frac{\alpha + \log(n^2)}{\log(n^2)} - 1\right] = \frac{\alpha}{2},$$

and the series converges for $\beta = 1$, $\alpha > 2$, and diverges for $\beta = 1$, $\alpha < 2$. For the case $\beta = 1$, $\alpha = 2$, we go on to the third limit test,

$$\lim_{n\to\infty} \log \log n[\log n(n^2 \log R_n - 1) - 1] = 0,$$

and the series diverges.

The tests of Theorem 1 are valid if $\log R_n$ is replaced by $R_n - 1$, the tests of the resulting sequence being in some cases more convenient to apply than those of Theorem 1.

THEOREM 2. *Let* $\{u_n\}$ *be a sequence of positive terms with ratios* $r_n = u_{n+1}/u_n$, $R_n = r_{n+1}/r_n$, *such that* $\lim_{n\to\infty} r_n = 1$. *If of the limits*

$$\lim_{n\to\infty} n^2(R_n - 1) = b_0,$$

$$\lim_{n\to\infty} \log n\left[n^2(R_n - 1) - 1\right] = b_1,$$

$$\lim_{n\to\infty} \log\log n\left\{\log n\left[n^2(R_n - 1) - 1\right] - 1\right\} = b_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,$$

$b_k$ *is the first which is finite and different from* 1, *or the first to be positively or negatively infinite, the series* $\sum_{n=\nu}^{\infty} u_n$ *converges if* $b_k > 1$ *and diverges if* $b_k < 1$.

PROOF. Proceeding as in the proof of Theorem 1, we are led to the following inequalities:

(a) If $b_k > 1$, then for any number $\beta_1$ such that $1 < \beta_1 < b_k$, and for $n$ greater than a suitably chosen number $N_1'$,

$$R_n - 1 > M_k(n, \beta_1). \tag{8}$$

(b) If $b_k < 1$, then for any positive number $\beta_2$ such that $b_k < \beta_2 \leqq 1$, and for $n$ greater than a suitably chosen number $N_2'$,

$$R_n - 1 < M_k(n, \beta_2). \tag{9}$$

In case (a), consider the series $\sum_{n=\nu}^{\infty} u_n''$,

$$u_n'' = \exp\left\{-\sum_{j=\nu}^{n-1}\sum_{i=j}^{\infty}\left[M_k(i, \beta_1) - (M_k(i, \beta_1))^2\right]\right\}, \qquad \nu \geqq N_1'.$$

It can be shown that this series satisfies the conditions of Brink's Theorem VI with $R''(x) = \exp\{M_k(x, \beta_1) - (M_k(x, \beta_1))^2\}$. The test integral has the form

$$\int_\nu^\infty \exp\left\{-\int_\nu^x\int_x^\infty\left[M_k(x, \beta_1) - (M_k(x, \beta_1))^2\right]dxdx\right\}dx$$

$$\leqq \int_\nu^\infty \exp\left\{-\int_\nu^x\int_x^\infty M_k(x, \beta_1)dxdx + \int_\nu^\infty\int_x^\infty (M_k(x, \beta_1))^2dxdx\right\}dx$$

$$= K'\int_\nu^\infty \exp\left\{-\int_\nu^x\int_x^\infty M_k(x, \beta_1)dxdx\right\}dx,$$

where $K'$ is a constant. Since $\beta_1 > 1$, the latter integral converges, and hence the series $\sum_{n=\nu}^{\infty} u_n''$ converges. From (8),

$$R_n > 1 + M_k(n, \beta_1), \qquad n \geqq \nu.$$

Hence if $\nu$ is sufficiently large so that $M_k(\nu, \beta_1) < 1$,

$$\log R_n > \log [1 + M_k(n, \beta_1)] > M_k(n, \beta_1) - (M_k(n, \beta_1))^2 = \log R_n'', \qquad n \geqq \nu,$$

$R_n > R_n''$, and the series $\sum_{n=\nu}^{\infty} u_n$ of our theorem converges by part 1 of the lemma.

In case (b), set $\alpha = \beta_2$ in series (7), and take $\nu = N_2'$ and sufficiently large so that $|R_n' - 1| < 1$ for $n \geqq \nu$. $\sum_{n=\nu}^{\infty} u_{n;\beta_2}'$ diverges since $\beta_2 \leqq 1$. From (9), we have

$$R_n - 1 < M_k(n, \beta_2) = \log R_n' < R_n' - 1, \qquad n \geqq \nu,$$

$R_n < R_n'$, and hence the series $\sum_{n=\nu}^{\infty} u_n$ diverges by part 2 of the lemma.

*Example* 2. Consider the series $\sum_{n=3}^{\infty} u_n$, $u_n = \prod_{k=2}^{n-1} \prod_{m=k}^{\infty} (1 - \alpha/m^{\beta})$, $(\alpha > 0, \beta > 1)$. Here

$$r_n = \sum_{m=n}^{\infty} \left(1 - \frac{\alpha}{m^{\beta}}\right), \qquad R_n = \frac{1}{1 - \alpha/n^{\beta}}, \qquad R_n - 1 = \frac{\alpha}{n^{\beta} - \alpha}.$$

$\text{Lim}_{n \to \infty} r_n = 1$, since $\prod_{m=3}^{\infty} (1 - \alpha/m^{\beta})$ converges. Applying the first test of Theorem 2, we find

$$\lim_{n \to \infty} n^2 (R_n - 1) = \lim_{n \to \infty} n^2 \cdot \frac{\alpha}{n^{\beta} - \alpha} = \begin{cases} +\infty, & \beta < 2, \\ 0, & \beta > 2, \\ \alpha, & \beta = 2. \end{cases}$$

Thus the series converges when $1 < \beta < 2$ and diverges when $\beta > 2$. If $\beta = 2$, the series converges for $\alpha > 1$ and diverges for $\alpha < 1$. If $\beta = 2$ and $\alpha = 1$, we apply the second test of the sequence,

$$\lim_{n \to \infty} \log n [n^2 (R_n - 1) - 1] = \lim_{n \to \infty} \log n \left[ n^2 \frac{1}{n^2 - 1} - 1 \right] = 0,$$

and hence the series diverges.

University of Minnesota

# THE DOUBLE COSETS OF A FINITE GROUP[1]

J. S. FRAME

1. **Introduction.** It is the purpose of this paper to study some of the properties of the double cosets of a finite group and to prove two main theorems which generalize the results of two previous papers by the author,[2] giving some relations between the double cosets and the irreducible components of the permutation group generated by a given subgroup. We let $H$ be an arbitrary but fixed subgroup of order $h$ of a finite group $G$ of order $g$, $g=nh$, and we let $G_H$ be the permutation group of degree $n$ induced by right multiplication of the cosets $HS_i$, $i=1, 2, \cdots, n$, by elements of $G$. When written as a group of permutation matrices and completely reduced, the group $G_H$ will have $r'$ distinct irreducible components $\Gamma_i$ of degree $n_i$ and multiplicity $\mu_i^H$, and we may write

$$G_H = \sum_{i=1}^{r'} \mu_i^H \Gamma_i. \tag{1.1}$$

Multiplication of a right coset $HG_k$ on the left by a single element of $G$ does not in general produce a right coset, but if each coset $HG_k$ is multiplied on the left by all the elements of a right coset $HS_t$ and the products are added, a transformation is obtained which carries each of the $n$ right cosets $HG_k$ into a collection of right cosets $\sum_{\theta=1}^{h} HS_tH_\theta G_k$ in which, as we shall see in §4, each of the $k_t$ cosets occurs $d_t=h/k_t$ times. Its matrix $d_tV_t$ is permutable with each of the matrices of $G_H$. Certain cosets, which we shall call associated cosets, are permuted among themselves when multiplied on the right by elements of $H$. Each of these produces the same matrix $V_t$. The totality $K_t$ of elements belonging to a complete set of $k_t$ associated cosets, each counted once, will be called a *double coset*, whereas the term *weighted double coset* will refer to the complex of $h^2$ elements $HS_tH$ in which each element of the double coset $K_t$ occurs $d_t=n/k_t$ times. The integer $d_t$ will be called the density. The number of distinct double cosets $K_t$ will be denoted by $r$, and the elements $S_t$, $t=1, 2, \cdots, r$, one from each, will be said to *generate* the double cosets. The double coset which contains the inverses of the elements of $K_t$ will be denoted by $K_{t'}$.

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[1] Presented to the Society, September 12, 1940.

[2] J. S. Frame, *The degrees of the irreducible components of simply transitive permutation groups*, Duke Mathematical Journal, vol. 3 (1937), pp. 8–17.

J. S. Frame, *On the decomposition of transitive permutation groups generated by the symmetric group*, Proceedings of the National Academy of Sciences, vol. 26 (1940), pp. 132–139.

In §2 a number of elementary properties of double cosets are stated, mostly without proof. In §3 a study is made of self-inverse double cosets, $K_t = K_{t'}$, and their number is shown to equal $(1/g)\sum\chi(R^2)$, where $\chi(R^2)$ is the trace of $R^2$ in the permutation group $G_H$, and where the sum is taken for all $R$ in $G$. Using the results of a paper of Frobenius[3] we then prove the first principal theorem, which generalizes a result obtained by the author for the symmetric group.[4]

THEOREM A. *The number of self-inverse double cosets of a finite group $G$ with respect to a subgroup $H$ is equal to the sum of the multiplicities of those irreducible components of $G_H$ which have a symmetric bilinear invariant minus the sum of the multiplicities of those which have an alternating bilinear invariant.*

The Hermitian invariants of $G_H$ associated with the matrices $V_t$ are studied in §4.[5] Two bases consisting of $r$ independent invariants are found, the one obtained directly from the $r$ double cosets, and the other from the $r = \sum_{i=1}^{r'}(\mu_i^H)^2$ Hermitian invariants[6] which come into evidence when $G_H$ is completely reduced. The complex multiplication of the double cosets plays an important role in the discussion which culminates in the proof of Theorem B, which includes as a special case a theorem conjectured but only partially proved in a previous paper.[7]

THEOREM B. *Given a transitive permutation group $G_H$ of degree $n$ in which the subgroup $H$ leaving one symbol fixed permutes the $n$ symbols in $r$ transitive sets of $k_t$ symbols, $t=1, 2, \cdots, r$. Let $K$ be the product $\prod_{t=1}^{r}(k_t)$ and let $N$, $N = \prod_{i=1}^{r'} n_i^{(\mu_i^H)^2}$, denote the product of the degrees $n_i$ of the distinct irreducible components $\Gamma_i$ of $G_H$ each raised to a power equal to the square of its multiplicity in $G_H$. Then $n^{r-2}K/N = \overline{P}_1 P_1$, where $P_1$ is an algebraic integer in the field of the characters of the components of $G_H$.*

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[3] G. Frobenius and I. Schur, *Über die reellen Darstellungen der endlichen Gruppen*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1906 (I), pp. 186–208, especially p. 197. The author is indebted to G. de B. Robinson for referring him to this article.

[4] J. S. Frame, Proceedings of the National Academy of Sciences, loc. cit.

[5] A study of these matrices was made by I. Schur, *Zur Theorie der einfach transitiven Permutationsgruppen*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, 1933, pp. 598–623.

[6] This relation is well known in group theory. See, for example, W. Burnside, *Theory of Groups*, 1911, p. 275.

[7] J. S. Frame, Duke Mathematical Journal, loc. cit.

In §5 a study of the bilinear invariants of the components of $G_H$ which are irreducible in the real domain leads to an alternate proof of Theorem A, based on methods similar to those of §4.

2. **Elementary properties of double cosets.** The set of elements permutable with a coset $HS_i$ form a group $N_i$ called the normalizer of the coset, whose intersection with $H$ is a group $D_i$ of order $d_i$ leaving $HS_i$ fixed in the permutation group $G_H$. We let $D_iH_{ij}$ be the $k_i=h/d_i$ cosets of $H$ with respect to $D_i$ and note that for each $j$ the elements $HS_iD_iH_{ij}=H(S_iH_{ij})$ form a different coset $HS_j$ associated with $HS_i$. Hence there are $k_t$ distinct cosets each occurring $d_t$ times in the weighted double coset $HS_tH$. *If $H$ is invariant in $G$*, then each $D_t=H$, each $k_t=1$, and *each coset is a double coset*. It is convenient to select the $S_j$ for associated cosets so that $S_iH_{ij}=S_j$ and then to define $H_{ji}=S_j^{-1}S_i=H_{ij}^{-1}$, when obtaining the cosets of $H$ with respect to its subgroup $D_j=H_{ij}^{-1}D_iH_{ij}$, which is permutable with $HS_j$. It is seen immediately that the subgroups $D_i, D_j, \cdots$ which are permutable with the right cosets $HS_i, HS_j, \cdots$ of a double coset $HS_tH$ are all of the same order $d_t$ and form a complete set of conjugate subgroups of $H$.

The following four properties of double cosets are simple enough to be given here without proof.[8]

THEOREM 2.1. *Each element of $G$ lies in one and only one double coset of $G$ with respect to a given subgroup $H$.*

THEOREM 2.2. *A double coset may be generated by any one of its elements.*

THEOREM 2.3. *The inverses $H_j^{-1}S_t^{-1}H_i^{-1}$ of the elements $H_iS_tH_j$ of the double coset $K_t$ form a double coset $K_{t'}$ generated by $S_t^{-1}$, which has the same density as $K_t$.*

If $K_t=K_{t'}$, the double coset is called self-inverse.

THEOREM 2.4. *A double coset which contains a self-inverse element is self-inverse. In particular the double coset $H=K_1$ is self-inverse.*

The next three theorems show that the elements of a class of conjugates, of a left coset, and of the set of inverses of a right coset, are equally distributed among the right cosets of their double coset.

THEOREM 2.5. *Each coset of a double coset $K_t$ contains the same num-*

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[8] Several of the theorems in §2 are implied in the discussion of cosets in the standard texts on group theory. Nowhere has the author found them collected as properties of double cosets as such. They are stated here for convenience of reference for the later proofs.

*ber* $g_{i\lambda}$ *of the* $g_\lambda$ *elements forming the class* $C_\lambda$ *of conjugate elements in* $G$.[9]

Proof. Let $H_\theta S_i$ be a common element of $HS_i$ and $C_\lambda$, and let the subgroup of $D_i$ which is permutable with the element $H_\theta S_i$ be of index $g_{i\lambda}$ in $D_i$. Then there will be $g_{i\lambda}$ distinct elements each counted $d_i/g_{i\lambda}$ times among the elements $\delta_\kappa^{-1}(H_\theta S_i)\delta_\kappa$, $\delta_\kappa$ in $D_i$. Each of these distinct elements of $HS_i$ is transformed by $H_{ij}$ into a different element of $HS_j$, so the numbers $g_{i\lambda}, g_{j\lambda}, \cdots$ are the same for all the cosets of $K_t$ and may be denoted by $g_{t\lambda}$.

Theorem 2.6. *The* $h$ *elements of the left coset* $(H_\theta S_i)H$ *are equally distributed among the* $k_i$ *right cosets associated with* $HS_i$, *just* $d_i$ *elements lying in each right coset.*

Proof. Each element of $H$ can be written uniquely in the form $\delta_\kappa H_{ij}$, where $\delta_\kappa$ is in $D_i$. For each $j$ we obtain $d_i$ elements $H_\theta S_i \delta_\kappa H_{ij} = (H_\theta S_i \delta_\kappa S_i^{-1})S_j$ in $HS_j$.

If we note further that the inverses of the elements of a right coset $HS_i$ form a left coset $S_i^{-1}H$, then we obtain the following:

Theorem 2.7. *The inverses of the elements of a right coset are equally distributed among the* $k_i$ *cosets of the inverse double coset, with* $d_i$ *elements in each. Furthermore each coset of a self-inverse double coset* $K_t$ *contains just* $d_t$ *of its own inverses.*

Theorem 2.8. *Each element of the normalizer* $N_1$ *of* $H$ *transforms a double coset* $K_t$ *into itself or into a double coset* $N_1^{-1}K_tN_1$ *having the same distribution of elements among the classes* $C_\lambda$ *as* $K_t$ *does.*

These double cosets may be called conjugate double cosets.

Proof. Each element is transformed into a conjugate element in the double coset generated by $N_1^{-1}S_tN_1$.

Theorem 2.9. (a) *The number of times,* $k_{tuv}$, *that the identity element* $E$ *occurs among the* $h^3$ *elements* $HS_tHS_uHS_v$ *depends only on the double cosets to which* $S_t$, $S_u$, *and* $S_v$ *belong.*

(b) *The constant* $k_{tuv}$ *is unchanged by an even permutation of the double cosets* $K_t$, $K_u$, $K_v$, *or by an odd permutation coupled with a change to inverse double cosets.*

(c) *The number of the* $h$ *elements* $S_uHS_v$ *which lie in the double coset* $K_{t'}$ *is* $k_{tuv}/d_t$, *and* $\sum_{t=1}^{r} k_{tuv}/d_t = h$.

(d) *For* $t=1$, *we have* $k_{1uv} = k_{uv1} = hd_u\delta_{uv'}$.

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[9] D. E. Littlewood, *Theory of Group Characters*, 1940, p. 149. Littlewood shows, if we change his notation to ours, that the characters in $G_H$ of the elements of $C_\lambda$ are $ng_{1\lambda}/g_\lambda$.

(e) *The product of the double cosets* $K_tK_u$ *is given by the equation*

$$K_tK_u = (h/(d_td_u))\sum_v k_{tuv'}K_v. \tag{2.10}$$

To prove the various parts of Theorem 2.9, let us assume that $H_a$, $H_b$, and $H_c$ are elements of $H$ which satisfy the equation $H_aS_tH_bS_uH_cS_v=E$. The equation is still valid if we permute the six elements cyclicly, or if we interpose between two adjacent ones the product of an element of $H$ and its inverse, thus replacing an element by another element in its double coset, or if we replace all six elements by their inverses in the opposite order. The remainder of the proof depends on eliminating the weighting factors from the weighted double cosets $HS_tH$, etc., but may be left to the reader.

## 3. The number of self-inverse double cosets.

We shall now apply the theorems of §2 to obtain expressions for the number of self-inverse double cosets of a group.

THEOREM 3.1. *The number* $N_H$ *of self-inverse double cosets* $K_t$ *of* $G$ *with respect to* $H$ *is* $1/g$ *times the number of solutions of the equation*

$$S_jR^2 = H_kS_j, \qquad j = 1, 2, \cdots, n;\ H_k \text{ in } H,\ R \text{ in } G.$$

We rewrite the equation in the form $H_k^{-1}(S_jRS_j^{-1})=(S_jRS_j^{-1})^{-1}$, which states that the inverse of the element $S_jRS_j^{-1}=T$ belongs to its own right coset. For each of the $k_t$ cosets of $K_t$ there are $d_t$ available values of $T$, by Theorem 2.7. For each of these $h=d_tk_t$ values of $T$ we may choose $n$ generators $S_j$ to determine an element $R=S_j^{-1}TS_j$ conjugate to $T$. Thus for each self-inverse double coset we obtain $g=nh$ solutions of the given equation, and the theorem is proved.

Now the permutation matrix corresponding to $R^2$ in $G_H$ has a unit in the principal diagonal for every right coset which remains fixed by $R^2$, that is, for every solution of the equation $S_jR^2=H_kS_j$, $H_k$ in $H$, $j=1, 2, \cdots, n$. Its trace is the number of these solutions for fixed $R$. When we sum these for all $R$ in $G$ and apply Theorem 3.1, we obtain the following theorem.

THEOREM 3.2. *The number* $N_H$ *of self-inverse double cosets of* $G$ *with respect to* $H$ *is given by the formula*

$$N_H = (1/g)\sum_{R \text{ in } G} \chi(R^2), \tag{3.3}$$

*where* $\chi(S)$ *is the trace of the matrix* $S$ *in the permutation group* $G_H$.

This theorem may be applied to the results of a paper by Frobenius and Schur[10] to obtain the fundamental Theorem A of §1. It is known in the theory of group characters that for an irreducible representation $\Gamma_i$ with character $\chi^i(R)$ the quantity

$$\sum_R \{[\chi^i(R)]^2 + \chi^i(R^2)\}/(2g) \tag{3.4}$$

is 1 or 0 according as $\Gamma_i$ has or has not a symmetric bilinear invariant, and that

$$\sum_R \{[\chi^i(R)]^2 - \chi^i(R^2)\}/(2g) \tag{3.5}$$

is 1 or 0 according as $\Gamma_i$ has or has not an alternating bilinear invariant. Following Frobenius[10] we set $c_i = 1, -1$, or 0 according as $\Gamma_i$ has a symmetric, an alternating, or no bilinear invariant. We shall call these respectively "symmetric," "quaternion," and "rotary" representations. Then

$$(1/g)\sum_R \chi^i(R^2) = c_i. \tag{3.6}$$

Now let $G_H$ be reduced into irreducible components $\Gamma_i$ with multiplicities $\mu_i^H$. Then

$$G_H = \sum_i \mu_i^H \Gamma_i, \qquad \chi(R) = \sum_i \mu_i^H \chi^i(R). \tag{3.7}$$

Hence, by (3.3), (3.7), and (3.6), we have

$$N_H = (1/g)\sum_R \chi(R^2) = (1/g)\sum_R \sum_i \mu_i^H \chi^i(R^2) = \sum_i c_i \mu_i^H. \tag{3.8}$$

This formula is equivalent to Theorem A of §1.

**4. A theorem derived from a unitary reduction of a group and its double cosets.** When a group $G$ is represented in two ways as a regular permutation group of degree $g$, using cosets with respect to the identity subgroup $E$, the one $G_E$ obtained by the right multiplication of cosets by the elements of $G$, and the other $G_E'$ by left multiplication, the matrices of $G_E'$ form a basis for all matrices permutable with those of $G_E$, and vice versa. In that case each of the elements of $G$ forms a double coset. But when cosets are taken with respect to a subgroup $H \neq E$, a basis for the matrices permutable with those of the right multiplication permutation group $G_H$ is found in the double cosets of the left multiplication group. The matrices $V_t$ described in §1 are those of a complete set of invariant Hermitian forms of $G_H$ on which all others are linearly dependent. For let $x_i$, $x_i = x(HS_i)$, $i = 1, 2, \cdots, n$, be the variables permuted by $G_H$. Then the matrix $R$ of $G_H$ transforms the product $\bar{x}(H)x(HS_t)$ into $\bar{x}(HR)x(HS_tR)$, and if we write $R = H_iG_k$ and sum over all $R$ in $G$, we obtain an Hermitian form invariant under the right multiplications of $G_H$. Since the subgroup $D_t$ of $H$ leaves $HS_t$ fixed, each term of this form will contain the factor $d_t$. Dividing by $d_t$ and writing $K_t = HS_tH/d_t$, we obtain the simpler invariant Hermitian form $\sum_k \bar{x}(HG_k)x(K_tG_k)$, whose matrix we have denoted by $V_t$. The matrix $V_1$ is the unit matrix. Each row and column of $V_t$ contains $k_t$ 1's and the rest 0's, and the sum of all the matrices $V_t$ is a matrix consisting entirely of 1's.[11] To the inverse double coset $K_{t'}$ corresponds the transposed matrix $V_t'$, which will be denoted by $V_{t'}$. Self-inverse double cosets have symmetric matrices $V_t$.

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[10] G. Frobenius and I. Schur, loc. cit.

Now let $U$ be a unitary matrix which completely reduces the group $G_H$ into its irreducible components $\Gamma_i$ of degree $n_i$ and multiplicity $\mu_i^H$, so that all equivalent components of $U^{-1}G_HU$ are actually identical and so that the invariant Hermitian form for each component is a diagonal form with unit matrix $E_i$ of degree $n_i$. Then for the set of $\mu_i^H$ equal components $\Gamma_i$ we have $(\mu_i^H)^2$ linearly independent Hermitian forms whose matrices $\sum_{\alpha\beta} E_i^{\alpha\beta}\rho_i^{\alpha\beta}$ are obtained as the direct product of $E_i$ with an arbitrary matrix $(\rho_i^{\alpha\beta})$ of degree $\mu_i^H$. The $r$ matrices $M_t = U^{-1}V_tU$, obtained by transforming the Hermitian invariants of $G_H$, must be expressible as linear combinations of the $r$ matrices $E_i^{\alpha\beta}$. It is convenient to arrange the symbols $E_i^{\alpha\beta}$ in some arbitrary order starting with $E_1^{11} = E_1$, and to assign to each a single subscript $\gamma$. We write

$$M_t = \sum_\gamma E_\gamma\rho_{\gamma t}; \qquad \text{where } E_\gamma = E_i^{\alpha\beta} \quad \text{and } \rho_{\gamma t} = \rho_{i\,\cdot}^{\alpha\beta}(K_t); \quad \gamma, t = 1, 2, \cdots, r. \tag{4.1}$$

Since the matrices $M_t$ and $E_\gamma$ each form a basis for the invariant Hermitian forms of $U^{-1}G_HU$, the matrix $(\rho_{\gamma t})$ is nonsingular.

The $M_t$ combine according to a rule similar to the rule of combination for the inverse double cosets $K_{t'}$, as given in 2.10, namely,

$$M_tM_u = \sum_v c_{tuv}M_v, \qquad c_{tuv} = k_{t'u'v}/(d_td_u). \tag{4.2}$$

Since this is also the rule of combination for the matrices $\rho_i^{\alpha\beta}(K_t)$ corresponding to a given $\Gamma_i$, these matrices give that representation for the left multiplication of double cosets which is associated with $\Gamma_i$.

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[11] J. S. Frame, Duke Mathematical Journal, loc. cit.

Now when $G_B$ and $G_B'$ are simultaneously reduced, the component of $G_B'$ on the same variables as the $n_i$ components $\Gamma_i$ of $G_B$ consists of the direct product of $E_i$ with a representation equivalent to $\Gamma_i'$. When the matrices of this representation are summed over all elements in each double coset, the resulting set of $r$ matrices when reduced will contain the component of degree $\mu_i^H \leq n_i$, expressible in terms of the variables of $G_H$, which is equivalent to the matrices $\rho_i^{\alpha\beta}(K_t)$. Hence the $\rho_i^{\alpha\beta}(K_t)$ are expressible in terms of the matrices of $\Gamma_i'$.

The rule (4.2) is also the rule of combination for the matrices $V_t$,[12] so the coefficients $c_{tuv}$ must be integers, and the character of the product $M_{t'}M_u$ is given by the formula

$$\chi(M_{t'}M_u) = nc_{t'u1} = nk_t\delta_{tu}, \tag{4.3}$$

in view of Theorem 2.9 (d). Since $U$ is unitary, the matrix $M_{t'}$ is the transposed conjugate $\bar{M}_t'$ of $M_t$. We shall denote by $E_{\gamma'}$ the transposed conjugate of $E_\gamma$, and by $n_\gamma$ the degree of the corresponding irreducible representation.

Next consider the matrices $(M_{t'u})$ and $(E_{\gamma'\delta})$ of degree $r$ whose elements are

$$M_{t'u} = \chi(M_{t'}M_u) = \chi(V_{t'}V_u) = nk_t\delta_{tu}, \tag{4.4}$$

$$E_{\gamma'\delta} = \chi(E_{\gamma'}E_\delta) = n_\gamma\delta_{\gamma\delta}. \tag{4.5}$$

We obtain a relation between $(M_{t'u})$ and $(E_{\gamma'\delta})$ as follows:

$$M_{t'u} = \chi\left(\sum_\gamma \bar{\rho}_{t\gamma}E_{\gamma'}\sum_\delta E_\delta\rho_{\delta u}\right) = \sum_{\gamma,\delta}\bar{\rho}_{t\gamma}E_{\gamma'\delta}\rho_{\delta u}. \tag{4.6}$$

Denoting the determinant of $(\rho_{\delta u})$ by $P$, we have, by (4.4), (4.5), (4.6),

$$\prod_{t=1}^{r}(nk_t) = \left(\prod_{\gamma=1}^{r} n_\gamma\right)\bar{P}P = \left(\prod_{i=1}^{r'} n_i^{(\mu_i^H)^2}\right)\bar{P}P. \tag{4.7}$$

Since $\sum_t\rho_{\gamma t} = n\delta_{\gamma 1}$, the determinant $P$ may be written in the form $nP_1$, where $P_1$ is the minor of $\rho_{11}$ in $P$. Factoring $n^2$ from both sides of (4.7) and using the notation of Theorem B, we have

$$n^{r-2}K = N\bar{P}_1P_1, \quad \text{where} \quad K = \prod_{t=1}^{r} k_t, \quad N = \prod_{i=1}^{r'} n_i^{(\mu_i^H)^2}. \tag{4.8}$$

Since for each $i$ the matrices $\rho_i^\alpha\ (K_t)$ give a representation of the ring of matrices $V_t$ in which the coefficients of combination are integers, the matrix $P_1$ is an algebraic integer. It belongs to the field of characters of the $\Gamma_i$. This completes the proof of Theorem B. It will be noted that if the representations $\Gamma_i$ have rational characters, which will certainly be true if their degrees are all distinct, then the quotient $n^{r-2}K/N$ will be a perfect square.

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[12] J. S. Frame, Duke Mathematical Journal, loc. cit.

5. **A theorem derived from a real orthogonal reduction of a group and its double cosets.** Let us now consider the reduction of $G_H$ by a real orthogonal matrix $O$ into a form which is irreducible in the real domain. The component representations are of three types: (1) Those "symmetric" representations $\Gamma_i^{(+)}$ which are absolutely irreducible in the complex domain and have a symmetric bilinear invariant. (2) Those "quaternion" representations $2\Gamma_i^{(-)}$ which consist of two equivalent complex components of even degree each with real characters and each with an alternating but not a symmetric bilinear invariant. The pair together have a third alternating bilinear invariant and a symmetric bilinear invariant. The matrices of the four invariants, suitably normalized, combine like the quaternion units. (3) Those "rotary" representations $\Gamma_i^{(0)}+\Gamma_{i'}^{(0)}$ which have two non-equivalent conjugate complex absolutely irreducible components, each having complex characters, but having no bilinear invariant. Taken together they have an alternating and a symmetric bilinear invariant which, when suitably normalized, combine like the real and imaginary units.

The matrices $O^{-1}V_tO$ form a basis for the invariant bilinear forms, but in place of the other basis matrices $E_\gamma$ used in §4, we now use matrices $E_\gamma$ which for the symmetric representations of type 1 are like the old $E_\gamma$, for the quaternion representations of type 2 come in sets of four which multiply like the quaternion units, and for the rotary representations of type 3 come in pairs which multiply like the real and imaginary units. Those of type 2 are as follows:

$$(5.1)\qquad \begin{pmatrix} e_i & 0 & 0 & 0 \\ 0 & e_i & 0 & 0 \\ 0 & 0 & e_i & 0 \\ 0 & 0 & 0 & e_i \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & e_i & 0 \\ 0 & 0 & 0 & e_i \\ -e_i & 0 & 0 & 0 \\ 0 & -e_i & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -e_i & 0 & 0 \\ e_i & 0 & 0 & 0 \\ 0 & 0 & 0 & e_i \\ 0 & 0 & -e_i & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 & e_i \\ 0 & 0 & -e_i & 0 \\ 0 & e_i & 0 & 0 \\ -e_i & 0 & 0 & 0 \end{pmatrix},$$

where $e_i$ denotes the unit matrix of degree $n_i/2$.

We next normalize each of the matrices $V_t$ and $E_\gamma$ by dividing by the square root of the sum of the squares of its coefficients. From the new normalized matrices $V_t^*$ and $E_\gamma^*$, thought of as of degree $n$, we form a new set of matrices of degree $r$ defined by

$$(5.2)\qquad \begin{aligned} V^*_{tu} &= \chi(V^*_t V^*_u), & V^*_{t'u} &= \chi(V^{*\prime}_t V^*_u),\\ E^*_{\gamma\delta} &= \chi(E^*_\gamma E^*_\delta), & E^*_{\gamma'\delta} &= \chi(E^{*\prime}_\gamma E^*_\delta). \end{aligned}$$

Then $(V^*_{t'u})$ and $(E^*_{\gamma'\delta})$ are each the unit matrix, whereas $(V^*_{tu}) = (\delta_{t'u})$ is a matrix whose trace is the number $N_H$ of self-inverse double cosets, and $(E^*_{\gamma\delta})$ is a matrix whose trace is $\sum_{i=1}^{r'} c_i\mu_i^H$, since each component $\Gamma_i^{(+)}$ of type 1 contributes $+1$, each real component $2\Gamma_i^{(-)}$ of type 2 containing a pair of equivalent irreducible representations contributes $+1\ -1\ -1\ -1 = -2$, and each real component of type 3 contributes $+1\ -1 = 0$. Now let

$$(5.3)\qquad O^{-1}V^*_t O = \sum_\gamma E^*_\gamma q_{\gamma t}$$

define a matrix $Q$ which changes the basis of the bilinear forms of $O^{-1}G_H O$. Then

$$(5.4)\qquad \begin{aligned} V^*_{tu} &= \chi(V^*_t V^*_u) = \chi(O^{-1}V^*_t O\cdot O^{-1}V^*_u O) = \chi\Big(\sum_\gamma E^*_\gamma q_{\gamma t}\cdot \sum_\delta E^*_\delta q_{\delta u}\Big)\\ &= \sum_{\gamma,\delta} q_{\gamma t}E^*_{\gamma\delta}q_{\delta u}. \end{aligned}$$

Similarly, since $(V^*_{t'u})$ and $(E^*_{\gamma'\delta})$ are unit matrices, we have

$$(5.5)\qquad V^*_{t'u} = \sum_{\gamma,\delta} q_{\gamma' t'}E_{\gamma'\delta}q_{\delta u} = \sum q_{\delta t'}q_{\delta u} = \delta_{t'u}.$$

Hence $Q$ is an orthogonal matrix, and (5.4) may be written in the form

$$(5.6)\qquad (V^*_{tu}) = Q^{-1}(E^*_{\gamma\delta})Q.$$

By equating the traces of $(V^*_{tu})$ and $(E^*_{\gamma\delta})$ given above, an alternate proof of Theorem A is obtained.

BROWN UNIVERSITY

# CONDITIONS FOR THE CONTINUITY OF ARC-PRESERVING TRANSFORMATIONS[1]

D. W. HALL AND W. T. PUCKETT, JR.

1. **Introduction.** A single-valued transformation $T(A)=B$, where $A$ and $B$ are topological spaces, is said to be *arc-preserving*[2] provided that the image of every simple arc in $A$ is either a simple arc or a single point in $B$. Even when $A$ is a simple arc, an arc-preserving transformation may fail to be continuous; for example: on the unit interval $(x_0=0\leqq x\leqq 1=x_1)$ let $x_n=1/n$ $(n=1, 2, 3, \cdots)$. Define $T(x_0)=x_0$ and for each interval $A_n$ $(x_{n+1}\leqq x\leqq x_n)$ let $T(A_n)=A$ be a topological transformation such that $T(x_n)=x_0$ or $x_1$ according as $n$ is even or odd. Then the transformation $T(A)=A$ is arc-preserving, but fails to be continuous at $x_0$.

The results of this paper concern conditions under which an arc-preserving transformation is continuous, and the conclusions lead to homeomorphisms. We consider only the case where $A$ is a locally connected continuum. The transformation $T$ may be made continuous by putting conditions on the space $A$ or by putting added conditions on the transformation $T$ itself. In this paper we take both points of view. We shall say that $A$ is *strongly arcwise connected* provided every infinite subset of $A$ intersects some arc of $A$ in infinitely many points. Our principal theorem states that if $A$ is cyclic and $T(A)=B$ is arc-preserving then $T$ will be topological or $B$ will be an arc provided either $A$ is strongly arcwise connected or $T$ is tree-preserving[3] (that is, the image of every tree in $A$ is a tree or a single point in $B$). Moreover, we show that if $B$ is not an arc then $A$ must be strongly arcwise connected in order that a topological mapping be the only arc-preserving transformation of $A$ onto $B$.

Throughout the paper $A$ is a locally connected continuum and $T$ is a single-valued transformation, but not necessarily continuous. It is understood that a single point is to be regarded as an arc.

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[1] Presented to the Society in parts as follows: April 6, 1940, under the title *On arc-preserving transformations*, by Puckett; April 26, 1940, under the title *On arc and tree preserving transformations*, by D. W. Hall; and September 12, 1940, under the title *Arc-preserving transformations of a certain class of spaces*, by Hall and Puckett.

[2] See G. T. Whyburn, *Arc-preserving transformations*, American Journal of Mathematics, vol. 58 (1936), pp. 305–312. See also D. W. Hall and G. T. Whyburn, *Arc- and tree-preserving transformations*, Transactions of this Society, vol. 48 (1940), pp. 63–71.

[3] See R. G. Simond, Duke Mathematical Journal, vol. 4 (1938), pp. 575–589; also Hall and Whyburn, loc. cit.

2. **On strongly arcwise connected sets.** The set $A$ is said to be *strongly arcwise connected* provided that every infinite subset of $A$ contains infinitely many points which lie on an arc in $A$. The following are immediate consequences of this definition:

(2.1) *A strongly arcwise connected set is compact and locally connected.*

(2.2) *In order that a continuum be strongly arcwise connected it must be the sum of a finite number of cyclic chains.*

(2.3) *The property of a continuum's being strongly arcwise connected is cyclicly reducible.*[4]

(2.4) *The property of being strongly arcwise connected is invariant under an arc-preserving transformation.*

(2.5) *If $A$ is strongly arcwise connected and $T(A)=B$ is a one-to-one arc-preserving transformation, then $T$ is topological.*

PROOF. We need only show that $T(A)=B$ is continuous. To this end let $\{x_n\}$ be a sequence of points converging to a point $x$ in $A$. Since $A$ is strongly arcwise connected we lose no generality in assuming that all the points $\{x_n\}$ lie on an arc $\alpha$ in $A$. For each $n$ let $\alpha_n$ be the irreducible subarc of $\alpha$ containing $x+\sum_{i=n}^{\infty} x_i$; then $\prod \alpha_n = x$. Now $T(\prod \alpha_n)=\prod T(\alpha_n)$, since $T$ is one-to-one. Thus $T(x)=\prod T(\alpha_n)$ is the intersection of a monotone decreasing sequence of arcs. Thus $T(x_n)$ converges to $T(x)$, since $T(x_n)$ is contained in $T(\alpha_n)$ for every $n$.

The following example shows that condition of strong arcwise connectivity cannot be omitted:

EXAMPLE. *There exists a one-to-one arc-preserving transformation $T(A)=B$, where $A$ is a cyclicly connected continuum, which is not continuous.*

PROOF. The example will be constructed in the euclidean plane. Let $L$ be the unit interval and for every positive integer $n$ let $A_n$ be a line segment of length $1/n^2$ erected perpendicular to $L$ at the point $1/n$. Define $a_n$ as the end of $A_n$ not on the line $L$ and let $B_n$ be the segment joining the point $a_n$ to the origin. Then the cyclicly connected continuum of the example will consist of the unit interval $L$ together with all the segments $A_n$ and $B_n$.

To construct $B$ let $O$ denote the origin and for every positive integer $n$ define $Q_n$ as the point $(1/n, 0)$. Let $M$ denote the line $x=.7$, and $P_1=(1, 1)$. We may then define an infinite sequence of points

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[4] See Kuratowski and Whyburn, Fundamenta Mathematicae, vol. 16 (1930), pp. 305–331.

$\{P_n\}$ as follows: assuming that $P_{n-1}$ has been defined let $R_{n-1}$ be the point of intersection of the bisector of the angle $OP_{n-1}Q_{n-1}$ with the line $M$, and define $P_n$ as the midpoint of the segment $P_{n-1}R_{n-1}$. Let $C_n$ be the segment $OP_n$ and $D_n$ the segment $P_nQ_n$. The set $B$ is then defined as the sum of $L$ and all the line segments $C_n$ and $D_n$.

The transformation $T(A)=B$ is now easily set up. Define $T(x)=x$ for all $x$ in $L$, and let $T$ send the sum of $A_n$ and $B_n$ topologically into the sum of $C_n$ and $D_n$. Evidently $T(A)=B$ as thus defined is one-to-one and arc-preserving, but not continuous at the origin.

(2.6) *Any locally connected continuum A which is not strongly arcwise connected may be mapped onto the unit circle by an arc-preserving transformation.*

PROOF. Since $A$ is not strongly arcwise connected it contains a sequence of disjoint regions[5] $\{U_n\}$ such that no arc in $A$ intersects infinitely many of the $U_n$. Clearly, the set $M=A-\sum U_n$ and the points of $\sum U_n$ give an upper semi-continuous decomposition of $A$. This decomposition determines a continuous transformation $T_1(A)=A_1$ such that each $T_1(U_n)$ is a component of $A_1-T_1(M)=A_1-p_1$ and $A_1=p_1+\sum T_1(U_n)$. Now for each positive integer $n$ let $L_n$ be the line segment in the euclidean plane between the points $x_0=(0, 0)$ and $x_n=(1/n, 1/n^2)$ and define $A_2=\sum L_n$. Then there exists a continuous transformation $T_2(A_1)=A_2$ such that $T_2(\overline{T_1(U_n)})=T_2(T_1(U_n)+p_1)=L_n$. Finally, let $B$ be the unit circle $x=\cos\theta$, $y=\sin\theta$ $(0\leqq\theta<2\pi)$ and let $B_n$ be the subarc of $B$ given by $0\leqq\theta\leqq(2n-1)\pi/n$. A sequence of topological transformations $T_3(L_n)=B_n$ which in every case maps $x_0$ onto the point $(1, 0)$ of $B$ defines a transformation $T_3(A_2)=B$. Let $\alpha$ be any arc of $A$ and consider its image $T(\alpha)=T_3T_2T_1(\alpha)$ in $B$. Since $T_2T_1(A)=A_2$ is continuous and $\alpha$ can intersect at most a finite number of the regions $U_n$, it follows that $T_2T_1(\alpha)$ is a connected subset of $A_2$ and is contained in some $\sum_{i=1}^{k}L_{n_i}$ $(n_{i+1}>n_i)$. Moreover, by construction $T_3(\sum_{i=1}^{k}L_{n_i})=B_{n_k}$ is continuous and, consequently, $T_3T_2T_1(\alpha)$ is an arc. Thus $T(A)=T_3T_2T_1(A)=B$ is arc-preserving. However, it will be noted that $T$ fails to be continuous at any point in the limit superior of $\{U_n\}$, but is continuous on every arc $\alpha$ of $A$.

3. **Lemmas.** In this section we obtain some preliminary results concerning arc-preserving transformations defined on cyclic locally connected continua. The set $B$ is a topological space.

(3.1) *Let A be cyclic, let $T(A)=B$ be arc-preserving, and let $G=[\alpha]$*

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[5] See Hall and Puckett, *Strongly arcwise connected spaces*, to appear in American Journal of Mathematics.

*be the set of all arcs with endpoints in* $T^{-1}(p')$ *and* $T^{-1}(q')$ *respectively. If the image of every simple closed curve of* $A$ *is an arc, then* $\prod T(\alpha)$ *contains an arc joining* $p'$ *and* $q'$.

Proof. Let $G^*=[\alpha^*]$ be the subcollection of arcs of $G$ which have $p^*\in T^{-1}(p')$ and $q^*\in T^{-1}(q')$ for their endpoints, and let $\alpha_1^*$ and $\alpha_2^*$ be any two arcs of this collection. Now let $\beta_1'$ and $\beta_2'$ be subarcs of $T(\alpha_1^*)$ and $T(\alpha_2^*)$ respectively which have $p'$ and $q'$ as endpoints. Suppose there exists a point $x'$ of $\beta_1'$ not contained in $\beta_2'$; then $\beta_1'+\beta_2'$ contains a simple closed curve $J'$. Therefore, since $T^{-1}(x')$ is disjoint with $\alpha_2^*$, $\alpha_1^*+\alpha_2^*$ contains a simple closed curve $J$ such that $T(J)$ contains $J'$, contrary to hypothesis. Define $\beta'=\beta_1'=\beta_2'$, and suppose there exists a point $p\neq p^*$ of $T^{-1}(p')$. Since $A$ is cyclic it contains an arc $p^*q^*+q^*p$. Now $T(p^*q^*)$ contains $\beta'$, since $p^*q^*$ is an arc of $G^*$. Moreover, $T(q^*p)$ must contain $\beta'$, for otherwise $T(p^*q^*+q^*p)$ would contain a simple closed curve. Because of the symmetry of the above argument it follows that $\prod T(\alpha)$ contains $\beta'$.

(3.2) *If* $T(A)=B$ *is arc-preserving and* $J$ *is a simple closed curve, then* $T(J)=J'$ *is topological or* $J'$ *is an arc.*

Proof. Suppose $J'$ is not an arc. Then, since $J$ is strongly arcwise connected, we need only show that $T(J)=J'$ is one-to-one, by virtue of (2.5). If $T(J)=J'$ is not one-to-one there exist two points $x$ and $y$ of $J$ such that $T(x)=T(y)$. Express $J=\alpha+\beta$, where $\alpha$ and $\beta$ are arcs such that $\alpha\cdot\beta=x+y$. There exist in $\alpha$ distinct points $p$ and $q$ whose images are the endpoints of $T(\alpha)$. (In case $T(\alpha)$ is degenerate these points may be $x$ and $y$.) Let the points be so named that $\alpha=xp+pq+qy$, where any two of the arcs on the right have at most a common endpoint. Now the endpoints of $T(\alpha)$ lie in its subarcs $T(xp)$ and $T(qy)$, which have a common point. Consequently, $T(xp+qy)=T(xp)+T(qy)=T(\alpha)$. Thus the arc $\gamma=px+\beta+qy$ is such that $T(\gamma)=J'$, contrary to the hypothesis that $J'$ is not an arc.

(3.3) *If* $T(A)=B$ *is arc-preserving, where* $A$ *is cyclic, and if there exists a simple closed curve* $J$ *in* $A$ *such that* $T(J)=J'$ *is not an arc, then* $T$ *is one-to-one on* $A$.

Proof. Since $J'$ is not an arc, it follows from (3.2) that $T(J)=J'$ is topological. Let $z$ be any point of $J$ and suppose $z\neq T^{-1}T(z)$. Then there exists a point $z_1$ in $A-J$ such that $T(z_1)=T(z)$. Let $cz_1d$ be an arc in $A$ spanning[6] $J$, and suppose $d\neq z$. Write $J$ as the sum of two simple arcs $\alpha$ and $\beta$ having precisely the points $z$ and $d$ in common.

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[6] An arc $axb$ is said to *span* a point set $M$ provided $M\cdot axb=a+b$.

Now $T(z)=T(z_1)$ and $T$ is topological on $\alpha$. Consequently, since $T(z_1d+\alpha)$ is an arc, we have $T(z_1d)$ contains $T(\alpha)$. Therefore, $T(z_1d+\beta)$ contains $J'$, contrary to the fact that $J'$ is not an arc. Hence for every point $z$ of $J$ we have $z=T^{-1}T(z)$.

Now let $z$ be a point of $A-J$ and let $\alpha$ be an arc through $z$ spanning $J$ and dividing $J$ into two arcs $\beta$ and $\gamma$. From the above and (3.2) it follows that $T(\alpha+\beta)$ is a simple closed curve. Consequently, $z=T^{-1}T(z)$ and, therefore, $T$ is one-to-one on $A$.

(3.4) *Under the hypotheses of* (3.3) *either of the following conditions suffices to make $T$ topological*: (a) *$A$ is strongly arcwise connected, or* (b) *$T$ sends trees into compact sets.*

PROOF. That (a) suffices is immediate from (2.5). To show that (b) suffices we need only establish the continuity of $T$. Assume $T$ is not continuous. Then there exists a sequence of points $\{x_n\}$ converging to a point $x$ of $A$ such that either the sequence $T(x_n)=x_n'$ converges to a point $y'\neq T(x)$ or the set $T(x_n)$ has no limit point. Now there exists a tree $t$ containing infinitely many of the $x_n$, but not $y=T^{-1}(y')$ if $y'$ exists. It follows in either case that $T(t)$ is not compact.[7] This contradiction completes the proof.

(3.5) *Let $J$ be any simple closed curve in the cyclic continuum $A$ and let $T(A)=B$ be arc-preserving but not topological. Then either of the following conditions suffices to make $T(J)=J'$ a free arc*[7] *of $B$*: (a) *$A$ is strongly arcwise connected, or* (b) *the image of each tree in $A$ is a locally connected continuum.*

PROOF. From (3.2) we see that $T(J)=J'$ is topological or $J'$ is an arc. If $T(J)=J'$ is topological, then by (3.4) we see that $T(A)=B$ is topological and the theorem is established. Hence assume that $J'$ is an arc $a'x'b'$ of $B$ which is not a free arc of $B$. To obtain the desired contradiction we first establish the following assertion:

(i) *There exists an arc $uv$ in $A$ such that $T(uv)$ contains a nondegenerate subarc $u'v'$ having exactly the point $v'$ in common with $J'$, where $v'$ is an interior point of the arc $J'$.*

To prove (i) we observe that since $J'$ is not a free arc of $B$ there must exist a sequence of points $x_n'$ of $B-J'$ converging to an interior point $x'$ of $J'$. If $A$ is strongly arcwise connected there exists an arc $N$ in $A$ intersecting infinitely many of the sets $T^{-1}(x_n')$. Then $T(N)$ contains a nondegenerate subarc $u'v'$ satisfying the conditions of (i) and we immediately obtain the arc $uv$ as a subarc of $N$ joining a point of $T^{-1}(u')$ to a point of $T^{-1}(v')$. This proves (i) under our first hypothesis. Assume next that (b) holds. Then there exists a tree $t$ in $A$ intersecting $T^{-1}(x')$ and infinitely many of the sets $T^{-1}(x_n')$. Thus $T(t)$ is a locally connected continuum in $B$ containing $x'$ and infinitely many of the points $x_n'$. Hence $T(t)$ is locally arcwise connected and thus contains an arc $u'v'$ satisfying (i). The arc $uv$ is then obtained as any arc in $t$ joining a point of $T^{-1}(u')$ to a point of $T^{-1}(v')$. This completes the proof of (i).

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[7] An arc $\alpha$ is a *free* arc of $M$ provided $\alpha$ spans $\overline{M-\alpha}$.

Now since $A$ is cyclic there exists an arc $H$ in $A$ intersecting $T^{-1}(u')$ and having its endpoints in $T^{-1}(a')$ and $T^{-1}(b')$. It follows at once from (3.1) that $T(H)$ contains $J'$. Now $T(H)$ contains both $u'$ and $v'$ and hence a subarc joining these points. Since $T(H)$ is an arc, this subarc must contain either $a'$ or $b'$, hence we assume that it contains $a'$ and denote it by $u'a'v'$. From (3.1) it follows that if $M$ is any arc in $A$ having its endpoints in $T^{-1}(u')$ and $T^{-1}(v')$ then $T(M)$ contains $u'a'v'$. But this tells us at once that the image of the arc $uv$ given by (i) must contain a simple closed curve. This contradiction completes the proof.

4. **Principal theorem.** We shall now prove our principal theorem.

(4.1) *Let $T(A)=B$ be arc-preserving, where $A$ is a cyclic locally connected continuum and $B$ is not an arc. Then $T$ is topological if either $A$ is strongly arcwise connected or $T$ is tree-preserving.*

PROOF. Assume $T$ is not topological. Then by (3.5) the image of every simple closed curve $J$ of $A$ is a free arc of $B$ and, consequently, every two points of $B$ lie on a free arc of $B$. Thus to show that $B$ is a simple closed curve it is certainly sufficient to show that $B$ is a locally connected continuum. This follows at once from (2.1) and (2.4) if $A$ is strongly arcwise connected. Hence we need only establish it in the case where $T$ is tree-preserving. That $B$ is compact follows at once from this condition since every convergent sequence of points in $A$ lies on a tree in $A$ and the image of this tree is a tree. Assuming $B$ not locally connected we can find two points $a$ and $b$ in $B$ at which $B$ fails to be locally connected. Let $axb$ be a free arc in $B$ and $d$ a point of $B$ not on this free arc. Then there exists a free arc $dx$ in $B$ where $x$ is an interior point of $axb$. Thus either $a$ or $b$ is interior to the free arc $dx$ of $B$ which is impossible since $B$ is not locally connected at either $a$ or $b$. Hence we have established the fact that under either of our hypotheses $B$ must be a simple closed curve.

We show first that if $z'$ is any point of $B$ there exists an arc $\beta$ of $A$

such that $z'$ is an interior point of $T(\beta)$. Since $B$ is a simple closed curve, it contains an arc $p'z'q'$ having $z'$ as an interior point. Let $\{p_n'\}$ be a sequence of points in the arc $p'z'$ converging monotonically to $z'$. Let $t$ be a tree in $A$ intersecting infinitely many of the sets $T^{-1}(p_n')$. If $T$ is tree-preserving then $T(t)$ is an arc $t'=a'b'$. There exists in $t$ an arc $\alpha$ intersecting both $T^{-1}(a')$ and $T^{-1}(b')$. Since $T(\alpha)=t'$, it follows that $T(\alpha)$ must contain a subarc $\gamma_p=z'x'$ of $z'p'$. If $A$ is strongly arcwise connected the same result may be obtained by taking $\alpha$ as an arc in $A$ which intersects infinitely many of the sets $T^{-1}(p_n')$. Likewise we obtain an arc $\gamma_q=z'y'$ which is a subarc of both $z'q'$ and the image of an arc of $A$. Let $\beta$ be an arc in $A$ intersecting $T^{-1}(x')$, $T^{-1}(y')$, and $T^{-1}(z')$. By virtue of (3.1), $T(\beta)$ contains $\gamma_p+\gamma_q$, an arc which has $z'$ as an interior point.

It follows from the above and the Heine-Borel theorem that there exists a finite number of arcs $\alpha_1, \alpha_2, \cdots, \alpha_n$ $(n\geqq 2)$ in $A$ such that $\sum_{i=1}^{n} T(\alpha_i)=\sum\alpha_i'=B$. Moreover, the $\alpha_i$ may be so selected and named that $\alpha_i'\cdot\alpha_k'$ is empty except for $k=i-1, i, i+1$ $(n+1=1)$. Let $p_i'$ and $q_i'$ be the endpoints of $\alpha_i'$, and assume them so named that $p_{i+1}'$ is a point of $\alpha_i'$ while $q_{i+1}'$ is not. Select in $A$ an arc $\beta_1=p_1q_1+q_1q_2$, where the points $p_1$, $q_1$, and $q_2$ are in $T^{-1}(p_1')$, $T^{-1}(q_1')$, and $T^{-1}(q_2')$ respectively. By (3.1), $T(p_1q_2)$ contains $\alpha_1'$, and consequently a point $p_2$ of $T^{-1}(p_2')$. The subarc $p_2q_2$ of $\beta_1$ has an image which contains $\alpha_2'$, by (3.1). Therefore $T(\beta_i)$ contains $\alpha_1'+\alpha_2'$. To complete the induction assume that an arc $\beta_r$ has been obtained in $A$ such that $T(\beta_r)$ contains $\alpha_1'+\alpha_2'+\cdots+\alpha_{r+1}'$. Select in $A$ an arc $\beta_{r+1}=p_1q_{r+1}+q_{r+1}q_{r+2}$, where the points $p_1$, $q_{r+1}$, and $q_{r+2}$ are in $T^{-1}(p_1)$, $T^{-1}(q_{r+1})$, and $T^{-1}(q_{r+2})$ respectively. By applying (3.1) as above it follows that $T(p_1q_{r+1})$ contains $\alpha_1+\cdots+\alpha_{r+1}$, and, finally, that $T(\beta_{r+1})$ contains $\alpha_1+\cdots+\alpha_{r+2}$. Therefore, by induction, there exists an arc $\beta_{n-1}$ in $A$ such that $T(\beta_{n-1})=B$, which, as a consequence of our supposition, is a simple closed curve. This contradiction completes the proof of the theorem.

(4.11) COROLLARY. *Let $A$ be a locally connected continuum having no local separating points. If $T(A)=B$ is arc-preserving, then either $B$ is an arc or $T$ is topological.*

PROOF. Since $A$ has no local separating point it can contain no cut point. Moreover, $A$ is strongly arcwise connected, since every closed and totally disconnected set of $A$ is contained in an arc.[8] Consequently, $A$ satisfies the hypotheses of (4.1).

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[8] See G. T. Whyburn, *On disconnected sets*, Fundamenta Mathematicae, vol. 18 (1931), pp. 48–60.

The following theorem is an immediate consequence of (4.1) and (2.6):

(4.2) *The class of cyclic strongly arcwise connected continua consists exactly of all cyclic locally connected continua $A$ such that every arc-preserving transformation $T(A)=B$, where $B$ is not an arc, is topological.*

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UNIVERSITY OF CALIFORNIA AT LOS ANGELES

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# A NOTE ON SUBGEOMETRIES OF PROJECTIVE GEOMETRY AS THE THEORIES OF TENSORS[1]

T. L. WADE

Klein's viewpoint (A) of a geometry as the invariant theory of a transformation group, as formulated in the Erlanger Programm in 1870,[2] has played an important part in the study of geometry during the past half century. A number of explicit utilizations of this viewpoint in invariant aspects of algebraic geometry have been made.[3] In the last decade the viewpoint (B) of a geometry as the theory of a tensor has received considerable theoretical discussion and utilization in connection with the new differential geometries.[4] While the adjunction argument, whereby subgeometries of projective geometry result from the latter by holding certain forms latent, has had considerable use,[5] and is closely related to tensor algebra, there seems to have been no explicit treatment of algebraic invariants for subgeometries of projective geometry from the viewpoint (B) with the use of tensor algebra. To indicate how this might be done is the purpose of this paper. The material here is largely an application and continuation of the basic paper by Cramlet.[6]

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[1] Presented to the Society, April 27, 1940.

[2] F. Klein, *Gesammelte Mathematische Abhandlungen*, Berlin, 1921, vol. 1, p. 460.

[3] C. C. MacDuffee, *Euclidean invariants of second degree curves*, American Mathematical Monthly, vol. 33 (1926), pp. 243–252; *Covariants of r-parameter groups*, Transactions of this Society, vol. 39 (1933).

[4] J. A. Schouten and J. Haantjes, *On the theory of the geometric object*, Proceedings of the London Mathematical Society, vol. 42 (1937), pp. 356–376.

[5] H. Weyl, *The Classical Groups: Their Invariants and Representations*, Princeton University Press, 1939, pp. 254–258; H. W. Turnbull, *The Theory of Determinants, Matrices, and Invariants*, Blackie and Son, 1929, chap. 21.

[6] C. M. Cramlet, *The derivation of algebraic invariants by tensor algebra*, this Bulletin, vol. 34 (1928), pp. 334–342.

If in the $(n-1)$ projective geometry we hold the linear form $L_iX^i$, where $L_i=(0, \cdots, 0, 1)$ latent, we have the special numerical covariant tensor $L_i$. Then from Theorems 1 and 2 of Cramlet we have the following theorem:

THEOREM 1. *Affine geometry as a subgeometry of projective geometry is the theory of the tensor* $L_i$.

Similarly, $U_i$ being dual covariant coordinates, holding the quadratic form

$$E^{ij}U_iU_j = 0$$

latent, where

$$E^{ij} = \begin{bmatrix} 1 & 0 & 0\cdots 0 & 0 \\ 0 & 1 & 0\cdots 0 & 0 \\ \cdot & \cdot & \cdot\cdot\cdot\cdot\cdot & \cdot \\ 0 & 0 & 0\cdots 1 & 0 \\ 0 & 0 & 0\cdots 0 & 0 \end{bmatrix},$$

we have a second theorem:

THEOREM 2. *Euclidean geometry as a subgeometry of affine geometry is the theory of the tensor* $E^{ij}$, *and as a subgeometry of projective geometry is the theory of the tensors* $L_i$ *and* $E^{ij}$.

THEOREM 3. *Every euclidean concomitant for a set of ground forms in* $X^i$ *and dual variables* $U_i$, *where* $X^iU_i=0$, *is expressible by composition as a tensor of order zero with the use of the coefficient tensors of the ground forms and the tensors* $\epsilon^{i_1\cdots i_n}$, $\epsilon_{i_1\cdots i_n}$, $L_i$, $E^{ij}$, $X^i$, *and* $U_i$. *The first two of these tensors are the commonly used skew symmetrical numerical tensors.*

In applications it is advantageous to introduce the tensor $A^{i_1\cdots i_{n-1}}$ where $A^{i_1\cdots i_{n-1}} = \epsilon^{i_1\cdots i_n}L_{i_n}$. For $n=3$,

$$A^{ij} = \epsilon^{ijk}L_k = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

As an illustration of an application we give a corallary:

COROLLARY. *An algebraically complete system of euclidean invariants for the ternary cubic curve*

$$C_{ijk}X^iX^jX^k = 0, \qquad i, j, k = 1, 2, 3; C_{ijk} = C_{ikj} = C_{jik},$$

*consists of the seven invariants*

$$I_1 = C_{a_1b_1c_1}C_{a_2b_2c_2}A^{a_1a_2}A^{b_1b_2}E^{c_1c_2},$$

$$I_2 = C_{a_1a_2c_1}C_{b_1b_2c_2}E^{a_1a_2}E^{b_1b_2}E^{c_1c_2},$$

$$I_3 = C_{a_1b_1c_1}C_{a_2b_2d_1}C_{a_3c_2d_2}\epsilon^{a_1a_2a_3}A^{b_1b_2}A^{c_1c_2}A^{d_1d_2},$$

$$I_4 = C_{a_1b_1e_1}C_{a_2b_2f_1}C_{c_1d_1e_2}C_{c_2d_2f_2}A^{a_1a_2}A^{b_1b_2}A^{c_1c_2}A^{d_1d_2}A^{e_1e_2}A^{f_1f_2},$$

$$I_5 = C_{a_1b_1c_1}C_{b_2c_2d_1}C_{c_3d_2a_2}C_{d_3a_3b_3}\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}\epsilon^{d_1d_2d_3},$$

$$I_6 = C_{a_1b_1c_1}C_{a_2b_2d_1}C_{b_3c_2e_1}C_{c_3a_3f_1}C_{d_2e_2f_2}C_{d_3e_3f_3}$$
$$\cdot\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}\epsilon^{d_1d_2d_3}\epsilon^{e_1e_2e_3}\epsilon^{f_1f_2f_3},$$

$$I_7 = C_{a_1b_1c_1}C_{d_1e_1f_1}A^{a_1d_1}\epsilon^{b_1e_1g_1}E^{c_1f_1}$$
$$\cdot C_{a_2b_2c_2}C_{d_2e_2f_2}A^{a_2d_2}\epsilon^{b_2e_2g_2}E^{c_2f_2}$$
$$\cdot C_{a_3b_3c_3}C_{d_3e_3f_3}A^{a_3d_3}\epsilon^{b_3e_3g_3}E^{c_3f_3}C_{g_1g_2g_3}.$$

In expanded form

$$I_1 = 2[C_{111}C_{122} - (C_{112})^2 - (C_{122})^2 + C_{112}C_{222}],$$

$$I_2 = (C_{111} + C_{122})^2 + (C_{112} + C_{222})^2,$$

$$I_3 = 6\begin{vmatrix} C_{111} & C_{112} & C_{113} \\ C_{211} & C_{221} & C_{231} \\ C_{221} & C_{222} & C_{223} \end{vmatrix}.$$

The expansions of the other invariants are considerably longer; $I_5$ contains twenty-five terms and $I_6$ one hundred and three terms. These seven invariants are, except in some cases for a constant factor, the members of the algebraically complete system of euclidean invariants for the cubic curve found through geometric means by Thomae.[7]

An invariant study of the geometry associated with the Galilei-Newton group of ten parameters has been made by Weitzenböck using the symbolic notation.[8] In the Galilei-Newton geometry, with $X^i$ $(i=1, \cdots, 5)$ the homogeneous coordinates of a point, the basic latent forms may be represented by

$$K_iX^i,$$

where $K_i=(0, 0, 0, 0, 1)$;

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[7] J. Thomae, *Ueber orthogonale Invarianten der Curven dritter Ordnung*, Berichte der Saechsischen Akademie der Wissenschaften, Leipzig, vol. 51 (1899), pp. 317–353.

[8] R. Weitzenböck, *Die Invarianten der Galilei-Newton-Gruppe*, Mathematische Annalen, vol. 80 (1920).

$$L_i X^i,$$

where $L_i = (0, 0, 0, 1, 0)$;

$$Q^{ij} U_i U_j,$$

where

$$Q^{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Here $X^i U_i = 0$.

From this approach we are led to the following theorem:

THEOREM 4. *Every concomitant in Galilei-Newton geometry for a set of ground forms in $X^i$ and dual variables $U_i$ is expressible by composition as a tensor of order zero with the coefficient tensors of the ground forms and the tensors*

$$\epsilon^{i_1 \cdots i_5},\ \epsilon_{i_1 \cdots i_5},\ K_i,\ L_i,\ Q^{ij},\ X^i,\ U_i.$$

This theorem, in directness of approach and in simplicity of application, seems to have advantages over the concluding statement of Weitzenböck, which concerns itself with twelve basic symbolic factors for the Galilei-Newton group.

UNIVERSITY OF ALABAMA

# A NOTE ON QUASI-METRIC SPACES[1]

GEORGE E. ALBERT

1. **Introduction.** In a recent abstract[2] the author introduced an important application of a class of neighborhood spaces in which pairs of points are required to satisfy only a very weak separation axiom (due to Kolmogoroff):

(K) *If $x$ and $y$ are distinct points, then at least one of these points has a neighborhood which does not contain the other.*

In the present note it is shown that a very useful generalized distance function may be defined in certain of these spaces.

Clearly, any such distance function must be an asymmetric one. W. A. Wilson[3] considered the definition of asymmetric distances in certain spaces which satisfy stronger separation axioms than K. It is shown here that a slight modification of one of the axioms in [W] allows the extension of a large part of the theory developed there to spaces subject to K.

Since many of the theorems and proofs in [W] remain valid here with only very obvious changes, this note will be limited to a mere sketch concerning new properties which arise under the weaker axioms used. The reader should have no difficulty in adapting the more complete discussion given in [W] to the case studied here.

2. **The distance function.** The symbol 1 will denote a space of points; points and point sets in 1 will be denoted by small and capital letters respectively.

The space 1 will be said to be *quasi-metric* if for every pair of points $x$, $y$ in 1 there are defined two non-negative numbers $xy$ and $yx$, not necessarily equal, which satisfy the following postulates:

I. $xy=yx=0$ *if and only if* $x=y$;[4]
II. *for any three points*, $xy\leqq xz+zy$.

If for some pair of points $xy=0\neq yx$, then the point $x$ will be said to be *adjacent* to the point $y$. It is to be emphasized that, regardless of the connotation of symmetry familiar with the word adjacent, the relationship to which it refers here is an asymmetric one.

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[1] Presented to the Society, April 13, 1940.

[2] Abstract 46-3-138, this Bulletin.

[3] W. A. Wilson, *On quasi-metric spaces*, American Journal of Mathematics, vol. 53 (1931), p. 675. Hereafter this paper will be referred to as [W].

[4] In [W] the stronger axiom: $xy=0$ if and only if $x=y$ was used. This excludes the case $xy=0\neq yx$ allowed here.

3. **Limits, closed and open sets.** The nature of the function $xy$ leads to the definition of three separate types of limit points and the corresponding classes of closed and open sets. A point $p$ will be called a *u-limit*, *l-limit*, or a *c-limit* of a point set $A$ (of a sequence $\{x_n\}$) if for every positive number $\epsilon$ there exists a point $x$ in $A-p$ such that $px<\epsilon$, $xp<\epsilon$, or both simultaneously, respectively (if there is a number $N$ such that $px_n<\epsilon$, and so on, for $n>N$).

The point set $S_u(a, r)$ consisting of all points $x$ such that $ax<r$ will be called a *u-sphere* of center $a$ and radius $r$. Similarly one defines the spheres $S_l(a, r)$ and $S_c(a, r)$ by the relations $xa<r$ and both $ax<r$ and $xa<r$.

A point set will be termed *u-open*, *l-open*, or *c-open* if for every point $p$ in the set some sphere $S_u$, $S_l$, or $S_c$, respectively, about $p$ is contained entirely within the set. Such sets will be denoted by $G_u$, $G_l$, and $G_c$ respectively. A point set will be termed *u-closed*, *l-closed*, or *c-closed* if it contains all of its limit points of the respective class. These sets will be denoted by $F_u$, $F_l$, and $F_c$ respectively.

The space 1 will be called *separable* if there exists a denumerable sequence of points which is everywhere dense in 1 in the sense of $c$-limits.

Just as in the theory of ordinary topological spaces, one proves that the complement of a set $G_u$ ($G_l$ or $G_c$) is a set $F_u$ ($F_l$ or $F_c$). It is evident that every set $F_u$ or $F_l$ is also a set $F_c$ and that every set $G_u$ or $G_l$ is also a set $G_c$; however the converses are not true.

It is to be noted that a point $p$ may be a $u$-limit and at the same time an $l$-limit of a point set $A$ without being a $c$-limit of that set. On the other hand, for sequences one has the following theorem:

THEOREM 1. *If* $x=u\text{-lim}\, x_n$ *and* $x=l\text{-lim}\, x_n$ *then* $x=c\text{-lim}\, x_n$. *Moreover, there exists no other c-limit of the sequence* $\{x_n\}$ *and the Cauchy criterion of convergence is satisfied by the sequence.*

PROOF. The first clause is immediate from the definitions.

Suppose that $x$ and $y$ are both $c$-limits of the sequence $\{x_n\}$. It follows that for an arbitrary positive number $\epsilon$ there exist numbers $n'$ and $n''$ such that $xy\leqq xx_n+x_ny<\epsilon$ for $n>n'$ and $yx\leqq yx_n+x_nx<\epsilon$ for $n>n''$. Thus $xy=0=yx$ and by I, $x=y$.

Again, $x=c\text{-lim}\, x_n$ implies that for arbitrary $\epsilon>0$ there is an $n'$ such that $x_nx_m\leqq x_nx+xx_m<\epsilon$ for $n>n'$ and the Cauchy criterion is satisfied.

The following theorems indicate the relations between the several limits of a sequence and the property of adjacency.

THEOREM 2. *If* $x=l$-lim $x_n$ *and* $y=u$-lim $x_n$ *for the same sequence* $\{x_n\}$, *then either* $x=y$ *or else* $y$ *is adjacent to* $x$.

The proof is immediate from $yx \leqq yx_n + x_n x$ and the fact that $yx_n \to 0$ and $x_n x \to 0$.

COROLLARY. *If* $x=c$-lim $x_n$ *and* $y=u$-lim $x_n$ *then either* $y=x$ *or* $y$ *is adjacent to* $x$; *if* $y=l$-lim $x_n$, *then either* $y=x$ *or* $x$ *is adjacent to* $y$.

It is worthy of note that if a point $x$ is adjacent to a point $y$ then $x$ is a $u$-limit of the point set consisting of the single point $y$. Thus, if the usual definition of connected set be adopted with limits taken in the $u$-sense, the point set $(x+y)$ forms a connected set.

4. **Topological versus quasi-metric spaces.** Hausdorff[5] stated the following axioms for topological spaces:

(A) Every point $p$ has a neighborhood $U_p$. For every $U_p$, $p \in U_p$.

(B) For every two neighborhoods $U_p$ and $V_p$ of the point $p$, there is a third $W_p \subset U_p V_p$.

(C) For every point $q$ in $U_p$, there is a neighborhood $U_q \subset U_p$.

(D) To each point $p$ there corresponds a denumerable set of neighborhoods such that every neighborhood $U_p$ contains a neighborhood of that set.

(E) There exists a denumerable set of neighborhoods $\{U_n\}$ such that if $U_p$ is any neighborhood of the point $p$, then there is an $n$ such that $p \in U_n \subset U_p$.

THEOREM 3. *Let* 1 *denote a quasi-metric space and, for each point* $x$ *in* 1, *let the class of spheres* $S_u(x, r)$ *of rational radii* $r$ *be termed neighborhoods of* $x$. *Then* 1 *is a topological space satisfying the axioms* A, B, C, D, *and* K. *The u-limits of sequences of points in* 1 *are the topological limits of the sequences and conversely. If the space is separable, Axiom* D *may be replaced by the stronger Axiom* E. *The theorem remains valid if the spheres* $S_u$ *are replaced by the spheres* $S_l$ *and u-limits by l-limits.*

The proof is evident.

THEOREM 4. *Let* 1 *be a neighborhood space satisfying the axioms* A, B, C, E, *and* K. *Then distances can be defined in* 1 *so that the space is a separable, quasi-metric space. A topological limit of a sequence of*

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[5] Hausdorff, *Mengenlehre*, Berlin, 1935, pp. 228, 229.

*points in* 1 *is a u-limit of the sequence and conversely. The theorem remains valid if u-limit is replaced by l-limit provided that a corresponding change be made in the distance function.*

Since the proof of this theorem is long but follows that of Theorem IV, §7, in [W] with only a few minor changes, it seems sufficient to state the definition of the distance function for the case of $u$-limits.

Let the denumerable system of neighborhoods postulated by Axiom E be denoted by $\{U_n\}$. For any pair of points $x$ and $y$ in 1 and for each $n=1, 2, 3, \cdots$, define the function $f_n(x, y)$ to be equal 1 if $x$ is in $U_n$ and $y$ is in $1-U_n$, or equal to 0 in case $x$ is in $1-U_n$ or in case $x+y$ is in $U_n$. Then define the function

$$xy = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x, y).$$

It should be remarked that a quasi-metric is not necessarily a continuous function of either variable.

5. **Applications.** The distance function described above furnishes an exceedingly simple topologization for a very general type of collection of closed point sets. Let $S$ denote a compact metric space and let $S=\sum X$ be any decomposition of $S$ into closed point sets. It is not assumed that the sets of the decomposition are disjoint. However it is assumed that if $X_1$ and $X_2$ are distinct elements of the decomposition which overlap, then $X_1X_2=X$ is again an element of the decomposition. Moreover, if $X_3$ is any element which intersects $X$, then $X_1X_3=X=X_2X_3$. Such a collection is afforded by the *upper semi-continuous collection of the second type* introduced by R. L. Moore[6] and, in particular, by the so-called hyperspace of cyclic elements of a continuous curve. If, now, the elements of the collection are thought of as points of a new space 1, the topologization is introduced as follows: if $X$ and $Y$ are two elements of 1, the distance function $XY=$ the greatest lower bound of the numbers $r$ such that, as seen in the space $S$, every point $x$ of the set $X$ has a distance from the set $Y$ which is less than $r$. The space is easily shown to be quasi-metric in the sense described in the preceding paragraphs.

OHIO STATE UNIVERSITY

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[6] The Rice Institute Pamphlet, vol. 23, no. 1, 1936, p. 42.

# CESÀRO SUMMABILITY OF A CLASS OF SERIES

M. S. MACPHAIL

The following theorem has recently been proved by H. L. Garabedian:[1] If $a_n$ is a polynomial in $n$ of degree $k-1$, the series $\sum_{n=0}^{\infty}(-1)^n a_n$ is summable $(C, k)$ but not $(C, k-1)$ to the value $\sum_{i=0}^{k-1}2^{-i-1}\Delta^i a_0$. Here and elsewhere in this paper $k$ is understood to be a fixed positive integer.

Our present object is to obtain some extensions of this result. One of these may be stated at once, the proof being given at the end of the paper.

THEOREM 2. *Let $a_n$ be a polynomial in $n$ of degree $k-1$, and let $z$ be a complex number such that $|z|=1$, $z\neq 1$. Then the series $\sum_{n=0}^{\infty}a_n z^n$ is summable $(C, k)$ but not $(C, k-1)$ to the value $-\sum_{m=0}^{k-1}z^m(z-1)^{-m-1}\Delta^m a_0$.*

Before stating our first theorem we require the following definitions. Let $f_n$ be a periodic function of the integer $n$; that is to say, let there be an integer $p$ such that $f_{n+p}=f_n$ for all values of $n$. Let $M(f)$ denote the mean value of $f_n$ over a period, thus $M(f)=(1/p)\sum_{n=0}^{p-1}f_n$. Suppose now that $f_n$ is a periodic function of mean value zero; that is, let $M(f)=0$. Writing $f(n)$, $a(n)$ in place of $f_n$, $a_n$, set

$$\text{(a)}\qquad \begin{array}{lll} f_1(n) = \sum_{j=0}^{n} f(j), & c_1 = M(f_1), & F_1(n) = f_1(n) - c_1, \\ f_2(n) = \sum_{j=0}^{n} F_1(j), & c_2 = M(f_2), & F_2(n) = f_2(n) - c_2, \\ f_3(n) = \sum_{j=0}^{n} F_2(j), & c_3 = M(f_3), & F_3(n) = f_3(n) - c_3, \end{array}$$

and so on; this procedure ensures that $F_i(n)$ is periodic with mean value zero $(i=1, 2, \cdots)$. In terms of these definitions we prove the following theorem:

THEOREM 1. *Let $a(n)$ be a polynomial in $n$ of degree $k-1$, and $f(n)$ a periodic function with mean value zero. Then the series $\sum_{n=0}^{\infty}f(n)a(n)$ is summable $(C, k)$ but not $(C, k-1)$ to the value $\sum_{i=0}^{k-1}c_{i+1}\Delta^i a_0$.*

If $f(n)=(-1)^n$ it is easily verified that $c_{i+1}=2^{-i-1}$, which shows that the present theorem includes that of Garabedian as a special case. The method of proof is entirely different.

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[1] This Bulletin, vol. 45 (1939), pp. 592–596. In the theorem as here stated it is required that $a_n$ satisfy the conditions $\Delta^{k-1}a_0\neq 0$, $\Delta^i a_0=0$ $(i\geqq k)$. It is easily seen that this is equivalent to requiring $a_n$ to be a polynomial in $n$ of degree $k-1$.

We shall make use of the notation of finite differences. We recall that $\Delta a(n) = a(n) - a(n+1)$, $\Delta^2 a(n) = \Delta(\Delta a(n))$, $\cdots$, and $E^j a(n) = a(n+j)$. We shall require the well known equation, in symbolic form,

$$\Delta^k = (1 - E)^k = 1 - C_{k,1}E + C_{k,2}E^2 - \cdots + (-1)^k E^k, \tag{1}$$

where the $C_{k,i}$ are the binomial coefficients. We also write $n^{(k)} = n(n-1)(n-2)\cdots(n-k+1)$, and by convention $n^{(0)} = 1$, $\Delta^0 a(n) = a(n)$; then the successive differences of $n^{(k)}$ are given by

$$\Delta^i n^{(k)} = (-1)^i k^{(i)} n^{(k-i)}, \qquad 0 \leqq i \leqq k. \tag{2}$$

We require two algebraic identities, which are stated as lemmas.

LEMMA 1. *We have, for $r$ a positive integer,*

$$\sum_{n=0}^{r} f(n)a(n) = \sum_{i=0}^{k-1} c_{i+1}\Delta^i a(0) + \sum_{i=0}^{k-1} F_{i+1}(r)\Delta^i a(r+1) + \sum_{n=0}^{r} F_k(n)\Delta^k a(n).$$

For,

$$\begin{aligned}
\sum_{n=0}^{r} f(n)a(n) &= f(0)a(0) + f(1)a(1) + \cdots + f(r)a(r) \\
&= f_1(0)a(0) + [f_1(1) - f_1(0)]a(1) + \cdots \\
&\quad + [f_1(r) - f_1(r-1)]a(r) \\
&= [F_1(0) + c_1]a(0) + [F_1(1) - F_1(0)a](1) + \cdots \\
&\quad + [F_1(r) - F_1(r-1)]a(r) \\
&= c_1 a(0) + F_1(0)[a(0) - a(1)] + \cdots \\
&\quad + F_1(r-1)[a(r-1) - a(r)] \\
&\quad + F_1(r)[a(r) - a(r+1)] + F_1(r)a(r+1) \\
&= c_1 a(0) + F_1(r)a(r+1) + \sum_{n=0}^{r} F_1(n)\Delta a(n).
\end{aligned}$$

The lemma follows by repeated applications of this device.

LEMMA 2. *The expressions*

$$\frac{(r+1-n)(r+2-n)\cdots(r+k-n)}{(r+1)(r+2)\cdots(r+k)} \tag{3}$$

*and*

$$\begin{aligned}(4)\qquad & 1 - C_{k,1}\frac{n}{r+1} + C_{k,2}\frac{n(n+1)}{(r+1)(r+2)} \\ & - C_{k,3}\frac{n(n+1)(n+2)}{(r+1)(r+2)(r+3)} + \cdots \\ & + (-1)^k\frac{n(n+1)\cdots(n+k-1)}{(r+1)(r+2)\cdots(r+k)}\end{aligned}$$

*are algebraically identical.*

To prove this, set

$$G(t) = \frac{n(n+1)\cdots(n+t-1)}{(r+1)(r+2)\cdots(r+t)}, \qquad t \text{ a positive integer,}$$
$$G(0) = 1.$$

Then, if $EG(t)=G(t+1)$, the expression (4) may be written

$$\begin{aligned}G(0) - C_{k,1}EG(0) + C_{k,2}E^2G(0) - \cdots + (-1)^kE^kG(0) \\ = (1-E)^kG(0) = \Delta^kG(0),\end{aligned}$$

by (1). We have therefore to show that $\Delta^kG(0)$ is equal to the expression (3). We write

$$\begin{aligned}\Delta G(t) &= \frac{n(n+1)\cdots(n+t-1)}{(r+1)(r+2)\cdots(r+t)} - \frac{n(n+1)\cdots(n+t)}{(r+1)(r+2)\cdots(r+t+1)} \\ &= \frac{n(n+1)\cdots(n+t-1)}{(r+1)(r+2)\cdots(r+t+1)}(r+1-n), \qquad t \geqq 1.\end{aligned}$$

In particular

$$\Delta G(1) = \frac{n}{(r+1)(r+2)}(r+1-n).$$

Also, since $G(0)=1$ and since $G(1)=n/(r+1)$, we have $\Delta G(0) = (r+1-n)/(r+1)$. Similarly we find

$$\Delta^2G(t) = \frac{n(n+1)\cdots(n+t-1)}{(r+1)(r+2)\cdots(r+t+2)}(r+1-n)(r+2-n),$$

$$\Delta^2G(1) = \frac{n}{(r+1)(r+2)(r+3)}, \quad \Delta^2G(0) = \frac{(r+1-n)(r+2-n)}{(r+1)(r+2)}.$$

Proceeding in this way, we obtain the desired result by induction.

PROOF OF THEOREM 1. The series $\sum_{n=0}^{\infty} f(n)a(n)$ is said to converge $(C, k)$ to the value $S$ provided the expression

$$\sum_{n=0}^{r} \frac{(r+1-n)(r+2-n)\cdots(r+k-n)}{(r+1)(r+2)\cdots(r+k)} f(n)a(n)$$

tends to $S$ as $r$ becomes infinite. Now any polynomial $a(n)$ of degree $k-1$ can be expressed in the form $A_1(n-1)^{(k-1)}+A_2(n-1)^{(k-2)}+\cdots$ where $A_1, A_2, \cdots$ are constants. Hence we need only consider

$$\sum_{n=0}^{r} \frac{(r+1-n)(r+2-n)\cdots(r+k-n)}{(r+1)(r+2)\cdots(r+k)} (n-1)^{(k-1)}f(n),$$

which by Lemma 2 is equal to

$$\sum_{n=0}^{r}\left\{1-C_{k,1}\frac{n}{r+1}+C_{k,2}\frac{n(n+1)}{(r+1)(r+2)}-\cdots\right\}(n-1)^{(k-1)}f(n). \tag{5}$$

We consider the terms of this expression separately. By (2) and Lemma 1 we have

$$\begin{aligned}
\sum_{n=0}^{r}(n-1)^{(k-1)}f(n) &= \sum_{i=0}^{k-1} c_{i+1}(-1)^i(k-1)^{(i)}(-1)^{(k-1-i)} \\
&\quad + \sum_{i=0}^{k-1} F_{i+1}(r)(-1)^i(k-1)^{(i)}r^{(k-1-i)}, \\
\sum_{n=0}^{r} n^{(k)}f(n) &= \sum_{i=0}^{k-1} F_{i+1}(r)(-1)^i k^{(i)}(r+1)^{(k-i)} \\
&\quad + (-1)^k k!\sum_{n=0}^{r} F_k(n), \\
\sum_{n=0}^{r}(n+1)^{(k+1)}f(n) &= \sum_{i=0}^{k-1} F_{i+1}(r)(-1)^i(k+1)^{(i)}(r+2)^{(k+1-i)} \\
&\quad + (-1)^k(k+1)^{(k)}\sum_{n=0}^{r}(n+1)F_k(n), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
\sum_{n=0}^{r}(n+k-1)^{(2k-1)}f(n) &= \sum_{i=0}^{k-1} F_{i+1}(r)(-1)^i(2k-1)^{(i)}(r+k)^{(2k-1-i)} \\
&\quad + (-1)^k(2k-1)^{(k)}\sum_{n=0}^{r}(n+k-1)^{(k-1)}F_k(n).
\end{aligned}$$

Multiply the above equations by

$$1, \quad -\frac{C_{k,1}}{r+1}, \quad \frac{C_{k,2}}{(r+1)(r+2)}, \cdots,$$

respectively, and add. We then have on the left the expression (5) and on the right

$$\begin{aligned}\sum_{i=0}^{k-1} c_{i+1}(-1)^i(k-1)^{(i)}(-1)^{(k-1-i)} \\ &+ \sum_{i=0}^{k-1} F_{i+1}(r)(-1)^i r^{(k-1-i)}(1-E)^k(k-1)^{(i)} \\ &+ \text{terms involving } \sum_{n=0}^{r}.\end{aligned}$$

Now, by (1), $(1-E)^k(k-1)^{(i)}=\Delta^k(k-1)^{(i)}=0$ for $0 \leqq i \leqq k-1$. Also, the terms involving $\sum_{n=0}^{r}$ can easily be shown to approach zero as $r$ becomes infinite. We have therefore proved that the series $\sum_{n=0}^{\infty}(n-1)^{(k-1)}f(n)$ converges $(C, k)$ to the value $\sum_{i=0}^{k-1}c_{i+1}(-1)^i \cdot (k-1)^{(i)}(-1)^{(k-1-i)}$, or, setting $b(n)=(n-1)^{(k-1)}$, $\sum_{n=0}^{\infty}b(n)f(n)$ converges $(C, k)$ to the value $\sum_{i=0}^{k-1}c_{i+1}\Delta^i b(0)$.

Theorem 1 now follows without difficulty. It is easily seen on examining the argument that no order of summation less than $k$ will serve.

We now prove Theorem 2. If am $z$ is a rational multiple of $\pi$, say am $z=2\pi q/p$, where $p$ and $q$ are integers, the function $f(n)=z^n$ is periodic with period $p$ and mean value zero. It is easily verified that, for $m \geqq 0$, the equations (a) become

$$\begin{aligned} f_{m+1}(n) &= \frac{z^m}{(z-1)^m}\,\frac{z^{n+1}-1}{z-1}, \qquad c_{m+1} = -\frac{z^m}{(z-1)^{m+1}}, \\ F_{m+1}(n) &= \frac{z^{m+1}}{(z-1)^{m+1}}\,z^n, \end{aligned} \tag{b}$$

whence Theorem 2 follows at once, as a special case of Theorem 1. If am $z$ is not a rational multiple of $\pi$, then we define $M(f)$ as $\lim_{r\to\infty}(1/r)\sum_{n=0}^{r-1}f(n)$. Then the equations (b) still hold, and the theorem follows by the same argument as Theorem 1.

ACADIA UNIVERSITY

# A CHARACTERIZATION OF THE GROUP OF HOMOGRAPHIC TRANSFORMATIONS

P. O. BELL

1. **Introduction.** The objectives of this note are three-fold: (1) to present a new differential geometric characterization of the group of homographic transformations of a complex variable, (2) to interpret in geometrical language the significance of the invariance of the Schwarzian derivative under a homographic transformation, and (3) to characterize a general homographic transformation by its unique association with two families of concentric circles.

2. **Preliminaries.** Let the equation

$$w = w(z) \tag{1}$$

denote a conformal representation of the points $z=x+iy$ of a region $R$ of the $z$-plane on the points $w=u+iv$ of a region $\bar{R}$ of the $w$-plane, whereby a general curve $C$ is transformed into a curve $\bar{C}$. Let $\gamma$ and $\bar{\gamma}$ denote the curvatures of $C$ and $\bar{C}$ at corresponding points $z$ and $w$, and let $s$ and $\bar{s}$ denote corresponding lengths of arc of $C$ and $\bar{C}$. For a given transformation (1) it is well known that the rate of variation $ds/d\bar{s}$ is a function $\lambda(x, y)$ which may be expressed in any one of the following forms $(u_x^2+u_y^2)^{-1/2}$, $(u_x^2+v_x^2)^{-1/2}$, $(u_y^2+v_y^2)^{-1/2}$, $(v_x^2+v_y^2)^{-1/2}$.

Comenetz[1] (using a different notation) has obtained, by elementary methods, the formula

$$\bar{\gamma} = \gamma\lambda + \lambda_y \cos\theta - \lambda_x \sin\theta, \tag{2}$$

wherein $\theta = \arctan(dy/dx)$, which is *the law of transformation of curvature in conformal mapping*, and the formula

$$d\bar{\gamma}/d\bar{s} = \lambda^2 d\gamma/ds + \lambda[\lambda_{xy}\cos 2\theta + \tfrac{1}{2}(\lambda_{yy} - \lambda_{xx})\sin 2\theta], \tag{3}$$

which is *the law of transformation of the rate of change of curvature with respect to arc length, under conformal mapping.*

3. **A new characterization of the group of homographic transformations.** It is known that the most general directly conformal transformation which carries circles into circles (including straight lines) is a homographic transformation. The transformation (1) will be such a transformation if and only if

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[1] Comenetz, *Kasner's invariant and trihornometry*, The American Mathematical Monthly, vol. 45 (1938), pp. 82–87.

$$(4) \qquad \lambda_{xx} \equiv \lambda_{yy}, \qquad \lambda_{xy} \equiv 0.$$

For if these relations hold, it follows from (3) that

$$d\bar{\gamma}/d\bar{s} = \lambda^2 d\gamma/ds.$$

Hence circles are transformed into circles. Conversely, if circles are transformed by (1) into circles, we must have at corresponding points of an arbitrarily selected circle $C$ and its correspondent $\bar{C}$

$$d\bar{\gamma}/d\bar{s} = \lambda^2 d\gamma/ds = 0.$$

Equation (3) also holds. Hence, no matter what the direction of $C$ at $z$ may be, the following equation must hold:

$$(5) \qquad \lambda(\lambda_{yy} - \lambda_{xx})(\sin 2\theta)/2 + \lambda\lambda_{xy} \cos 2\theta = 0.$$

This equation must, therefore, be satisfied independently of $\theta$, and conditions (4) necessarily follow. We may state, therefore, the following theorem.

THEOREM 1. *A necessary and sufficient condition that a conformal transformation be a homographic transformation is that the associated function* $\lambda(x, y)$ *satisfy both of the following identities*

$$\lambda_{xx} - \lambda_{yy} \equiv 0, \qquad \lambda_{xy} \equiv 0.$$

*A geometric interpretation of this condition is that at a general point* $w$ *of the curve* $\bar{C}$ *the rate of variation* $d\bar{\gamma}/d\bar{s}$ *of the curvature of* $\bar{C}$ *per unit length of arc* $\bar{s}$ *is independent of the direction of the curve* $C$ *at the corresponding point* $z$.

Equation (3) shows that this is equivalent to the following characterization.

THEOREM 2. *The group of homographic transformations consists of all of the conformal transformations under which the differential form* $d\gamma ds$ *is absolutely invariant.*

The following theorem may be deduced, similarly, in consideration of equation (2).

THEOREM 3. *A necessary and sufficient condition that a transformation* (1) *be a directly conformal collineation is that the associated function* $\lambda(x, y)$ *satisfy both of the identities*

$$\lambda_x \equiv 0, \qquad \lambda_y \equiv 0.$$

*Under this condition the differential form* $\gamma ds$ *is an absolute invariant of the transformation* (1).

Since $\gamma = d\theta/ds$ where $dz/ds = e^{i\theta}$, the differential form $d\gamma ds$ may be written in the form

$$(d^2\theta/ds^2)(ds)^2 \tag{6}$$

wherein $\theta = -i \log (dz/ds)$.

4. **The Schwarzian derivative.** Consider a curve defined by

$$z = x(t) + iy(t),$$

wherein $x$ and $y$ are functions of a real variable $t$. It is known that the Schwarzian derivative

$$\{z, t\} \equiv (d^3z/dt^3)/(dz/dt) - \tfrac{3}{2}[(d^2z/dt^2)/(dz/dt)]^2$$

is an absolute invariant under the homographic transformations. Let us investigate the geometric significance of the invariance of this derivative.

We find that

$$z''/z' = i\gamma \tag{7}$$

where accents indicate differentiation with respect to $s$. On differentiating the members of equation (7) with respect to $s$ we obtain

$$z'''/z' - (z''/z')^2 \equiv i d\gamma/ds. \tag{8}$$

Making use of (7) and (8) we deduce

$$\{z, s\} \equiv i d\gamma/ds + \gamma^2/2. \tag{9}$$

If we make a change of variable by the formula[2]

$$\{z, t\} \equiv \{z, s\}(ds/dt)^2 + \{s, t\}, \tag{10}$$

we obtain

$$\{z, t\} \equiv (i d\gamma/ds + \gamma^2/2)(ds/dt)^2 + \{s, t\}. \tag{11}$$

The real and imaginary components of $\{z, t\}$,

$$R \equiv (\gamma^2/2)\,(ds/dt)^2 + \{s, t\}, \quad I \equiv (d\gamma/ds)\,(ds/dt)^2,$$

are, themselves, absolute invariants of the group of homographic transformations. Thus, if (1) is homographic, we have

$$(\gamma^2/2)(ds/dt)^2 + \{s, t\} \equiv (\bar{\gamma}^2/2)(d\bar{s}/dt)^2 + \{\bar{s}, t\}, \tag{12}$$

$$(d\gamma/ds)(ds/dt)^2 \equiv (d\bar{\gamma}/d\bar{s})(d\bar{s}/dt)^2. \tag{13}$$

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[2] For the change of variable formula see, for example, Ford, *Automorphic Functions*, McGraw-Hill, 1929, p. 99.

If now we put $t=s$ in (12), we find

$$2\{\bar{s}, s\} \equiv \gamma^2 - \bar{\gamma}^2(d\bar{s}/ds)^2. \tag{14}$$

Likewise equation (12) yields

$$2\{s, \bar{s}\} \equiv \bar{\gamma}^2 - \gamma^2(ds/d\bar{s})^2 \tag{15}$$

on substituting $\bar{s}$ for $t$.

Equation (13) expresses the invariance of the form $d\gamma ds$. Making use of this equation and equations (14) and (15), we deduce

$$2\{\bar{s}, s\} \equiv \gamma^2 - \bar{\gamma}^2(d\gamma/d\bar{\gamma})^2, \tag{16}$$

$$2\{s, \bar{s}\} \equiv \bar{\gamma}^2 - \gamma^2(d\bar{\gamma}/d\gamma)^2. \tag{17}$$

These equations express the significance of the invariance of the Schwarzian derivatives $\{z, s\}$ and $\{z, \bar{s}\}$ as intrinsic geometric relations between any pair of curves $C$, $\overline{C}$ which correspond under a homographic transformation. To complete the geometric interpretations of (16) and (17) let us recall the significance of the Schwarzian derivative of a real function.[3] Consider two real functions $\sigma=\sigma(s)$ and $\bar{\sigma}=\bar{\sigma}(s)$ which are chosen to satisfy

$$\{\sigma, s\} \equiv \{\bar{\sigma}, s\}$$

identically in $s$. This relation is necessary and sufficient that $\sigma(s)$ and $\bar{\sigma}(s)$ be connected by a homographic transformation

$$\bar{\sigma}(s) = [a\sigma(s) + b]/[c\sigma(s) + d], \tag{18}$$

wherein $a$, $b$, $c$, $d$ are constants and $ad-bc\neq 0$. The relation (18) is also necessary and sufficient that corresponding to any set of four values $s=s_j$, $(j=1, 2, 3, 4)$, the cross-ratios

$$(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \qquad (\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3, \bar{\sigma}_4)$$

are identical. If as $s$ varies, points $P$ and $\overline{P}$ describe curves whose corresponding lengths of arc are defined by $\sigma=\sigma(s)$ and $\bar{\sigma}=\bar{\sigma}(s)$, the movements of these points will be called *projectively applicable*. This designation is suggested by the property that the development of these movements along a straight line produces projectively equivalent rectilinear movements.

Corresponding to a real single-valued differentiable function $\sigma=\sigma(s)$ there exists a *class* $\mathfrak{S}_{\sigma(s)}$ *of projectively applicable movements* to which a movement defined by $\bar{\sigma}=\bar{\sigma}(s)$ will be said to belong if $\{\bar{\sigma}, s\}\equiv\{\sigma, s\}$. We shall call the Schwarzian $\{\sigma, s\}$ *the absolute projective acceleration*

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[3] Cartan, *Leçons sur la Théorie des Espaces*, Paris, Gauthier-Villars, 1937, p. 3.

*of the movements of the class* $\mathfrak{S}_{\sigma(s)}$ or simply the *absolute projective acceleration*[4] $\{\sigma, s\}$.

We may now state the following theorem:

THEOREM 4. *The invariance of the Schwarzian derivative* $\{z, t\}$ *yields the identities* (16) *and* (17) *which express the absolute projective accelerations* $\{\bar s, s\}$ *and* $\{s, \bar s\}$ *algebraically in terms of the squares of the curvatures* $\gamma$ *and* $\bar\gamma$ *and the square of their rate of variation* $d\gamma/d\bar\gamma$.

Other interesting identities may be obtained by forming various combinations of (13), (14) and (15). One of these has the surprisingly simple form

$$\bar\gamma^2/\{s, \bar s\} + \gamma^2/\{\bar s, s\} \equiv 2. \tag{19}$$

5. **The magnimetric circles.** Let us consider a homographic transformation

$$w = (az + b)/(cz + d), \tag{20}$$

where $a, b, c, d$ are constants and $ad-bc=1$. Since $\lambda(x, y)=|dz/dw|$ and for (20) $dz/dw$ is defined by $dz/dw=(cz+d)^2$, we may write

$$\lambda(x, y) \equiv (cz + d)(\bar c\bar z + \bar d), \tag{21}$$

where $\bar z$, $\bar c$, $\bar d$ denote the conjugate imaginaries of $z$, $c$, $d$. Let the value of $\lambda$ defined at the point $z_1=x_1+iy_1$ be denoted by $\lambda_1$. By making use of (20), (21) and the equation

$$z = (-dw + b)/(cw - a), \tag{22}$$

for the inverse of (20), the proof of the following theorem may be supplied by the reader.

THEOREM 5. *Through a point* $z_1$, *in the z-plane (excluding* $z_1=\infty$ *and* $z_1=-d/c$), *there passes just one circle which by* (20) *is magnified in all of its elements of arc length by the constant multiple* $1/\lambda_1$. *Similarly, through the point* $w_1$, *which corresponds by* (20) *to the point* $z_1$, *there passes just one circle which by* (22) *is magnified in all of its elements of arc-length by the constant multiple* $\lambda_1$.

These circles will be called *magnimetric circles of the transformations* (20) *and* (22). The totality of the magnimetric circles in the $z$-plane form a family of concentric circles, a general one of which is defined by

$$(cz + d)(\bar c\bar z + \bar d) = \lambda, \qquad \lambda = \text{const.} \tag{23}$$

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[4] Loc. cit., Footnote 3, p. 3. Cartan, in considering rectilinear motion defined by $x=x(t)$ has called the Schwarzian $\{x, t\}$ *"l'accélération projective du mouvement."*

The corresponding magnimetric circles in the $w$-plane form the family of concentric circles, a general one of which is defined by

$$(cw - a)(\bar{c}\bar{w} - \bar{a}) = 1/\lambda. \tag{24}$$

Let $r(z)$ and $\rho(w)$ denote the radii of the circles (23) and (24), respectively. We have, clearly, that

$$r^2(z) = \lambda/c\bar{c}, \qquad \rho^2(w) = 1/\lambda c\bar{c}. \tag{25}$$

When $\lambda = 1$, equations (24) and (25) represent the isometric[5] circles.

We shall refer to families in the $z$- and $w$-planes as $z$- and $w$-families, respectively. By making use of equations (23) and (24) in connection with (20), the following theorem is obtained.

THEOREM 6. *There are* $\infty^2$ *transformations of the form* (20) *which transform a* $z$*-family of concentric circles with an arbitrarily selected center* $z_0$ *into a* $w$*-family of concentric circles with an arbitrarily selected center* $w_0$. *There are* $\infty^1$ *of these which transform a selected circle of the* $z$*-family into a selected circle of the* $w$*-family. On requiring this pair of circles to correspond, a one-to-one correspondence among the other members of the two families is established. Finally, there is just one transformation of this infinite system which transforms a selected point* $z_1$ *on any circle of the* $z$*-family into a selected point* $w_1$ *on the corresponding circle of the* $w$*-family. The circles of these* $z$*- and* $w$*-families are the magnimetric circles of the transformation, and the product of any corresponding pair of radii is equal to* $1/|c|^2$.

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[5] Loc. cit., Footnote 2, pp. 23–27.

# ON THE MAPPING OF QUADRATIC FORMS[1]

LLOYD L. DINES

The development of this paper was suggested by a theorem proposed by Bliss, proved by Albert,[2] by Reid,[3] and generalized by Hestenes and McShane.[4] That theorem had to do with two quadratic forms $P(z)$ and $Q(z)$ in real variables $z^1, z^2, \cdots, z^n$ with real coefficients, and may be stated as follows:

*If $P(z)$ is positive at each point $z \neq (0)$ at which $Q(z)=0$, then there is a real number $\mu$ such that the quadratic form $P(z)+\mu Q(z)$ is positive definite.*[5]

If one considers the set of points $\mathfrak{M}$ in the $xy$-plane into which the $z$-space is mapped by the transformation

$$(1) \qquad x = P(z), \qquad y = Q(z),$$

he will note that the above theorem may be interpreted as asserting the existence of a *supporting line* of the map $\mathfrak{M}$ which has contact with $\mathfrak{M}$ only at $(x, y) = (0, 0)$. This suggests that the theorem is related to the theory of convex sets.

In the present paper it is proven (Theorem 1) that $\mathfrak{M}$ is a *convex* set. Furthermore it is proven (Theorem 2) that if $P(z)$ and $Q(z)$ have no common zero except $z=(0)$, then $\mathfrak{M}$ is *closed*, and is either the entire $xy$-plane or an angular sector of angle less than $\pi$. Immediate corollaries include not only the theorem quoted above, but also statements of criteria for the existence of (1) semi-definite, and (2) definite linear combinations $\lambda P(z)+\mu Q(z)$. The author hopes in a subsequent paper to obtain analogous results for the general case of $m$ quadratic forms.

Throughout the paper it is to be understood without further statement that $P(z)$ and $Q(z)$ are quadratic forms in $z^1, z^2, \cdots, z^n$, with real coefficients, the variables $z^i$ being restricted to real values.

1. **Convexity, and the condition for $\lambda P(z)+\mu Q(z) \geqq 0$.** We give first the following theorem.

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[1] Presented to the Society, December 31, 1940.

[2] This Bulletin, vol. 44 (1938), p. 250.

[3] This Bulletin, vol. 44 (1938), p. 437.

[4] Transactions of this Society, vol. 47 (1940), p. 501.

[5] While the present paper was in press, Professor N. H. McCoy kindly called the author's attention to the fact that this theorem was proven first by Paul Finsler: *Über das Vorkommen definiter und semidefiniter Formen in Scharen quadratischer Formen*, Commentarii Mathematici Helvetici, vol. 9 (1937), pp. 188–192. Apparently this work had been overlooked by the authors referred to above.

THEOREM 1. *Under the transformation* (1), *the map* $\mathfrak{M}$ *of the z-space onto the xy-plane is convex.*

If $A$ is a point of the map, distinct from the origin $O$, every point of the ray $OA$ belongs to the map, since $P(rz)=r^2P(z)$ and $Q(rz)=r^2Q(z)$ for every real number $r$. Hence, if $A$ and $B$ are two points collinear with $O$, and each belongs to $\mathfrak{M}$, then all points of the line segment $AB$ belong to $\mathfrak{M}$.

We will therefore assume that $A(x_1, y_1)$ and $B(x_2, y_2)$ are points of $\mathfrak{M}$, not collinear with the origin, defined by

$$\begin{aligned} x_1 &= P(z_1), & x_2 &= P(z_2), & & \\ y_1 &= Q(z_1), & y_2 &= Q(z_2), & z_i &= (\overset{1}{z}_i, \overset{2}{z}_i, \cdots, \overset{n}{z}_i), \end{aligned} \tag{2}$$

and attempt to show that every point on the line segment $AB$ belongs to $\mathfrak{M}$. Without loss of generality we will further assume that

$$x_2y_1 - x_1y_2 = k^2 > 0. \tag{3}$$

It will suffice to show analytically that if $\bar{t}$ is any *given* number such that $0<\bar{t}<1$, then the equations

$$P(z) = x_1 + \bar{t}(x_2 - x_1), \qquad Q(z) = y_1 + \bar{t}(y_2 - y_1) \tag{4}$$

admit a real simultaneous solution $z=(z^1, z^2, \cdots, z^n)$.

In (4) we make the substitution

$$z = \rho(z_1 \cos\theta + z_2 \sin\theta) \tag{5}$$

where $\rho$ and $\theta$ are real variables, and write the results in the form

$$\begin{aligned} \rho^2 p(\cos\theta, \sin\theta) &= x_1 + \bar{t}(x_2 - x_1), \\ \rho^2 q(\cos\theta, \sin\theta) &= y_1 + \bar{t}(y_2 - y_1), \end{aligned} \tag{6}$$

where $p$ and $q$ are quadratic forms in $\cos\theta$, $\sin\theta$, defined by

$$\begin{aligned} p(\cos\theta, \sin\theta) &\equiv P(z_1 \cos\theta + z_2 \sin\theta), \\ q(\cos\theta, \sin\theta) &\equiv Q(z_1 \cos\theta + z_2 \sin\theta). \end{aligned} \tag{7}$$

Elimination of $\rho^2$ from the two equations (6) imposes upon $\theta$ the condition

$$y_1p(\cos\theta, \sin\theta) - x_1q(\cos\theta, \sin\theta) = \bar{t}T(\theta) \tag{8}$$

where

$$T(\theta) \equiv (y_1 - y_2)p(\cos\theta, \sin\theta) - (x_1 - x_2)q(\cos\theta, \sin\theta). \tag{9}$$

The function $T(\theta)$ is a quadratic form in $\cos\theta$, $\sin\theta$, which has the positive value $k^2$ at $\theta=-\pi/2$, $\theta=0$, and $\theta=\pi/2$; as can be easily verified from (7), (2), and (9). Since it can vanish for at most two values of $\theta$ between $-\pi/2$ and $\pi/2$, and must be negative between any two such values if they exist, the function $T(\theta)$ will be positive on at least one of the two intervals $-\pi/2 \leqq \theta \leqq 0$ or $0 \leqq \theta \leqq \pi/2$. We will suppose, for definiteness, that it is the latter, the argument being similar in the two cases.

We define a function $f(\theta)$ by the formula

$$f(\theta) = \frac{y_1 p(\cos\theta, \sin\theta) - x_1 q(\cos\theta, \sin\theta)}{T(\theta)}, \qquad 0 \leqq \theta \leqq \pi/2,$$

which is obviously continuous on the range indicated, and which has the further properties $f(0)=0$ and $f(\pi/2)=1$. Hence it takes on all values between 0 and 1, and in particular there is a value $\bar\theta$ such that $f(\bar\theta)=\bar t$. This $\bar\theta$ is then a solution of (8).

The compatibility condition (8) being satisfied by $\theta=\bar\theta$, we easily satisfy the two equations (6) by taking $\rho^2=\bar\rho^2=k^2/T(\bar\theta)$. And the resulting

$$z = \bar z = \bar\rho(z_1 \cos\bar\theta + z_2 \sin\bar\theta)$$

given by (5) provides the required solution for (4).

COROLLARY. *A necessary and sufficient condition that there exist real* $\lambda$, $\mu$, *such that for all real* $z$

$$\lambda P(z) + \mu Q(z) \geqq 0$$

*is that there exist real* $a$, $b$, *such that the two equations* $P(z)=a$, $Q(z)=b$ *are inconsistent for real* $z$.

The condition is necessary, since in its absence the map $\mathfrak{M}$ is the entire $xy$-plane, and every line $\lambda x+\mu y=0$ *separates* the plane into a positive half-plane and a negative half-plane, each of which contains points determined by $x=P(z)$, $y=Q(z)$.

However, if the point $(a, b)$ does not belong to the map, no point on the ray from the origin to $(a, b)$ belongs to the map. Hence the origin is a boundary point of the convex set $\mathfrak{M}$, and through this boundary point there passes a *supporting line* $\lambda x+\mu y=0$, such that $\lambda P(z)+\mu Q(z)\geqq 0$ for all real $z$.

2. **Closure, and the conditions for $\lambda P(z)+\mu Q(z)>0$.** We now prove the following theorem.

THEOREM 2. *If* $P(z)$ *and* $Q(z)$ *have no common zero except* $z=(0)$, *then* $\mathfrak{M}$ *is closed as well as convex, and is either the entire xy-plane or an angular sector of angle less than* $\pi$.

Since $\mathfrak{M}$ is convex, if it is not the entire $xy$-plane it lies entirely in some half-plane

$$ax + by \geq 0, \qquad a^2 + b^2 = 1. \tag{10}$$

We first show that, under the stated hypothesis, $\mathfrak{M}$ cannot contain both rays of the boundary line $ax+by=0$. Suppose it did contain the two symmetrical points $A(b, -a)$, $B(-b, a)$, and more explicitly that

$$P(z_1) = b, \qquad Q(z_1) = -a, \qquad P(z_2) = -b, \qquad Q(z_2) = a.$$

Since either $a$ or $b$ is certainly different from zero, we may assume the notation so chosen that $a>0$. Then $Q(z_1)<0$ and $Q(z_2)>0$. Hence[6] there are, in the hyperplane defined by $z=z_1u+z_2v$, two linearly independent points $z_0=z_1u_0+z_2v_0$, $z_0'=z_1u_0'+z_2v_0'$, such that

$$Q(z_0) = Q(z_0') = 0. \tag{11}$$

Consider now the quadratic form

$$\phi(u, v) = aP(z_1u + z_2v) + bQ(z_1u + z_2v)$$

in the two real variables $u$, $v$. It is easily verified that $\phi$ vanishes at $(u, v)=(1, 0)$ and at $(u, v)=(0, 1)$. These, together with the dependent points $(c, 0)$ and $(0, c)$, are its only possible zeros unless it vanishes identically. It does not vanish identically, since it does not vanish at $(u_0, v_0)$ or $(u_0', v_0')$ in view of (11) and our hypothesis. Hence, by (10), $\phi(u, v)>0$ except at $(c, 0)$ and $(0, c)$. This is clearly impossible, and the contradiction proves that the map $\mathfrak{M}$ cannot contain both points $A(b, -a)$ and $B(-b, a)$.

We now let $X(x, y)$ denote any point of $\mathfrak{M}$, and consider the angle $AOX$, where $A\equiv A(b, -a)$ and $O\equiv O(0, 0)$. Then $\cos AOX = (bx-ay)/(x^2+y^2)^{1/2}$. And as the point $z$ varies over the unit hypersphere $\|z\|=1$, $\cos AOX$ is represented by the function

$$\psi(z) = \frac{bP(z) - aQ(z)}{[P^2(z) + Q^2(z)]^{1/2}}, \qquad \|z\| = 1.$$

In view of the hypothesis, $\psi(z)$ is continuous on this hypersphere; and since its values are bounded below by $-1$ and above by $+1$, it attains a minimum value $m \geq -1$ and a maximum value $M \leq 1$. It is impossible that $m=-1$ *and* $M=1$, since then the map $\mathfrak{M}$ would contain both points $A(b, -a)$ and $B(-b, a)$. Hence $\mathfrak{M}$ consists of a closed sector bounded by rays $OA'$ and $OB'$ such that $\cos AOA' = M$ and $\cos AOB' = m$. And *angle* $A'OB' <$ *angle* $AOB = \pi$.

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[6] Reference may be made to Bôcher, *Introduction to Higher Algebra*, p. 151, Theorem 2.

COROLLARY 1. *Necessary and sufficient conditions that there exist real* $\lambda$, $\mu$, *such that for all real* $z \neq (0)$

$$\lambda P(z) + \mu Q(z) > 0 \tag{12}$$

*are that*: (1) *there exist real* $a$, $b$, *such that the two equations* $P(z) = a$, $Q(z) = b$ *are inconsistent for real* $z$; *and* (2) $P(z)$ *and* $Q(z)$ *have no common zero except* $z = (0)$.

The necessity is obvious. The sufficiency follows from Theorem 2. For if $(\lambda, \mu) \neq (0, 0)$ is a point of $\mathfrak{M}$ on the bisector of its angular sector, then (12) is satisfied.

COROLLARY 2. (Bliss-Albert theorem.) *If, whenever* $Q(z) = 0$ *and* $z \neq (0)$, $P(z) > 0$; *then there exists a real number* $\mu$ *such that* $P(z) + \mu Q(z)$ *is positive definite.*

The conditions of Corollary 1 are obviously satisfied with $(a, b) = (-1, 0)$. Hence there exist $\lambda$, $\mu$, satisfying (12). If $Q(z)$ actually vanishes for some $z \neq (0)$, $\lambda$ is necessarily positive and hence may be taken equal to 1.

If, on the contrary, $Q(z)$ is definite, then the map $\mathfrak{M}$ is a *closed* sector of which only the vertex (0, 0) is on the $x$-axis. Hence there is a line $x + \mu y = 0$ such that $x + \mu y > 0$ for all points of $\mathfrak{M}$ except (0, 0). Then $P(z) + \mu Q(z)$ is positive definite.

It is perhaps worthy of note that the two conditions of Corollary 1 are completely independent. This is shown by the following four examples.

*Example* 1, in which both (1) and (2) are satisfied:

$$P(u, v) \equiv u^2, \qquad Q(u, v) \equiv v^2.$$

*Example* 2, in which (1) is satisfied but (2) is not:

$$P(u, v) \equiv u^2, \qquad Q(u, v) \equiv uv.$$

*Example* 3, in which (1) is not satisfied but (2) is:

$$P(u, v) \equiv u^2 + 2uv, \qquad Q(u, v) \equiv 2uv + v^2.$$

*Example* 4, in which neither (1) nor (2) is satisfied:

$$P(u, v, w, t) \equiv u^2 + 2uv + w^2, \qquad Q(u, v, w, t) \equiv 2uv + v^2 + wt.$$

CARNEGIE INSTITUTE OF TECHNOLOGY

# ON THE REPRESENTATIONS, $N_3(n^2)$ [1]

C. D. OLDS

1. **Introduction.** Let the symbol $N_r(n)$ denote the number of representations of the positive integer $n$ in the form $n=x_1^2+x_2^2+\cdots+x_r^2$, where $x_1, x_2, \cdots, x_r$ are positive or negative integers or zero. We will agree to count the two representations $n=x_1^2+x_2^2+\cdots+x_r^2$, $n=y_1^2+y_2^2+\cdots+y_r^2$, as distinct unless simultaneously $x_\nu=y_\nu$, $\nu=1, 2, \cdots, r$. Notice that in a given representation the signs of the roots, as well as their arrangement, are relevant. A zero square, however, is supposed to have only one root.

In a letter written in 1884 to Ch. Hermite, T. J. Stieltjes[2] proved by means of elliptic functions that if $n=p^k$, $p\equiv 1 \pmod 8$, $p$ prime, then $N_3(n^2)=6p^k$. Later in 1907, A. Hurwitz[3] stated without proof that if

$$n = 2^k m = 2^k PQ, \qquad P = \prod_{\nu=1}^{r} p_\nu^{a_\nu}, \qquad Q = \prod_{\nu=1}^{s} q_\nu^{b_\nu}, \tag{1}$$

where each $p_\nu$ is a prime $\equiv 1 \pmod 4$, and each $q_\nu$ is a prime $\equiv 3 \pmod 4$, then

$$N_3(n^2) = 6P\prod_{\nu=1}^{s}\left[q_\nu^{b_\nu} + 2\,\frac{q_\nu^{b_\nu}-1}{q_\nu-1}\right]. \tag{2}$$

This result is also implicitly contained in Stieltjes' letter mentioned above.

In 1940, G. Pall[4] showed that (2) could be derived arithmetically by an application of certain divisibility properties of the Lipschitz integral quaternions. It is the purpose of this paper to give a simple arithmetical proof of (2) by a method which has been evolved from the study of a paper by Hurwitz[5] in which he derived the analogous formula for $N_5(n^2)$.[6]

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[1] This is the first part of a paper presented to the Society April 6, 1940, under the title *On the number of representations of the square of an integer as the sum of an odd number of squares.*

[2] T. J. Stieltjes, "Lettre 45," *Correspondence d'Hermite et de Stieltjes*, vol. 1, Paris, 1905, pp. 89–94.

[3] A. Hurwitz, *Mathematische Werke*, vol. 2, Basel, 1933, p. 751.

[4] G. Pall, Transactions of this Society, vol. 47 (1940), pp. 487–500. See also G. Pall, Journal of the London Mathematical Society, vol. 5 (1930), pp. 102–105. In this paper Pall gives analytical proofs of the formula for $N_r(cn^2)$, $r=3, 5, 7, 11$, $c$ an integer.

[5] A. Hurwitz, Comptes Rendus de l'Académie des Sciences, Paris, vol. 98 (1884), pp. 504–507; *Mathematische Werke*, vol. 2, pp. 5–7. Notice that Hurwitz makes use of certain results announced by Liouville and some formulas of Stieltjes.

[6] The author wishes to acknowledge the assistance rendered him by Professor J. V. Uspensky.

## 2. Two lemmas.

The arithmetical derivation of (2) depends upon the following propositions:

Lemma 1. *Let $f(n)$ be an arbitrary arithmetical function, and suppose that $f(nn')=f(n)f(n')$ for any two integers $n$, $n'$, and that $f(n)\neq 0$ for all $n$. If*

$$F(n) = \sum_{d|n} f(d)$$

*where, in particular, $f(1)=F(1)=1$, then*

$$F(nn') = \sum_{d|n,n'} \mu(d)f(d)F(n/d)F(n'/d), \tag{3}$$

*where $\mu(n)$ is the Möbius function,*[7] *and the summation extends over all divisors common to both $n$ and $n'$. If we agree to set $F(x)=0$ if $x$ is not equal to an integer, then the summation can be extended to $d=1, 2, \cdots$ ending with a number greater than $n$ or $n'$.*

Proof. Set $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, $n'=p_1^{\beta_1}p_2^{\beta_2}\cdots p_r^{\beta_r}$, where $\alpha_\nu\geqq 0$, $\beta_\nu\geqq 0$, and $p_1, p_2, \cdots, p_r$ are distinct primes. Then it follows from the definition of $F(n)$ that

$$F(p_\nu^{\alpha_\nu+\beta_\nu}) = F(p_\nu^{\alpha_\nu})F(p_\nu^{\beta_\nu}) - f(p_\nu)F(p_\nu^{\alpha_\nu-1})F(p_\nu^{\beta_\nu-1}),$$

for $\nu=1, 2, \cdots, r$, and for $\alpha_\nu\geqq 0$, $\beta_\nu\geqq 0$. It is also clear that $F(p^\alpha q^\beta)=F(p^\alpha)F(q^\beta)$ provided $p$ and $q$ are distinct primes. Hence

$$\begin{aligned} F(nn') = \prod_{\nu=1}^{r} F(p_\nu^{\alpha_\nu+\beta_\nu}) &= \prod_{\nu=1}^{r} [F(p_\nu^{\alpha_\nu})F(p_\nu^{\beta_\nu}) - f(p_\nu)F(p_\nu^{\alpha_\nu-1})F(p_\nu^{\beta_\nu-1})] \\ &= \sum_{d=1,2,3,\cdots} \mu(d)f(d)F(n/d)F(n'/d). \end{aligned}$$

The second lemma we need is derived as follows: Let $f(x)=f(-x)$ be an arbitrary even function defined for integral values of $x$; then it is possible to show by purely arithmetical reasoning that[8]

$$\sum_{(a)} [f(d'-d'') - f(d'+d'')] = \sum_{(b)} d[f(0) - f(2d)], \tag{4}$$

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[7] The Möbius function is defined as follows: $\mu(1)=1$, $\mu(n)=0$ if $n$ has a squared factor; $\mu(p_1p_2\cdots p_r)=(-1)^r$, if all the primes $p_1, p_2, \cdots, p_r$ are different.

[8] M. J. Liouville, Journal de Mathématique, (2), vol. 3 (1858), p. 194. Although this identity was first proved arithmetically by T. Pepin, Journal de Mathématique, (4), vol. 4 (1888), p. 94, it is more convenient to refer to the exposition in Uspensky and Heaslet, *Elementary Number Theory*, New York, 1939, p. 462.

the summations extending respectively over all positive integral solutions of the equations

(a) $2n=d'\delta'+d''\delta''=s'+s''$, $d'$, $\delta'$, $d''$, $\delta''$ odd,

(b) $n=d\delta$, $\delta$ odd.

If in (4) we replace $f(x)$ by $(-1)^{x/2}f(x)$, $x$ even, we obtain after a few simple reductions,

$$\sum_{(a)}(-1)^{(\delta'-1)/2+(\delta''-1)/2}\left[f(d'-d'')+f(d'+d'')\right] = \sum_{(b)} d\left[f(2d)-(-1)^d f(0)\right]. \tag{5}$$

Now define two arithmetical functions $\sigma_k(n)$ and $\rho_k(n)$ as follows:

$$\sigma_k(n) = \sum_{d|n} d^k, \qquad \sigma_0(n) = \sigma(n), \quad 1 \leqq d \leqq n;$$
$$\rho_k(n) = \sum_{n=d\delta,\,\delta \text{ odd}} (-1)^{(\delta-1)/2}d^k, \quad \rho_0(n) = \rho(n),$$

where in the second function the summation extends over all positive integral solutions $d$, $\delta$ of the equation $n=d\delta$, where $\delta$ is odd.

On setting $f(x)=1$ in (5), and supposing that $n$ is odd $(=m)$ we obtain the following.

LEMMA 2.

$$\sum_{2m=s'+s''} \rho(s')\rho(s'') = \sigma_1(m),$$

*where the summation extends over all positive, odd integers $s'$, $s''$, satisfying the equation $2m=s'+s''$.*

3. **The formula for $N_3(n^2)$.** Using the definition of $n$ given by (1), and noticing that $N_3(2^{2k}m^2)=N_3(m^2)$, we see that we need only seek an expression for the number of solutions of the equation $m^2=x^2+y^2+z^2$. Since $m^2\equiv 1 \pmod 4$, then one of the roots of this equation must be odd, while the other two must be even. Denote by $R$ the number of solutions in which $x$ is even. Then it is a simple matter to verify that

$$N_3(m^2) = \tfrac{3}{2}R.$$

On the other hand, $R$ can be expressed by the sum

$$R = \sum_{\nu=0,\pm 1,\pm 2,\cdots} N_2(m^2-4\nu^2),$$

which is extended over all integers $\nu$ rendering the argument nonnegative. Knowing this, and using the well known result that[9]

$$N_2(n) = 4\rho(n), \qquad n \geqq 1,$$

we obtain at once

$$\begin{aligned} N_3(n^2) = N_3(m^2) &= \tfrac{3}{2} \sum_{\nu=0,\pm1,\pm2,\cdots} N_2(m^2 - 4\nu^2) \\ &= \tfrac{3}{2}\cdot 4 \sum_{\nu=0,\pm1,\pm2,\cdots} \rho(m^2 - 4\nu^2) \\ &= 6 \sum_{\nu=0,\pm1,\pm2,\cdots} \rho((m - 2\nu)(m + 2\nu)) = 6 \sum_{2m=a+b} \rho(ab), \end{aligned}$$

where the last sum extends over all positive odd integers $a$, $b$ which satisfy the equation $2m=a+b$.

The problem is now reduced to the evaluation of the expression $\sum\rho(ab)$. To this end we use Lemma 1. Define $f(n)$ as follows:

$$\begin{aligned} f(n) &= 0 \text{ if } n \text{ is even,} \\ f(n) &= (-1)^{(n-1)/2} \text{ if } n \text{ is odd.} \end{aligned}$$

Then $f(nn')=f(n)f(n')$ for any two integers $n$, $n'$ as required. For odd $n$

$$F(n) = \sum_{d\mid n} f(d) = \sum_{d\mid n} (-1)^{(d-1)/2} = \sum_{n=d\delta,\,\delta \text{ odd}} (-1)^{(\delta-1)/2} = \rho(n).$$

Setting $n=ab$, we obtain from Lemma 1,

$$\rho(ab) = \sum_{d=1,3,5,\cdots} \mu(d)(-1)^{(d-1)/2}\rho(a/d)\rho(b/d),$$

where $\rho(x)=0$ if $x$ is not an integer. It follows that

$$\begin{aligned} \sum_{2m=a+b} \rho(ab) &= \sum_{2m=a+b} \sum_{d=1,3,5\cdots} \mu(d)(-1)^{(d-1)/2}\rho(a/d)\rho(b/d) \\ &= \sum_{d=1,3,5,\cdots} \mu(d)(-1)^{(d-1)/2} \sum_{2m=a+b} \rho(a/d)\rho(b/d). \end{aligned}$$

Now let $d$ be any common divisor of $a$ and $b$, and set $a=\alpha d$, $b=\beta d$, $\alpha$, $\beta$ odd. Then, using Lemma 2, we see that

$$\sum_{2m=a+b} \rho(a/d)\rho(b/d) = \sum_{2m/d=\alpha+\beta} \rho(\alpha)\rho(\beta) = \sigma_1(m/d), \qquad m/d \text{ odd}.$$

Consequently,

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[9] For an arithmetical proof of this result see, for example, Hardy and Wright, *The Theory of Numbers*, Oxford, 1938, p. 241.

$$\sum_{2m=a+b} \rho(ab) = \sum_{d|m} \mu(d)(-1)^{(d-1)/2}\sigma_1(m/d)$$
$$= \left[\sum_{d|P} \mu(d)(-1)^{(d-1)/2}\sigma_1(P/d)\right]$$
$$\cdot\left[\sum_{d|Q} \mu(d)(-1)^{(d-1)/2}\sigma_1(Q/d)\right], \qquad m = PQ.$$

Now using the properties of $\mu(n)$, and being careful of the sign of $(-1)^{(d-1)/2}$ according as $d|P$ or $d|Q$, we can easily show, if we notice that $\sigma_1(p^a)=(p^{a+1}-1)/(p-1)$, $p$ prime, that

$$\sum_{d|P} \mu(d)(-1)^{(d-1)/2}\sigma_1(P/d) = \prod_{\nu=1}^{r} \left[\sigma_1(p_\nu^{a_\nu}) - \sigma_1(p_\nu^{a_\nu-1})\right]$$
$$= \prod_{\nu=1}^{r}\left[\frac{p_\nu^{a_\nu+1}-1}{p_\nu-1} - \frac{p_\nu^{a_\nu}-1}{p_\nu-1}\right]$$
$$= \prod_{\nu=1}^{r} p_\nu^{t_\nu} = P.$$

Likewise,

$$\sum_{d|Q} \mu(d)(-1)^{(d-1)/2}\sigma_1(Q/d) = \prod_{\nu=1}^{s} \left[\sigma_1(q_\nu^{b_\nu}) + \sigma_1(q_\nu^{b_\nu-1})\right]$$
$$= \prod_{\nu=1}^{s}\left[\frac{q_\nu^{b_\nu+1}-1}{q_\nu-1} + \frac{q_\nu^{b_\nu}-1}{q_\nu-1}\right].$$

Combining these two results, we obtain the required expression for $N_3(n^2)$, namely,

$$N_3(n^2) = 6P\prod_{\nu=1}^{s}\left[q_\nu^{b_\nu} + 2\,\frac{q_\nu^{b_\nu}-1}{q_\nu-1}\right].$$

STANFORD UNIVERSITY

# SOME MORE UNIFORMLY CONVEX SPACES[1]

MAHLON M. DAY

Let $\{B_i, i=1, 2, \cdots\}$ be a sequence of Banach spaces, and define $B=\mathfrak{P}^p\{B_i\}$ to be the space of sequences $b=\{b_i\}$ with $b_i \in B_i$ and $\|b\|=(\sum_i\|b_i\|^p)^{1/p}<\infty$, $1<p<\infty$. It is known that $B$, normed in this way, is also a Banach space. Boas[2] showed that if $B_i=l^p$ for all $i$ or if $B_i=L^p$ for all $i$ then $B$ is uniformly convex. Since then I have shown[3] that if $B_i=l^{p_i}$ or $L^{p_i}$, $1<p_i<\infty$, and if the sequence $\{p_i\}$ is not bounded away from 1 and $\infty$ (that is, if there do not exist $1<m \leqq M<\infty$ with $m \leqq p_i \leqq M$ for all $i$), then $B$ is not uniformly convex and cannot be renormed to an isomorphic uniformly convex space. One purpose of this note is to fill in the gap between these results by means of the following theorem.

THEOREM 1. *If $B_i=l^{p_i}$ or $L^{p_i}$, $1<p_i<\infty$, and if the sequence $\{p_i\}$ is bounded away from* 1 *and* $\infty$, *then $B=\mathfrak{P}^p\{B_i\}$ is uniformly convex.*

Another question which is answered here is raised by Boas (loc. cit., Footnote 3). In Boas' notation $l^p(B_0)$ is the space $\mathfrak{P}^p\{B_i\}$ with $B_i=B_0$ for all $i$. The space $L^p(B_0)$ is the space of all Bochner integrable functions[4] $f$ on, say, $[0, 1]$ with values in $B_0$ and with $\|f\|=[\int_0^1\|f(x)\|^p dx]^{1/p}<\infty$. Boas' conjecture is verified by the next theorem.

THEOREM 2. *$l^p(B_0)$ and $L^p(B_0)$ are uniformly convex if (and, obviously, only if) $B_0$ is.*

Both of these results follow from the remaining theorem of this note. If all $B_i$, $i<\infty$, are uniformly convex there exists for each $\epsilon$, $0<\epsilon \leqq 2$, a positive number $\delta_i(\epsilon)$ such that $\|b_i\|=\|b_i'\|=1$ and $\|b_i-b_i'\|>\epsilon$ implies $\|b_i+b_i'\|<2(1-\delta_i(\epsilon))$. The sequence $\{B_i, i<\infty\}$ will be said to have a common modulus of convexity if there is one function $\delta(\epsilon)>0$ which can be used here in place of all $\delta_i(\epsilon)$. It is clear that if we define $\delta_i(\epsilon)=\frac{1}{2}\inf[2-\|b_i+b_i'\|]$, where the infimum is taken over $b_i$, $b_i'$ with $\|b_i\|=\|b_i'\|=1$ and $\|b_i-b_i'\|>\epsilon$, then such a $\delta(\epsilon)$ exists if and only if $\inf_i\delta_i(\epsilon)>0$ for every $\epsilon>0$. It is clear that $\delta(\epsilon)$ may be assumed to be a non-decreasing function of $\epsilon$.

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[1] Presented to the Society, November 23, 1940.

[2] R. P. Boas, Jr., *Some uniformly convex spaces*, this Bulletin, vol. 46 (1940), pp. 304–311.

[3] M. M. Day, *Reflexive Banach spaces not isomorphic to uniformly convex spaces*, this Bulletin, vol. 47 (1941), pp. 313–317.

[4] S. Bochner, *Integration von Funktionen, deren Werte die Elemente eines Vektorraumes sind*, Fundamenta Mathematicae, vol. 20 (1933), pp. 262–276.

THEOREM 3: $B=\mathfrak{P}^p\{B_i\}$ *is uniformly convex if and only if the* $B_i$ *have a common modulus of convexity.*

The space $B_i$ is equivalent to the closed linear subspace of those points of $B$ of the form $\{0, 0, \cdots, 0, b_i, 0, \cdots\}$. If $B$ is uniformly convex, it follows that the function $\delta(\epsilon)$ existing in $B$ will be suitable as a common modulus for all the $B_i$.

The proof of the second part of Theorem 3 falls naturally into two cases. First we assume $\|b\|=\|b'\|=1$, $\|b-b'\|>\epsilon$ and $\|b_i\|=\|b_i'\|$ for all $i$. For convenience let $\|b_i\|=\beta_i$ and $\|b_i-b_i'\|=\gamma_i$. Then for any $i$, $\|b_i+b_i'\|<2(1-\delta(\gamma_i/\beta_i))\beta_i$ since $b_i$ and $b_i'$ both lie on the sphere of radius $\beta_i$ about the origin in $B_i$. Hence

$$\begin{aligned} \|b+b'\| &= \left(\sum_{i<\infty}\|b_i+b_i'\|^p\right)^{1/p} \\ &\leqq 2\left\{\sum_i\left[(1-\delta(\gamma_i/\beta_i))\beta_i\right]^p\right\}^{1/p}. \end{aligned} \tag{1}$$

Also $\|b_i-b_i'\|\leqq\|b_i\|+\|b_i'\|$ so $\gamma_i\leqq 2\beta_i$ for all $i$; let $E$ be the set of those $i$'s such that $\gamma_i/\beta_i>\epsilon/4$ and let $F$ be the complement of $E$. $\beta_i\geqq 4\gamma_i/\epsilon$ if $i\in F$ so

$$1=\left(\sum_{i<\infty}\beta_i^p\right)^{1/p}\geqq\left(\sum_{i\in F}\beta_i^p\right)^{1/p}\geqq(4/\epsilon)\left(\sum_{i\in F}\gamma_i^p\right)^{1/p};$$

that is,

$$\left(\sum_{i\in F}\gamma_i^p\right)^{1/p}\leqq\epsilon/4$$

or

$$\left(\sum_{i\in E}\gamma_i^p\right)^{1/p}=\left(\sum_{i<\infty}\gamma_i^p-\sum_{i\in F}\gamma_i^p\right)^{1/p}\geqq\left(\epsilon^p-(\epsilon/4)^p\right)^{1/p}\geqq 3\epsilon/4.$$

Hence

$$\alpha=\left(\sum_{i\in E}\beta_i^p\right)^{1/p}\geqq 3\epsilon/8.$$

Using this and (1)

$$\begin{aligned} \|b+b'\| &\leqq 2\left[(1-\delta(\epsilon/4))^p\sum_{i\in E}\beta_i^p+\sum_{i\in E}\beta_i^p\right]^{1/p} \\ &\leqq 2\left[(1-\delta(\epsilon/4))^p\alpha^p+1-\alpha^p\right]^{1/p} \\ &\leqq 2\left\{1-\left[1-(1-\delta(\epsilon/4))^p\right]\alpha^p\right\}^{1/p} \\ &\leqq 2\left\{1-\left[1-(1-\delta(\epsilon/4))^p\right](3\epsilon/8)^p\right\}^{1/p}. \end{aligned}$$

The coefficient of $\alpha^p$ is positive and depends only on $\epsilon$, so, if we set the right-hand side equal to $2(1-\delta_0(\epsilon))$, then $\delta_0(\epsilon)>0$ if $\epsilon>0$, thus disposing of the first case.

In the general case we suppose only that $\|b\|=\|b'\|=1$ and that $\|b+b'\|>2(1-\delta_1(\alpha))$, where $0<\alpha\leqq 2$ and $\delta_1(\alpha)$ is any modulus of convexity which can be used in the definition of uniform convexity in $l^p$. Then

$$2(1-\delta_1(\alpha)) \leqq \left(\sum_i \|b_i+b_i'\|^p\right)^{1/p} \leqq \left(\sum_i (\|b_i\|+\|b_i'\|)^p\right)^{1/p} \leqq 2.$$

Since the sequences $\{\|b_i\|\}$ and $\{\|b_i'\|\}$ are in $l^p$, we have

$$\left(\sum_i (\|b_i\|-\|b_i'\|)^p\right)^{1/p} \leqq \alpha.$$

Let $b''=\{b_i''\}$ be the point with $b_i''=b_i'\|b_i\|/\|b_i'\|$; then $\|b_i''\|=\|b_i\|$ for each $i$ and $\|b'-b''\|=(\sum_i(\|b_i'\|-\|b_i\|)^p)^{1/p}<\alpha$; hence, by Case I, $\|b-b''\|\leqq\epsilon/2$ if $\|b+b''\|\geqq 2(1-\delta_0(\epsilon/2))$, but $\|b+b''\|\geqq\|b+b'\|-\|b''-b'\|>2(1-\delta_1(\alpha)-\alpha/2)$.

Take $\alpha$ so that $0<\alpha<\epsilon/2$ and $\delta_1(\alpha)+\alpha/2<\delta_0(\epsilon/2)$; then $\|b+b'\|>2(1-\delta_1(\alpha))$ implies $\|b+b''\|>2(1-\delta_0(\epsilon/2))$ which in turn implies $\|b-b''\|<\epsilon/2$, leading finally to $\|b-b'\|\leqq\|b-b''\|+\|b'-b''\|<\epsilon/2+\epsilon/2$. Let the $\delta_1(\alpha)$ defined by this process be $\delta_2(\epsilon)$; this is greater than zero and $\|b-b'\|<\epsilon$ if $\|b\|=\|b'\|=1$ and $\|b+b'\|>2(1-\delta_2(\epsilon))$. This proves Theorem 3.

To show that Theorem 1 follows from this, use Clarkson's evaluations[5] of $\delta_p(\epsilon)$ in $l^p$ or $L^p$: $\delta_p(\epsilon)=1-[1-(\epsilon/2)^r]^{1/r}$ where $r=p$ if $p\geqq 2$ and $r=p/(p-1)$ if $1<p\leqq 2$. The sequence $\{p_i\}$ with $1<p_i<\infty$ is bounded away from 1 and $\infty$ if and only if the sequence $\{r_i\}$ associated with $\{p_i\}$ is bounded. A little elementary calculus will convince the reader that if $\epsilon$ is fixed $\delta_p(\epsilon)$ decreases as $r$ increases; hence if $r\geqq r_i$ for all $i$, $\delta_r(\epsilon)\leqq\delta_{p_i}(\epsilon)$ for all $i$ and the sequence of spaces $\lambda^{p_i}$ satisfies the hypothesis of Theorem 3 for $\lambda^{p_i}=l^{p_i}$ or $L^{p_i}$ provided only that $\{p_i\}$ is bounded away from 1 and $\infty$.

The first part of Theorem 2, that $l^p(B_0)$ is uniformly convex if $B_0$ is, is a trivial consequence of Theorem 3. The proof for $L^p(B_0)$ involves an embedding argument similar to that used by Clarkson in analogous circumstances in $L^p$.

Let $H$ be the class of simple functions in $L^p(B_0)$; that is, $h\in H$ if there exist measurable sets $E_1, \cdots, E_n$, disjoint, whose sum is the unit interval, and points $h_1, \cdots, h_n \in B_0$ such that $h = \sum_{i \leq n} h_i \phi_{E_i}$ where $\phi_{E_i}$ is the characteristic function of the set $E_i$; that is, $\phi_{E_i}(t) = 1$ if $t \in E_i$ and is otherwise 0. $H$ is dense in $L^p(B_0)$. If $E_1, \cdots, E_n$ are fixed, set up a correspondence between all functions in $H$ of the form $h = \sum_{i \leq n} h_i \phi_{E_i}$ and all points $\mathbf{h}$ in $l^p(B_0)$ with coordinates $b_i = 0$ for $i > n$ as follows: $h$ corresponds to $\mathbf{h} = \{b_i\}$ if $b_i = h_i \beta_i^{1/p}$ for $i \leq n$ where $\beta_i = m(E_i)$, $b_i = 0$ otherwise. Then $\|\mathbf{h}\| = (\sum_{i \leq n} h_i^p \beta_i)^{1/p} = \|h\|$ so the transformation is 1-to-1 and norm preserving.

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[5] J. A. Clarkson, *Uniformly convex spaces*, Transactions of this Society, vol. 40 (1936), pp. 396–414.

If $h, h'$ are of the given form, if $\|h\| = \|h'\| = 1$ and $\|h - h'\| > \epsilon$, then the same relations are true for the points $\mathbf{h}, \mathbf{h}'$ in $l^p(B_0)$; so $\|h - h'\| = \|\mathbf{h} - \mathbf{h}'\| < 2(1 - \delta(\epsilon))$ where $\delta(\epsilon)$ is the modulus of convexity in $l^p(B_0)$. Now for any pair $h, h'$ of functions in $H$ there is a partition of the unit interval into sets $E_1, \cdots, E_n$ such that $h, h'$ are represented in the proper form using these sets. It follows that the inequality given holds for any $h, h' \in H$ with $\|h\| = \|h'\| = 1$ and $\|h - h'\| > \epsilon$. Hence by continuity of the norm in $L^p(B_0)$ and by the density of $H$ in $L^p(B_0)$, $L^p(B_0)$ is uniformly convex and the same modulus of convexity holds in it as in $l^p(B_0)$.

We may remark that the use of the interval (0, 1) and Lebesgue measure was obviously non-essential; any measure finite or infinite on any Borel field of measurable subsets of any set might be used with integration as might be expected. The essential property of the type of integration is that the simple functions in $L^p(B_0)$ shall be dense there.

It is also noteworthy that in the proof of the last part of Theorem 2 we gave the essential part of the proof of the following corollary.

COROLLARY 1. *For fixed $p$ with $1 < p < \infty$ and fixed uniformly convex $B_0$ the spaces $l^p(B_0)$ and $L^p(B_0)$ (no matter what measure is used in defining the space $L^p(B_0)$) have a common modulus of convexity, that of, for example, $l^p(B_0)$.*

In fact, since the evaluation of $\delta_2(\epsilon)$ in Theorem 3 used only $\delta(\epsilon)$ in the spaces $B_i$ and $\delta_1(\epsilon)$ in $l^p$, it follows for $p$ in any interval $1 < m \leq p \leq M < \infty$ and for any $\{B_i\}$ or $B$ from a family of spaces $\{B\}$ with common modulus of convexity, that $\mathfrak{P}^p\{B_i\}$ and hence $l^p(B)$ and $L^p(B)$ all have a common modulus of convexity.

UNIVERSITY OF ILLINOIS

# A FLAT KLEIN BOTTLE ISOMETRICALLY EMBEDDED IN EUCLIDEAN 4-SPACE

C. TOMPKINS

There is no information available as to whether any compact manifold which is equipped with an intrinsically consistent riemannian metric may be embedded in euclidean space so that this metric is induced on it by the metric of the euclidean space. The simplest manifold which seemed to be a possible counterexample is a flat Klein bottle. An example of a flat Klein bottle embedded in euclidean 4-space is given here. The manifold of this example intersects itself. The example is offered to remove the strongest contender from the list of possible counterexamples.

The embedding equations of the Klein bottle are

$$x_1 = \cos v \cos u, \qquad x_2 = \cos v \sin u,$$

$$x_3 = 2 \sin v \cos \frac{u}{2}, \qquad x_4 = 2 \sin v \sin \frac{u}{2} \cdot$$

The following identities are clearly satisfied:

$$x_i(u, v + 2\pi) = x_i(u, v), \qquad x_i(u + 2\pi, 2\pi - v) = x_i(u, v).$$

These are just the identifications of points which by definition of the Klein bottle convert the $u$, $v$-plane into a Klein bottle. The components of the metric tensor are

$$E = 1, \qquad F = 0, \qquad G = 1 + 3 \cos^2 v,$$

and these can be transformed into the more usual form

$$\bar{E} = 1, \qquad \bar{F} = 0, \qquad \bar{G} = 1$$

by an obvious transformation involving an elliptic function.

PRINCETON UNIVERSITY

# THE $R_\lambda$-CORRESPONDENT OF THE TANGENT TO AN ARBITRARY CURVE OF A NON-RULED SURFACE

P. O. BELL

In a recent paper[1] the author defined at a general point $y$ of a non-ruled analytic surface $S$ the tangent line which he calls the $R_\lambda$-*correspondent* of the tangent at $y$ to a general curve $C_\lambda$ of $S$. It was proved[2] that (i) *a curve* $C_\lambda$ *is a curve of Darboux if and only if at each of its points the* $R_\lambda$-*correspondent of the tangent to* $C_\lambda$ *coincides with this tangent*, (ii) *a curve* $C_\lambda$ *is a curve of Segre if and only if at each of its points the tangent to* $C_\lambda$ *and its* $R_\lambda$-*correspondent are conjugate tangents of* $S$.

The primary purpose of this note is to present the following simple construction for the $R_\lambda$-correspondent: Let $\Lambda$ denote a point of $C_\lambda$ distinct from $y$, let $U$, $V$ denote, respectively, the points of intersection of the asymptotic $u$- and $v$-curves passing through $y$ with the asymptotic $v$- and $u$-curves passing through $\Lambda$, and let $W$ denote the point of intersection of the tangent plane to $S$ at $y$ with the line joining the points $U$, $V$. *If* $y$ *is held fixed while* $\Lambda$ *tends toward* $y$ *along* $C_\lambda$, *the point* $W$ *describes a curve* $C_w$ *and, except when* $C_\lambda$ *is a curve of Segre or is tangent at* $y$ *to a curve of Segre, the limit of* $W$ *is the point* $y$. *The tangent at* $y$ *to* $C_w$ *is the* $R_\lambda$-*correspondent of the tangent to* $C_\lambda$ *at* $y$.

The validity of this construction will be proved, and in addition the following theorem will be demonstrated:

*A curve* $C_\lambda$ *is a curve of Segre if and only if for a general point* $y$ *of* $C_\lambda$ *the limit of* $W$ *as* $\Lambda$ *tends to* $y$ *along* $C_\lambda$ *is a point* $W_0$ *distinct from* $y$. *The point* $W_0$ *is the intersection of the directrix of the first kind of Wilczynski with the tangent at* $y$ *to the corresponding curve* $C_{-\lambda}$ *of Darboux.*

Let the homogeneous projective coordinates $y^{(1)}, \cdots, y^{(4)}$ of a general point $y$ on a non-ruled analytic surface $S$ in ordinary space be functions of asymptotic parameters $u$, $v$. The functions $y^{(i)}$ are solutions of a system of differential equations, which can be reduced by a suitable transformation to Wilczynski's canonical form

$$(1)\qquad y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + 2a'y_u + gy = 0.$$

The coefficients of these equations are functions of $u$, $v$ which are connected by three conditions of integrability. Moreover, the coordinates

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[1] P. O. Bell, *A study of curved surfaces by means of certain associated ruled surfaces.* Transactions of this Society, vol. 46 (1939), pp. 389–409.

[2] Loc. cit., p. 393.

$y^{(i)}$ are not solutions of any equation of the form $Ay_{uv}+By_u+Cy_v+Dy=0$ whose coefficients are functions of $u$, $v$ not all zero. This statement implies that the point $y_{uv}$, whose coordinates are the functions $y^{(i)}_{uv}$, does not lie in the tangent plane to $S$ at $y$.

An arbitrary one-parameter family $F_\lambda$ of curves of $S$ is defined by the curvilinear differential equation $dv-\lambda du=0$, where $\lambda$ is an arbitrary function of $u$, $v$. We denote by $C_\lambda$ the curve of $F_\lambda$ which passes through $y$. If $u$, $v$ be regarded as functions of a single parameter $t$, as $t$ varies, the point $y$, whose curvilinear coordinates are $u$, $v$, describes a curve of $S$. This curve will be the curve $C_\lambda$ if the functions $u=u(t)$, $v=v(t)$ are selected such that for a general value of $t$

$$\lambda(u, v) = v'/u', \tag{2}$$

where accents indicate differentiation with respect to $t$.

The curvilinear coordinates of the point $\Lambda$ are given by $u(t+\overline{\Delta t})$, $v(t+\overline{\Delta t})$. The points $U$ and $V$ are therefore given by $u(t+\overline{\Delta t})$, $v$ and $u$, $v(t+\overline{\Delta t})$. The general homogeneous coordinates of the points $U$ and $V$ are consequently functions of $t$, and may be represented by the developments

$$\begin{aligned} U = {} & y + y_u u'\overline{\Delta t} + (y_{uu}u'^2 + y_u u''u')\overline{\Delta t}^2/2 \\ & + (y_{uuu}u'^3 + 3u'^2u''y_{uu} + f_1 y_u)\overline{\Delta t}^3/6 \\ & + (y_{uuuu}u'^4 + 6u'^2u''y_{uuu} + f_2 y_{uu} + f_3 u_u)\overline{\Delta t}^4/24 + \cdots , \end{aligned} \tag{3}$$

$$\begin{aligned} V = {} & y + y_v v'\overline{\Delta t} + (y_{vv}v'^2 + y_v v''v')\overline{\Delta t}^2/2 \\ & + (y_{vvv}v'^3 + 3v'^2v''y_{vv} + g_1 y_v)\overline{\Delta t}^3/6 \\ & + (y_{vvvv}v'^4 + 6v'^2v''y_{vvv} + g_2 y_{vv} + g_3 y_v)\overline{\Delta t}^4/24 + \cdots \end{aligned} \tag{4}$$

wherein $f_i$, $g_i$, $i=1, 2, 3$, represent functions of $u$, $v$ which for our purpose do not require explicit determination.

By differentiating equations (1) we find that the coefficients of $y_{uv}$ in the expressions for $y_{uuu}$, $y_{vvv}$, $y_{uuuu}$, $y_{vvvv}$ are $-2b$, $-2a'$, $-4b_u$, $-4a'_v$, respectively. The coefficients of $y_{uv}$ in the expressions for the homogeneous coordinates of the points $U$, $V$ are, therefore,

$$- bu'^3\overline{\Delta t}^3/3 - (b_u u'^4 + 3bu'^2u'')\overline{\Delta t}^4/6 + \cdots ,$$

$$- a'v'^3\overline{\Delta t}^3/3 - (a'_v v'^4 + 3a'v'^2v'')\overline{\Delta t}^4/6 + \cdots ,$$

respectively. The point $W$, which is the intersection of the tangent plane to $S$ at $y$ with the line joining $U$, $V$ has homogeneous coordinates which may be obtained by forming a linear combination of those of $U$ and $V$ which contains no $y_{uv}$ term. Hence, such a combination is

$$(2a'v'^3 + [a_v' v'^4 + 3a'v'^2v'']\overline{\Delta t})U - (2bu'^3 + [b_u u'^4 + 3bu'^2u'']\overline{\Delta t})V.$$

Expanding this we obtain the expression

$$\begin{aligned} &2(a'v'^3 - bu'^3)y + (a_v' v'^4 + 3a'v'^2v'' - b_u u'^4 - 3bu'^2u'')\overline{\Delta t}y \\ &\qquad + 2(a'v'^3u'y_u - bu'^3v'y_v)\overline{\Delta t} + \text{terms of order } \overline{\Delta t^2} \end{aligned} \tag{5}$$

for the homogeneous coordinates of the point $W$. If $a'v'^3 - bu'^3 \neq 0$, the limit of $W$ as $\overline{\Delta t}$ tends to zero is, clearly, the point $y$. Moreover, the tangent to $C_w$ at $y$ has the direction defined by $dv/du = -bu'^2/a'v'^2$. This is the direction of the $R_\lambda$-correspondent of the tangent to $C_\lambda$ at $y$. This completes the proof for the general case in which $\lambda$ is not a direction of Segre.

The curve $C_\lambda$ is a curve of Segre if and only if at each of its points the direction defined by $\lambda = v'/u'$ satisfies the equation $a'v'^3 - bu'^3 = 0$. In this case it is clear from (5) that the limit of $W$ as $\overline{\Delta t}$ tends to zero is a point $W_0$, distinct from $y$, whose homogeneous coordinates are given by

$$\begin{aligned} &(3a'v'^2v'' - 3bu'^2u'' + a_v v'^4 - b_u u'^4)y \\ &\qquad + 2(a'v'^3u'y_u - bu'^3v'y_v), \qquad \text{where } a'v'^3 = bu'^3. \end{aligned} \tag{6}$$

If we divide this expression by $a'u'v'^3$, make use of the condition $a'v'^3 = bu'^3$, and make the following substitutions, $v'/u' = \lambda$, $-v'u''/u'^3 = \lambda_u$, $v''/u'v' = \lambda_v$, we obtain the simpler form

$$\begin{aligned} &(3[\lambda_u + \lambda\lambda_v]/\lambda + a_v'\lambda/a' - b_u/b)y \\ &\qquad + 2(y_u - \lambda y_v), \qquad \text{where } a'\lambda^3 = b, \end{aligned} \tag{7}$$

for the coordinates of $W_0$. It is, clearly, a simple matter to evaluate (7) explicitly for a direction $\lambda = \epsilon(b/a')^{1/3}$, wherein $\epsilon$ is a cube root of unity. The result is

$$2y_u - a_u' y/a' - \epsilon(b/a')^{1/3}(2y_v - b_v y/b). \tag{8}$$

This expression is a linear combination of the expressions $2y_u - a_u'y/a'$ and $2y_v - b_v y/b$ for the homogeneous coordinates of the points in which the directrix of the first kind intersects the asymptotic $u$- and $v$-tangents to $S$ at $y$. Moreover, the ratio of the coefficient of $y_v$ to that of $y_u$ is the direction $-\epsilon(b/a')^{1/3}$ of Darboux which corresponds to the direction $\epsilon(b/a')^{1/3}$ of Segre. This completes the demonstration of the theorem.

University of Kansas

# ON THE SIMULTANEOUS APPROXIMATION OF TWO REAL NUMBERS[1]

RAPHAEL M. ROBINSON

If $\xi_1, \xi_2, \cdots, \xi_n$ are any real numbers and $t$ is a positive integer, then it is well known that integers $a_1, a_2, \cdots, a_n, b$ can be found, such that $0<b\leqq t^n$ and

$$|b\xi_k - a_k| < 1/t, \qquad k = 1, 2, \cdots, n.$$

The proof is briefly the following.[2] Consider the $t^n+1$ points $(r\xi_1, r\xi_2, \cdots, r\xi_n)$, where $r=0, 1, \cdots, t^n$. Reduce mod 1 to congruent points in the unit cube $(0\leqq x_1<1, \cdots, 0\leqq x_n<1)$. If this cube is divided into $t^n$ cubes of edge $1/t$ (including the lower boundaries), then at least one of these small cubes must contain two of the reduced points, say those with $r=r'$ and $r=r''$. With $b=|r'-r''|$ and suitable $a$'s, we evidently satisfy the required inequalities.

For $n=1$, the inequality can be sharpened to

$$|b\xi - a| \leqq 1/(t+1),$$

$b$ satisfying the condition $0<b\leqq t$. For if we consider the points $r\xi$ $(r=0, 1, \cdots, t)$, and mark the points in the interval $0\leqq x\leqq 1$ which are congruent to them mod 1, we have at least $t+2$ points marked, since corresponding to $r=0$ we mark both 0 and 1. Some two of the marked points must lie within a distance $1/(t+1)$ from each other, so that the desired conclusion follows. This is the best result, as the example $\xi=1/(t+1)$ shows.

The present note solves the corresponding problem for $n=2$. For larger values of $n$ the problem appears more difficult.

THEOREM. *If $\xi_1$ and $\xi_2$ are any real numbers, and $s$ is a positive integer, then integers $a_1, a_2, b$ can be found, such that $0<b\leqq s$, and*

$$|b\xi_k - a_k| \leqq \max\left(\frac{[s^{1/2}]}{s+1}, \frac{1}{[s^{1/2}]+1}\right), \qquad k = 1, 2.$$

*For every $s$, values of $\xi_1$ and $\xi_2$ can be found for which the inequalities could not both be satisfied if the equality sign were omitted.*

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[1] Presented to the Society, November 23, 1940.

[2] The method used in this proof (*Schubfachprinzip* or "pigeonhole principle") was first used by Dirichlet in connection with a similar problem. We sketch the proof here in order to compare it with the proof of the theorem below, which also uses that method.

The inequalities may also be written

$$|b\xi_k - a_k| \leqq \begin{cases} t/(s+1) & \text{for } t^2 - 1 \leqq s \leqq t(t+1) - 1, \\ 1/(t+1) & \text{for } t(t+1) - 1 \leqq s \leqq (t+1)^2 - 1. \end{cases}$$

It will be noted that in some intervals the bound does not decrease as $s$ increases.

We show first that the theorem is the best possible. We shall think of the inequalities in the form just given. If $s<(t+1)^2$, then it is evident that $\xi_1=1/(t+1)$, $\xi_2=1/(t+1)^2$ are a pair of real numbers which cannot be approximated simultaneously with an error less than $1/(t+1)$; this settles the second case. For the first case, consider the pair of real numbers $\xi_1=1/(s+1)$, $\xi_2=t/(s+1)$. We are to show that not both errors can be made less than $t/(s+1)$. We note first that $b\xi_1$ and $b\xi_2$ differ from integers by the same amount as $(s+1-b)\xi_1$ and $(s+1-b)\xi_2$; hence we may suppose that $b\leqq(s+1)/2$, and therefore $0<b\xi_1\leqq 1/2$. In order to make $|b\xi_1-a_1|<t/(s+1)$, we must have $0<b<t$. Then $0<b\xi_2<1$. Since $b\xi_2\geqq\xi_2=t/(s+1)$ and $1-b\xi_2\geqq 1-(t-1)\xi_2=1-(t-1)t/(s+1)\geqq t/(s+1)$, we see that the inequality $|b\xi_2-a_2|<t/(s+1)$ cannot be satisfied.

The theorem evidently follows from the lemma below, by putting $t=[s^{1/2}]$.

LEMMA. *Let $s$ and $t$ be positive integers with $s\geqq t$. If $\xi_1$ and $\xi_2$ are any real numbers, then integers $a_1$, $a_2$, $b$ can be found, such that* $0<b\leqq s$, *and*

$$|b\xi_1 - a_1| \leqq t/(s+1), \qquad |b\xi_2 - a_2| \leqq 1/(t+1).$$

PROOF. Consider the points $(r\xi_1, r\xi_2)$ with $r=0, 1, \cdots, s$. Mark all the points congruent to these mod 1 which fall in the rectangle $0\leqq x_1\leqq t$, $0\leqq x_2<1$. There are $(s+1)t$ points to be marked with $x_1<t$; and in addition, the point $(t, 0)$ is marked, corresponding to $r=0$. If we divide our rectangle into $s+1$ rectangles of width $t/(s+1)$ (closed except at the top) by means of vertical lines, then at least one of them contains more than $t$ points, all corresponding to different values of $r$. The corresponding values of $x_2$ are $t+1$ or more numbers, some two of which differ mod 1 by not more than $1/(t+1)$. Thus we find two points $(r'\xi_1, r'\xi_2)$ and $(r''\xi_1, r''\xi_2)$, whose horizontal distance mod 1 does not exceed $t/(s+1)$ and whose vertical distance mod 1 does not exceed $1/(t+1)$. Putting $b=|r'-r''|$ gives the required result.

UNIVERSITY OF CALIFORNIA

# NOTE ON A THEOREM ON QUADRATIC RESIDUES

KAI-LAI CHUNG

In this note we shall give a short proof of a known result:

THEOREM. *For every prime* $p \equiv 3 \pmod 4$ *there are more quadratic residues* mod $p$ *between* 0 *and* $p/2$ *than there are between* $p/2$ *and* $p$.

An equivalent statement of this theorem is as follows (see E. Landau, *Vorlesungen über Zahlentheorie*, vol. 1, p. 129):

*Für* $p \equiv 3 \pmod 4$ *haben mehr unter den Zahlen* $1^2, 2^2, \cdots, (p-1)^2/4$ *ihren Divisionsrest* mod $p$ *unter* $p/2$ *als über* $p/2$.

For proof we shall use Fourier series with one of its applications, namely Gaussian sums.

Write $s^2 = qp + r$, $0 \leqq r < p$, so that

$$\left[\frac{s^2}{p}\right] = q.$$

It is evident that we have

$$\left[\frac{2s^2}{p}\right] - 2\left[\frac{s^2}{p}\right] = \begin{cases} 0 & \text{if} \quad r < p/2; \\ 1 & \text{if} \quad r > p/2. \end{cases}$$

Therefore we have to prove that $\sum_{s=1}^{(p-1)/2}([2s^2/p] - 2[s^2/p]) < (p-1)/4$, or $\leqq (p-1)/4$ since $p \equiv 3 \pmod 4$.

By a well known expansion in Fourier series, we have

$$x - [x] - \tfrac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n\pi},$$

so that

$$\lfloor x \rfloor = x - \tfrac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{n\pi}.$$

Substituting, we get

$$\begin{aligned} \left[\frac{2s^2}{p}\right] - 2\left[\frac{s^2}{p}\right] &= \frac{2s^2}{p} - \tfrac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin (4n\pi s^2/p)}{n\pi} \\ &\quad - 2\left\{\frac{s^2}{p} - \tfrac{1}{2} + \sum_{n=1}^{\infty} \frac{\sin (2n\pi s^2/p)}{n\pi}\right\} \\ &= \tfrac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}\left\{\sin \frac{4n\pi s^2}{p} - 2 \sin \frac{2n\pi s^2}{p}\right\}; \end{aligned}$$

$$\sum_{s=1}^{(p-1)/2}\left(\left[\frac{2s^2}{p}\right]-2\left[\frac{s^2}{p}\right]\right)$$
$$=\frac{p-1}{4}+\sum_{n=1}^{\infty}\frac{1}{n\pi}\left\{\sum_{s=1}^{(p-1)/2}\left(\sin\frac{4n\pi s^2}{p}-2\sin\frac{2n\pi s^2}{p}\right)\right\}.$$

Therefore we have to prove that

$$\sum_{n=1}^{\infty}\frac{1}{n\pi}\sum_{s=1}^{(p-1)/2}\sin\frac{4n\pi s^2}{p}\leqq\sum_{n=1}^{\infty}\frac{2}{n\pi}\sum_{s=1}^{(p-1)/2}\sin\frac{2n\pi s^2}{p}.$$

Now we have by the results on Gaussian sums,

$$\text{if }\left(\frac{2n}{p}\right)=1,\quad \sum_{s=1}^{(p-1)/2}e^{2n(2\pi is^2/p)}=\sum_{(r/p)=1}e^{2\pi ir/p}=+\frac{ip^{1/2}}{2};$$
$$\text{if }\left(\frac{2n}{p}\right)=-1,\quad \sum_{s=1}^{(p-1)/2}e^{2n(2\pi is^2/p)}=\sum_{(r/p)=1}e^{-2\pi ir/p}=-\frac{ip^{1/2}}{2};$$
$$\text{if }\left(\frac{2n}{p}\right)=0,\quad \sum_{s=1}^{(p-1)/2}e^{2n(2\pi is^2/p)}=\frac{p-1}{2}.$$

Taking imaginary parts, we obtain

$$\sum_{s=1}^{(p-1)/2}\sin\frac{4n\pi s^2}{p}=\left(\frac{2n}{p}\right)\frac{p^{1/2}}{2}.$$

Similarly,

$$\sum_{s=1}^{(p-1)/2}\sin\frac{2n\pi s^2}{p}=\left(\frac{n}{p}\right)\frac{p^{1/2}}{2}.$$

Therefore we have to prove that

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{2n}{p}\right)\frac{p^{1/2}}{2}\leqq\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{n}{p}\right)\frac{p^{1/2}}{2};$$

that is,

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{2n}{p}\right)\leqq 2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{n}{p}\right).$$

This is equivalent to the relation

$$\sum_{n=1}^{\infty}\frac{-1}{n}\left(\frac{n}{p}\right)\leqq 2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{n}{p}\right),\qquad \text{if } p\equiv 3 \pmod 8;$$

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{n}{p}\right) \leqq 2\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{n}{p}\right); \qquad \text{if } p \equiv 7 \pmod{8}.$$

Thus in each case we have to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{n}{p}\right) \geqq 0.$$

Now Landau would call this last result trivial. But for the sake of completeness we give its proof here; we have in fact, for $s > 1$,

$$\left\{\sum_{n=1}^{\infty} \frac{1}{n^s}\left(\frac{n}{p}\right)\right\} \prod_{p_1}\left(1 - \frac{1}{p_1^s}\left(\frac{p_1}{p}\right)\right) = 1,$$

where $p_1$ runs through the sequence of primes. The series being uniformly convergent for $s \geqq 1$ its sum is continuous at $s = 1$. Hence the result follows.

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# THE APRIL MEETING AT STANFORD UNIVERSITY

The three hundred seventy-seventh meeting of the American Mathematical Society was held at Stanford University, California, on Saturday, April 5, 1941. The attendance was about sixty including the following thirty-nine members of the Society:

H. M. Bacon, Clifford Bell, B. A. Bernstein, H. F. Blichfeldt, J. H. Bowman, D. G. Chapman, G. B. Dantzig, T. C. Doyle, G. C. Evans, A. L. Foster, E. C. Goldsworthy, C. A. Hayes, Lulu Hofmann, D. H. Lehmer, R. J. Levit, J. V. Lewis, C. F. Luther, Rhoda Manning, W. A. Manning, A. E. Marston, C. B. Morrey, A. P. Morse, W. H. Myers, M. L. Nelson, Jerzy Neyman, A. R. Noble, W. T. Puckett, T. M. Putnam, R. M. Robinson, E. B. Roessler, A. C. Schaeffer, W. H. Simons, Pauline Sperry, L. H. Swinford, Gabor Szegö, A. W. Tucker, J. V. Uspensky, Morgan Ward, W. F. Whitmore.

General sessions were held in the morning and in the afternoon. The meeting was called to order at 10 A.M., and was presided over at different times by Professors Gabor Szegö, G. C. Evans, and T. M. Putnam. By invitation of the Program Committee, Professor H. J. Stewart of the California Institute of Technology delivered an hour address on *Hydrodynamic problems arising from the investigation of the transverse circulation in the atmosphere.*

Following the meeting Professor and Mrs. Szegö served tea at their home for members of the Society and their friends.

The titles and cross references to abstracts of papers read at the meeting are given below. Abstracts whose numbers are followed by *t* were read by title. Dr. William Mersman was introduced by Professor E. B. Roessler.

1. L. H. Swinford: *On Abel's equation.* (Abstract 47-5-250.)

2. William Mersman: *Heat conduction in an infinite composite solid.* (Abstract 47-5-256.)

3. D. H. Lehmer: *On the values of certain Hurwitzian continued fractions.* (Abstract 47-5-213.)

4. A. W. Tucker: *Simplicial bands.* (Abstract 47-5-286.)

5. Morgan Ward: *Finite operators of formal power series.* (Abstract 47-5-206.)

6. G. B. Dantzig: *Variance of the error of a mean computed by grouping.* (Abstract 47-5-269.)

7. W. T. Puckett: *Concerning a transformation in the plane.* Preliminary report. (Abstract 47-5-282.)

8. Nathaniel Coburn: *Frenet formulas for curves in unitary space.* (Abstract 47-5-263-*t*.)

9. Henry Scheffé: *Note on the reduction of* $\chi^2$ *for fit of a frequency distribution.* (Abstract 47-5-274-*t*.)

10. Hans Lewy: *Fractional potentials.* (Abstract 47-5-231-*t*.)

11. H. M. Schwartz: *Sequences of Darboux-Stieltjes integrals.* (Abstract 47-5-246-*t*.)

12. R. K. Wakerling: *On the rational loci of* $\infty^1(\rho-1)$*-spaces in* $r$*-space.* (Abstract 47-5-267-*t*.)

13. Max Zorn: *Transcendental p-adic numbers related to roots of unity.* (Abstract 47-5-215-*t*.)

T. M. PUTNAM,
*Associate Secretary*

## THE APRIL MEETING IN CHICAGO

The three hundred seventy-eighth meeting of the American Mathematical Society was held at the University of Chicago, Chicago, Illinois, on Friday and Saturday, April 11–12, 1941. The attendance was about two hundred thirty including the following one hundred eighty-seven members of the Society:

A. A. Albert, G. E. Albert, W. L. Ayres, Reinhold Baer, R. M. Ballard, R. H. Bardell, R. W. Barnard, I. A. Barnett, Walter Bartky, E. F. Beckenbach, J. H. Bell, D. L. Bernstein, S. F. Bibb, D. H. Blackwell, G. A. Bliss, Morris Bloom, Henry Blumberg, L. M. Blumenthal, H. R. Brahana, Richard Brauer, Leonard Bristow, F. L. Brooks, H. K. Brown, O. E. Brown, P. B. Burcham, R. S. Burington, Herbert Busemann, L. E. Bush, J. W. Calkin, E. J. Camp, H. H. Campaigne, A. D. Campbell, E. W. Chittenden, R. V. Churchill, C. E. Clark, M. D. Clement, L. W. Cohen, B. H. Colvin, A. H. Copeland, J. J. Corliss, L. L. Cronvich, E. L. Crow, D. R. Curtiss, J. H. Curtiss, W. M. Davis, M. M. Day, J. J. DeCicco, L. L. Dines, J. M. Dobbie, D. M. Dribin, R. J. Duffin, W. D. Duthie, J. J. Eachus, J. M. Earl, Samuel Eilenberg, Benjamin Epstein, H. P. Evans, H. S. Everett, L. R. Ford, R. H. Fox, Bernard Friedman, Morris Friedman, H. L. Garabedian, E. R. Garbe, B. H. Gere, H. A. Giddings, J. S. Gold, H. H. Goldstine, Cornelius Gouwens, L. M. Graves, L. W. Griffiths, V. G. Grove, G. J. Haltiner, R. W. Hamming, G. E. Hay, O. C. Hazlett, A. E. Heins, E. D. Hellinger, M. R. Hestenes, J. F. Heyda, J. J. L. Hinrichsen, D. L. Holl, A. S. Householder, D. H. Hyers, M. H. Ingraham, Dunham Jackson, S. B. Jackson, R. N. Johanson, Wilfred Kaplan, William Karush, I. F. Keeler, J. L. Kelley, Lois Kiefer, H. S. Kieval, S. C. Kleene, W. C. Krathwohl, A. C. Ladner, E. P. Lane, R. E. Langer, W. G. Leavitt, R. A. Leibler, C. B. Lindquist, A. T. Lonseth, A. W. McGaughey, Saunders MacLane, Ralph Mansfield, Morris Marden, R. H. Marquis, C. W. Mendel, Herman Meyer, H. L. Meyer, E. W. Miller, R. A. Miller, W. E. Milne, W. L. Mitchell, Virginia Modesitt, C. W. Moran, Vladimir Morkovin, Max Morris, E. J. Moulton, J. P. Nash, A. L. Nelson, K. L. Nielsen, I. M. Niven, E. A. Nordhaus, E. N. Oberg, J. W. Odle, Rufus Oldenburger, E. W. Paxson, J. F. Paydon, E. R. Peck, G. H. Peebles, Sam Perlis, J. E. Powell, Tibor Radó, E. D. Rainville, W. C. Randels, R. B. Rasmusen, Maxwell Reade, W. T. Reid, C. E. Rickart, R. F. Rinehart, P. G. Robinson, Arthur Rosenthal, M. A. Sadowsky, R. G. Sanger, Max Sasuly, O. F. G. Schilling,

E. R. Schneckenburger, K. C. Schraut, G. E. Schweigert, W. T. Scott, M. E. Shanks, C. G. Shover, H. A. Simmons, A. H. Smith, F. C. Smith, W. S. Snyder, E. S. Sokolnikoff, I. S. Sokolnikoff, N. E. Steenrod, R. C. Stephens, B. M. Stewart, C. F. Strobel, Otto Szász, H. P. Thielman, J. M. Thompson, C. W. Topp, L. W. Tordella, H. C. Trimble, P. L. Trump, S. M. Ulam, Henry Van Engen, G. B. Van Schaack, J. I. Vass, H. S. Wall, M. S. Webster, K. W. Wegner, M. E. Wescott, G. W. Whaples, G. W. Whitehead, L. R. Wilcox, L. A. Wolf, M. C. Wolf, F. E. Wood, M. A. Woodbury, J. W. T. Youngs.

The meeting opened on Friday morning with three sections, Analysis, Professor E. W. Chittenden presiding; Analysis and Applied Mathematics, Professor R. V. Churchill presiding; and Algebra, Professor A. A. Albert presiding. On Friday afternoon Professor M. R. Hestenes gave the Symposium Lecture under the title *The problem of Bolza in the calculus of variations.* At this lecture Professor G. A. Bliss presided. On Saturday morning there were two sections, Analysis, Professor E. F. Beckenbach presiding; and Geometry and Topology, Professor L. M. Blumenthal presiding.

All sessions met in Eckhart Hall and the adjoining Ryerson Laboratory.

Friday afternoon after the Symposium Lecture the ladies of the Department of Mathematics of the University of Chicago served tea in the Commons Room of Eckhart Hall.

The dinner on Friday evening at Burton Court was attended by one hundred sixty-eight mathematicians and guests. The toastmaster, Professor Tibor Radó, pointed out that the University of Chicago was soon to have another claim to fame in having the youngest looking Professor Emeritus in the world in the person of Professor G. A. Bliss, who retires from the faculty at the end of the coming summer quarter. He then called on two former students of Professor Bliss, Professors L. L. Dines and L. M. Graves, who related many interesting incidents in the teaching career of Professor Bliss. ProProfessor Bliss then spoke of the earlier mathematicians at the University of Chicago and of his pleasure in being associated with them. Many former students of Professor Bliss, who could not be present, sent telegrams and letters of greetings which were presented to him at the dinner.

Titles and cross references to the abstracts of the papers read at this meeting follow below. Papers numbered 1 to 8 were read before the Analysis section Friday morning, papers 9 to 15 before the section for Analysis and Applied Mathematics, papers 16 to 26 before the Algebra section, papers 27 to 33 before the Analysis section on Saturday morning, and papers 34 to 41 before the section for Geometry and Topology. Papers 42 to 58, whose abstract numbers are followed by the letter *t*, were read by title. Mr. W. R. Scott was introduced by Professor Tibor Radó, Dr. R. S. Pate by Professor H. R. Brahana, Mr. L. J. Savage by Professor S. B. Myers, and Miss Harlan C. Miller by Professor R. L. Moore. Paper 31 was read by Professor Wall, 32 by Dr. Scott, and 39 by Dr. Youngs.

1. D. H. Hyers: *On the stability of the linear functional equation.* (Abstract 47-3-134.)

2. C. E. Rickart: *Integration in a convex linear topological space.* (Abstract 47-5-241.)

3. W. C. Randels: *On Bessel's inequality in abstract spaces.* (Abstract 47-5-240.)

4. H. H. Goldstine: *The parametric problem of the calculus of variations in general analysis.* (Abstract 47-5-224.)

5. W. R. Scott: *A lemma on the Weierstrass E-function.* Preliminary report. (Abstract 47-5-247.)

6. E. F. Beckenbach: *On functions having subharmonic logarithms.* (Abstract 47-5-216.)

7. Tibor Radó: *On the semi-continuity of double integrals in the calculus of variations.* (Abstract 47-3-140.)

8. J. F. Heyda: *A uniqueness property of general monogenic functions.* (Abstract 47-5-226.)

9. A. S. Householder: *A theory of steady-state activity in nerve-fiber networks.* II. *The simple circuit.* (Abstract 47-5-255.)

10. G. E. Hay: *The finite displacement of thin rods.* (Abstract 47-5-253.)

11. I. S. Sokolnikoff: *Deflection of non-isotropic elastic plates.* Preliminary report. (Abstract 47-5-257.)

12. A. E. Heins: *On the transformation theory of the solution of partial differential equations.* Preliminary report. (Abstract 47-5-254.)

13. E. L. Crow: *The expansion problem associated with an ordinary differential equation of first order in which the coefficient is a quadratic polynominal in the parameter.* (Abstract 47-5-221.)

14. A. T. Lonseth: *Dirichlet's principle for certain nonlinear elliptic equations.* (Abstract 47-5-232.)

15. G. H. Peebles: *On the behavior of series of polynomials orthogonal with respect to a weight function of changing sign.* (Abstract 47-5-238.)

16. H. P. Thielman: *Groups of linear fractional transformations.* (Abstract 47-5-203.)

17. M. A. Woodbury: *On ordered groups.* (Abstract 47-5-209.)

18. L. R. Wilcox: *Extensions of semi-modular lattices.* (Abstract 47-5-208.)

19. I. M. Niven: *Sums of nth powers of quadratic integers*. (Abstract 47-5-193.)

20. Reinhold Baer: *A unified theory of projective spaces and finite abelian groups*. (Abstract 47-5-182.)

21. B. M. Stewart: *Left-associated matrices with elements in an algebraic domain.* (Abstract 47-5-202.)

22. L. W. Griffiths: *The minimum number of variables in universal functions of polygonal numbers.* (Abstract 47-5-212.)

23. R. S. Pate: *Rings with multiple-valued operations.* (Abstract 47-5-194.)

24. H. A. Simmons: *Classes of maximum numbers associated with symmetric equations in n reciprocals.* IV. (Abstract 47-5-199.)

25. J. H. Bell: *Topics related to the factorization of matrices.* (Abstract 47-3-115.)

26. Richard Brauer: *On the connection between the ordinary and the modular characters of groups of finite order.* (Abstract 47-5-183.)

27. Benjamin Epstein: *Asymptotic problems associated with a Laplace-Mellin integral equation.* Preliminary report. (Abstract 47-5-223.)

28. R. J. Duffin: *Möbius transforms and Fourier transforms.* (Abstract 47-5-222.)

29. Dunham Jackson: *Generalization of a theorem of Korous.* (Abstract 47-5-227.)

30. Otto Szász: *On convergence and summability of trigonometric series.* (Abstract 47-5-251.)

31. W. T. Scott and H. S. Wall: *The transformation of series and sequences.* (Abstract 47-3-144.)

32. W. T. Scott and H. S. Wall: *Gronwall summability.* (Abstract 47-5-248.)

33. P. B. Burcham: *Some inclusion relations in the Hausdorff difference matrix.* (Abstract 47-5-219.)

34. V. G. Grove: *The transformation T of congruences.* (Abstract 47-5-264.)

35. J. J. DeCicco: *The differential geometry of the Laguerre group* $G_6$. (Abstract 47-3-154.)

36. L. J. Savage: *Distance spaces.* (Abstract 47-5-266.)

37. M. E. Shanks: *The space of all metrics on a compact space.* (Abstract 47-5-284.)

38. Samuel Eilenberg: *Banach space methods in topology.* I. (Abstract 47-5-278.)

39. G. E. Albert and J. W. T. Youngs: *The structure of locally connected topological spaces.* (Abstract 47-5-277.)

40. G. E. Schweigert: *Border transformations.* II. (Abstract 47-5-283.)

41. G. W. Whitehead: *Homotopy properties of rotation groups.* (Abstract 47-5-289.)

42. Edward Kasner and J. J. DeCicco: *Infinite groups generated by equilong involutions and symmetries.* (Abstract 47-3-160-*t*.)

43. J. J. DeCicco: *Lineal element transformations which preserve dual-isothermal families.* (Abstract 47-3-153-*t*.)

44. P. C. Hammer: *On fitting linear functions when all variables are subject to error.* (Abstract 47-5-272-*t*.)

45. Fred Kiokemeister: *The parastrophic criterion for the factorization of primes.* (Abstract 47-5-187-*t*.)

46. R. E. Lane: *The first three rational operations.* (Abstract 47-5-229-*t*.)

47. C. D. Olds: *Expansion of an arbitrary function in terms of certain polynomials.* (Abstract 47-5-237-*t*.)

48. C. D. Olds: *On some arithmetical identities.* (Abstract 47-5-214-*t*.)

49. G. E. Albert: *On separation spaces.* Preliminary report. (Abstract 47-5-276-*t*.)

50. H. A. Simmons: *Use of matrices in solving linear Diophantine equations.* (Abstract 47-5-200-*t*.)

51. Henry Scheffé: *Linear differential equations with two-term recurrence formulas.* (Abstract 47-5-244-*t*.)

52. Reinhold Baer: *Homogeneity of projective planes.* (Abstract 47-5-261-*t*.)

53. Lincoln La Paz: *Double integral variation problems with prescribed transversality coefficients.* (Abstract 47-5-230-*t*.)

54. Harlan C. Miller: *Concerning compact unicoherent continua.* (Abstract 47-5-281-*t*.)

55. N. E. Steenrod and A. W. Tucker: *Real n-quadrics as sphere-bundles.* (Abstract 47-5-285-*t*.)

56. D. M. Dribin: *A theorem on sets of ideals in solvable extensions of algebraic number fields.* (Abstract 47-5-186-*t*.)

57. Sam Perlis: *A characterization of the radical of an algebra.* (Abstract 47-5-195-*t*.)

58. W. T. Reid: *Green's lemma and related results.* (Abstract 47-7-313-*t*.)

W. L. AYRES,
*Associate Secretary*

# THE MAY MEETING IN WASHINGTON

The three hundred seventy-ninth meeting of the American Mathematical Society was held at The George Washington University, Washington, D. C., on Friday and Saturday, May 2–3, 1941. The attendance was approximately two hundred fifty including the following one hundred eighty-two members of the Society:

C. R. Adams, R. P. Agnew, Leonidas Alaoglu, C. B. Allendoerfer, P. H. Anderson, H. A. Arnold, J. V. Atanasoff, Y. K. Bal, Alfred Basch, P. T. Bateman, P. G. Bergmann, E. E. Betz, G. D. Birkhoff, Archie Blake, R. P. Boas, H. F. Bohnenblust, T. A. Botts, D. G. Bourgin, S. G. Bourne, R. H. Bruck, R. S. Burington, E. W. Cannon, Abraham Charnes, A. B. Coble, I. S. Cohen, R. M. Cohn, J. B. Coleman, Richard Courant, W. H. H. Cowles, M. J. Cox, M. T. Curran, H. B. Curry, Tobias Dantzig, C. H. Dowker, Arnold Dresden, F. G. Dressel, R. F. Dressler, J. C. Durand, Samuel Eilenberg, Benjamin Epstein, Paul Erdös, F. A. Ficken, E. J. Finan, M. M. Flood, Tomlinson Fort, R. M. Foster, F. H. Fowler, K. O. Friedrichs, T. C. Fry, Abe Gelbart, J. H. Giese, M. A. Girshick, J. W. Givens, Michael Goldberg, H. S. Grant, J. A. Greenwood, T. N. E. Greville, D. W. Hall, N. A. Hall, P. R. Halmos, W. J. Harrington, G. A. Hedlund, M. H. Heins, Edward Helly, H. C. Hicks, Einar Hille, M. P. Hollcroft, T. R. Hollcroft, Witold Hurewicz, Nathan Jacobson, R. F. Johnson, F. E. Johnston, H. A. Jordan, Mark Kac, Shizuo Kakutani, J. H. Kenna, R. B. Kershner, B. F. Kimball, J. W. Kitchens, J. R. Kline, H. N. Laden, O. E. Lancaster, A. E. Landry, M. E. Layne, Solomon Lefschetz, Joseph Lehner, Marie Litzinger, E. R. Lorch, Eugene Lukacs, D. T. McClay, N. H. McCoy, E. J. McShane, J. K. L. MacDonald, C. C. MacDuffee, H. M. MacNeille, Dorothy Maharam, P. T. Maker, Dorothy Manning, M. H. Martin, F. M. Mears, Don Mittleman, Elizabeth Monroe, T. W. Moore, Marston Morse, F. D. Murnaghan, F. J. Murray, F. G. Myers, Yael Naim, K. L. Nielsen, Ivan Niven, F. W. Owens, H. B. Owens, J. C. Oxtoby, Gordon Pall, G. W. Patterson, Everett Pitcher, George Polya, H. A. Rademacher, Tibor Radó, C. J. Rees, R. G. D. Richardson, E. K. Ritter, H. E. Robbins, J. H. Roberts, H. P. Robertson, M. S. Robertson, L. B. Robinson, W. J. Robinson, J. B. Rosser, Raphaël Salem, S. A. Schelkunoff, Ivor Schilansky, Abraham Schwartz, W. M. Scott, I. E. Segal, Abraham Seidenberg, I. M. Sheffer, Seymour Sherman, J. A. Shohat, Abraham Sinkov, M. F. Smiley, T. L. Smith, Virgil Snyder, V. E. Spencer, J. J. Stoker, M. H. Stone, J. D. Tamarkin, A. H. Taub, J. H. Taylor, B. J. Tepping, M. E. Terry, J. M. Thompson, C. J. Thorne, R. M. Thrall, E. W. Titt, C. B. Tompkins, W. R. Transue, W. J. Trjitzinsky, J. W. Tukey, J. H. Van Vleck, Oswald Veblen, T. L. Wade, G. L. Walker, A. D. Wallace, Henry Wallman, G. C. Webber, F. M. Weida, C. H. Wheeler, P. A. White, A. L. Whiteman, G. T. Whyburn, Norbert Wiener, J. D. Williams, M. H. Williams, W. L. G. Williams, John Williamson, H. A. Wood, D. W. Woodard, J. W. Wrench, J. W. T. S. Youngs, Oscar Zariski, Antoni Zygmund.

The meeting opened Friday morning in two sections, Analysis, Professor Tibor Radó presiding, and Algebra and Geometry, Professor Gordon Pall and Dr. R. M. Thrall presiding.

Two general sessions were held Friday afternoon. At the first, Vice President F. D. Murnaghan presiding, Professor F. J. Murray of

Columbia University gave an address entitled *The analysis of linear transformations.* The second, Professor A. B. Coble presiding, was for short papers on various subjects.

On Saturday morning at a general session, Professor J. A. Shohat presiding, Professor I. M. Sheffer of The Pennsylvania State College gave an address entitled *Some applications of certain polynomial classes.* This was followed by two sections, Algebra and Topology, Professor C. C. MacDuffee presiding, and Probability, Statistics, Applied Mathematics, Professor Norbert Wiener presiding.

Saturday afternoon was devoted to a symposium on the Rayleigh-Ritz method and its applications, at which President Marston Morse presided. The following addresses were given: *Variational methods for the solution of boundary value problems* by Professor Richard Courant of New York University; *Engineering applications of the Rayleigh-Ritz method* by Professor J. P. Den Hartog of Harvard University; *Applications of the Rayleigh-Ritz method in the theory of the structure of matter* by Professor H. M. James of Purdue University. The discussion was led by Professor J. H. Van Vleck of Harvard University and Professor P. M. Morse of Massachusetts Institute of Technology.

All sessions were held in the Hall of Government of The George Washington University.

On Friday afternoon tea was served at Columbian House.

The dinner Friday evening at the Cosmos Club was attended by one hundred thirty members and guests. The toastmaster, Professor C. R. Adams, introduced three speakers, Professor R. W. Bolwell, Chairman of the Graduate Council of The George Washington University, and Professors C. C. MacDuffee and H. P. Robertson.

At the beginning of the symposium Saturday afternoon Professor Gordon Pall presented a resolution expressing the appreciation and thanks of the Society to The George Washington University, the members of the local committee, and all others concerned for the excellent arrangements for the meeting. The resolution was unanimously adopted by a rising vote.

The Council met at 9:00 P.M., Friday, May 2, in the Holmes Room of the Cosmos Club.

The Secretary announced the election of the following fifty-nine persons to membership in the Society:

Mr. Milton Abramowitz, Brooklyn, N. Y.;
Miss Grace Karen Anderson, American Mathematical Society, New York, N. Y.;
Professor Tomás R. Bachiller, University of Madrid, Madrid, Spain;
Mr. Yusuf Kenan Bal, Columbia University;

Mr. James Douglas Bankier, Rice Institute;
Mr. Paul Trevier Bateman, University of Pennsylvania;
Professor Maurice Anthony Biot, Columbia University;
Professor Juan Blaquier, University of Buenos Aires, Buenos Aires, Argentina;
Professor Clotilde Bula, University of Rosario, Rosario, Argentina;
Mr. Philip Staats Carter, R.C.A. Communications Inc., Rocky Point, L. I., N. Y.;
Mr. Leo Cohen, Columbia University;
Mr. Charles Raymond DePrima, New York University;
Dr. Henry Chabot Dieckmann, Occidental College, Los Angeles, Calif.;
Dean Margaret Criswell Disert, Wilson College, Chambersburg, Pa.;
Mr. Robert Franklin Dressler, University of Pennsylvania;
Professor William Maurice Ewing, Lehigh University;
Mr. W. Eugene Ferguson, University of Missouri;
Professor Fernando L. Gaspar, University of Rosario, Rosario, Argentina;
Dr. John H. Giese, Rutgers University;
Dr. Basil E. Gillam, University of Missouri;
Mr. Thomas Franklin Glass, Rice Institute;
Dean John Alonzo Goff, Towne Scientific School, University of Pennsylvania;
Mr. Richard Albert Good, University of Wisconsin;
Mr. Lewis Greenwald, New York Evening High School;
Mr. Eli Grossman, United States Life Insurance Co., New York, N. Y.;
Professor Emil J. Gumbel, New School for Social Research, New York, N. Y.;
Professor Harrod Harold Hartzler, Goshen College, Goshen, Ind.;
Professor Clifton T. Hazard, Purdue University;
Dr. Aaron Paul Horst, Procter and Gamble Co., Cincinnati, Ohio;
Dr. Colin Munroe Hudson, Frankford Arsenal, Philadelphia;
Dr. Leroy Charles Hutchinson, Michigan College of Mining and Technology, Houghton, Mich.;
Professor Ralph William Jones, University of Delaware;
Major Paul Hanes Kemmer, Chief Aircraft Laboratory, Wright Field, Dayton, Ohio;
Dr. John Maurice Kingston, University of Washington;
Mr. Joseph Landin, New York University;
Miss Jeanne Starrett Le Caine, Cambridge, Mass.;
Mr. Max LeLeiko, Willard Contracting Corporation, New York, N. Y.;
Dr. Eugene Lukacs, Baltimore, Md.;
Dr. Rudolf Luneburg, Spencer Lens Company, Buffalo, N. Y.;
Dr. Szolem Mandelbrojt, Rice Institute;
Dr. Warren Perry Mason, Bell Telephone Laboratories, New York, N. Y.;
Dr. William Alvin Mersman, University of California, College of Agriculture;
Mr. Osborn Maitland Miller, American Geographical Society, New York, N. Y.;
Mr. Olen Alvin Nance, University of Missouri;
Dr. William Donald Phelps, R.C.A. Manufacturing Co., Camden, N. J.;
Mr. Harry Pollard, Dorchester, Mass.;
Dr. Virgil Nelson Robinson, Louisiana State University;
Professor Frederick William Rogers, University of Cincinnati;
Mr. Jerome Colbert Smith, University of Pennsylvania;
Dr. Ernst Snapper, Princeton University;
Mr. Herbert Solomon, Columbia University;
Dr. Thomas H. Southard, Wayne University;

Mr. Wolfgang Joseph Thron, Rice Institute;
Dr. Philip Russell Wallace, University of Cincinnati;
Mr. Edward Faith White, University of Cincinnati;
Mr. Albert Wiesenberg, Columbia University;
Mr. Herbert Wolfe, Brooklyn College (Evening Session);
Mr. Richard Schuyler Wolfe, Great Falls Commercial College, Great Falls, Mont.;
Mr. Max A. Woodbury, University of Michigan.

It was reported that the following had been elected as nominees on the Institutional Memberships of the institutions indicated:

University of Chicago: Messrs. William Caswell Carter, Gerhard Karl Kalisch, and Richard Donald Schafer, and Miss Alice Elizabeth Turner.
Lehigh University: Mr. Leonard Pattillo Burton
Metropolitan Life Insurance Company: Messrs. Malvin E. Davis and James Russell Herman.
Syracuse University: Mr. William John Strange.
Wayne University: Mr. Clifford Raymond Simms.
Wesleyan University: Messrs. Francis Escott and Joseph Wannemacher.

The following appointments by President Marston Morse were reported: as a standing Committee on Places of Meetings, Professors A. B. Coble (chairman), 3 years, W. B. Carver, 2 years, and E. J. McShane, 1 year; as auditors of the Society's books for 1941, Professors R. G. Archibald and A. E. Meder, Jr.; as an additional member of the Committee on Arrangements for the Annual Meeting of 1941, Professor W. D. Cairns; as Committee on Arrangements for the Summer Meeting of 1942 at Cornell University, Professors R. P. Agnew (chairman), W. D. Cairns, A. D. Campbell, T. R. Hollcroft, J. B. Rosser, R. J. Walker; as chairman of the Nominating Committee, to replace Professor G. A. Bliss, Professor A. B. Coble; as representative of the Society at the inauguration of Victor Raymond Edman as fourth President of Wheaton College, Wheaton, Illinois, on May 9, 1941, Professor L. M. Graves; as representative of the Society at the inauguration of Virgil Melvin Hancher as President of The State University of Iowa on May 24, 1941, Professor R. B. McClenon; as representative of the Society at the Fiftieth Anniversary Celebration of the University of Chicago on September 27, 28, and 29, 1941, Professor D. R. Curtiss. It was also reported that President Marston Morse will represent the Society at the Centenary Celebration of Fordham University on September 15–17, 1941.

The dates for the Annual Meeting of 1941 at Lehigh University were set as December 29–31, 1941.

The Secretary reported that Professor E. J. McShane had accepted the invitation to deliver the Colloquium Lectures at the Summer

Meeting of 1943; his subject will be *Existence theorems in the calculus of variations.*

The Secretary reported that the following had been chosen as the voting representatives on the Council of the editorial committees for the year 1941: Bulletin—Dean Tomlinson Fort; Transactions—Professor C. C. MacDuffee; Colloquium—Professor J. D. Tamarkin; American Journal of Mathematics—Professor T. H. Hildebrandt.

The Secretary reported that an unrestricted gift of $1,000 had been received from Dr. Robert Henderson. The following resolution was unanimously adopted and copies were ordered spread upon the records and sent to Dr. Henderson:

> The Council of the American Mathematical Society expresses its deep gratitude to Dr. Robert Henderson for his generous gift of $1,000 to the Society. It also wishes at this time to make grateful acknowledgment of his many years of faithful service on the Board of Trustees where his wise counsel has been invaluable. We recognize these acts as the natural expression of a life devoted to the study and application of mathematics.

It was announced that Professor Oscar Zariski had been elected as a member of the Transactions Editorial Committee to fill the unexpired term of the late Professor W. C. Graustein. Professor Hille reported that Professors Nelson Dunford and R. L. Wilder had been appointed Associate Editors.

Professors J. L. Walsh (chairman), E. W. Chittenden, and E. J. McShane were appointed as a committee to confer with the Williams and Wilkins Company on an agreement concerning the proposed new series of monographs for which the name *Mathematical Surveys* has been suggested. The Council had previously voted to approve the memorandum submitted by the Company in which the introduction of this series was suggested.

Titles and cross references to the abstracts of the papers read follow below. Papers whose abstract numbers are followed by the letter *t* were read by title. The papers numbered 1 to 11 were read before the section for Analysis; those numbered 12 to 22 before the section for Algebra and Geometry; those numbered 23 to 27 before the general session on Friday afternoon; those numbered 28 to 34 before the section for Algebra and Topology; those numbered 35 to 41 before the section for Probability, Statistics, Applied Mathematics; those numbered 42 to 67 were read by title. Paper 8 was read by Mr. Harrington, paper 14 by Dr. Smiley, paper 17 by Dr. Bruck, paper 18 by Professor Wade, paper 27 by Professor Polya, and paper 38 by Mr. Thorne. Dr. Bergman was introduced by Dr. D. C. Spencer, Mr. Gravalos by Professor G. D. Birkhoff, Professor Sutton by Professor C. B. Allendoerfer, Dr. Weinstein by Dr. H. E. Robbins, Dr. Bers by Dr. Abe Gelbart.

1. Joseph Lehner: *A partition function connected with the modulus five.* (Abstract 47-7-298.)

2. Stefan Bergman: *The method of the minimum integral and the analytic continuation of functions.* (Abstract 47-5-217.)

3. K. L. Nielsen: *Concerning boundary value problems for linear differential equations when the boundary conditions are given by Stieltjes integrals.* (Abstract 47-5-236.)

4. E. J. McShane: *Sufficient conditions for a weak relative minimum in the problem of Bolza.* (Abstract 47-7-309.)

5. F. A. Ficken: *Certain systems of subsets of quasi and partially ordered sets.* (Abstract 47-7-328.)

6. E. R. Lorch: *The spectrum of linear transformations.* (Abstract 47-5-235.)

7. W. J. Trjitzinsky: *Singular Lebesgue-Stieltjes integral equations.* (Abstract 47-7-316.)

8. W. J. Harrington and J. B. Rosser: *A study of certain functions auxiliary to Brun's method in number theory.* (Abstract 47-5-225.)

9. D. G. Bourgin: *On reflexive Banach spaces.* (Abstract 47-7-304.)

10. M. H. Heins: *A generalization of the Aumann-Carathéodory "Starrheitssatz."* (Abstract 47-7-306.)

11. Raphaël Salem: *On trigonometrical series whose coefficients do not tend to zero.* (Abstract 47-5-243.)

12. W. M. Scott: *On matrix algebras over an algebraically closed field.* (Abstract 47-5-198.)

13. R. M. Thrall: *On Young's semi-normal representation of the symmetric group.* (Abstract 47-5-204.)

14. Everett Pitcher and M. F. Smiley: *Transitivities of betweenness.* (Abstract 47-5-196.)

15. Gordon Pall: *The construction of positive ternary quadratic forms.* (Abstract 47-3-174.)

16. D. T. McClay: *Clifford numbers.* (Abstract 47-7-322.)

17. R. H. Bruck and T. L. Wade: *Bisymmetric tensor algebra.* I. (Abstract 47-5-184.)

18. R. H. Bruck and T. L. Wade: *Bisymmetric tensor algebra.* II. (Abstract 47-5-185.)

19. Y. K. Bal: *Study on graphs.* Preliminary report. (Abstract 47-5-262.)

20. F. G. Gravalos: *The algebraic integrals of Hill's equations.* (Abstract 47-7-318.)

21. J. B. Rosser: *A generalization of the euclidean algorithm to several dimensions.* (Abstract 47-7-299.)

22. Mark Kac: *Remark on the distribution of values of the arithmetic function* $d(m)$. (Abstract 47-7-297.)

23. Tibor Radó: *On convergence in length and convergence in area.* (Abstract 47-5-239.)

24. J. A. Shohat: *On the best polynomial approximation for functions possessing derivatives.* (Abstract 47-5-249.)

25. C. C. MacDuffee: *Products and norms of ideals.* (Abstract 47-5-188.)

26. G. T. Whyburn: *Orbit decompositions.* (Abstract 47-5-291.)

27. George Polya and Norbert Wiener: *On the oscillation of the derivatives of a periodic function.* (Abstract 47-7-312.)

28. G. L. Walker: *The elementary divisors of a direct product of matrices.* (Abstract 47-5-205.)

29. H. A. Arnold: *Homology in set-intersections, with an application to r-regular convergence.* (Abstract 47-7-325.)

30. I. M. Niven: *Equations in quaternions.* (Abstract 47-5-192.)

31. A. D. Wallace: *A fixed-point theorem for trees.* (Abstract 47-5-287.)

32. Samuel Eilenberg: *Banach space methods in topology.* II. (Abstract 47-5-279.)

33. P. A. White: *A decomposition of true cyclic elements by means of continua.* (Abstract 47-5-288.)

34. Witold Hurewicz: *On duality theorems.* (Abstract 47-7-329.)

35. Alfred Basch: *A contribution to the theory of multiple correlation.* (Abstract 47-7-324.)

36. J. A. Greenwood: *A theorem on probability assignments to events.* (Abstract 47-5-271.)

37. K. O. Friedrichs: *On the mathematical theory of the buckling of spherical shells.* (Abstract 47-7-317.)

38. C. J. Thorne and J. V. Atanasoff: *The application of a general functional approximation method to thin plate problems.* (Abstract 47-5-259.)

39. R. M. Sutton: *An instrument for drawing confocal conics.* (Abstract 47-5-258.)

40. Alexander Weinstein: *On the vibrations of a clamped plate under tension.* (Abstract 47-5-260.)

41. Richard Courant: *On a new type of variational problems, with physical demonstrations.* (Abstract 47-5-220.)

42. R. P. Agnew: *On limits of integrals.* (Abstract 47-7-301-*t*.)

43. H. A. Arnold: *On r-regular convergence of sets.* (Abstract 47-7-326-*t*.)

44. Stefan Bergman: *A method for summation of series of orthogonal functions of two variables.* (Abstract 47-7-302-*t*.)

45. Lipman Bers: *On a generalized harmonic measure.* (Abstract 47-5-218-*t*.)

46. Leonard Carlitz: *Some interpolation formulas connected with polynomials in* $GF(p^n)$. (Abstract 47-5-210-*t*.)

47. Leonard Carlitz: *The coefficients of the reciprocal of certain series.* (Abstract 47-5-211-*t*.)

48. Leonard Carlitz: *An analogue of the Bernoulli polynomials.* (Abstract 47-7-296-*t*.)

49. A. H. Clifford: *Matrix representations of completely simple semigroups.* (Abstract 47-7-293-*t*.)

50. A. H. Copeland: *If.* (Abstract 47-5-268-*t*.)

51. J. J. DeCicco: *Equilong geometry of differential equations of first order.* (Abstract 47-3-151-*t*.)

52. Samuel Eilenberg: *Irreducible transformations onto manifolds.* (Abstract 47-5-280-*t*.)

53. G. E. Forsythe: *Cesàro summability of random variables.* (Abstract 47-5-270-*t*.)

54. Einar Hille: *Notes on linear transformations.* III. *Semi-groups on Lebesgue spaces.* (Abstract 47-7-307-*t*.)

55. C. T. Hsu: *Two samples from normal bivariate populations.* (Abstract 47-5-273-*t*.)

56. L. H. Loomis: *A simple proof of the Fatou theorem.* (Abstract 47-5-234-*t*.)

57. L. H. Loomis: *A converse to the Fatou theorem.* (Abstract 47-5-233-*t*.)

58. Saunders MacLane and O. F. G. Schilling: *A formula for the direct product of cross product algebras.* (Abstract 47-5-189-*t*.)

59. Saunders MacLane and O. F. G. Schilling: *Group extensions characterizing complete fields.* (Abstract 47-5-190-*t*.)

60. Saunders MacLane and O. F. G. Schilling: *The principal divisor theorem for function fields.* (Abstract 47-5-191-*t*.)

61. H. J. Riblet: *Certain theorems for symmetric differential functions.* (Abstract 47-5-197-*t*.)

62. M. S. Robertson: *The partial sums of multivalently star-like functions.* (Abstract 47-7-314-*t*.)

63. L. B. Robinson: *Some complete systems of semitensors.* (Abstract 47-7-315-*t*.)

64. Raphaël Salem: *On some properties of symmetrical perfect sets.* (Abstract 47-5-242-*t*.)

65. Louis Weisner: *Polynomials whose roots lie in a sector.* (Abstract 47-5-207-*t*.)

66. G. T. Whyburn: *Almost periodicity.* (Abstract 47-5-290-*t*.)

67. G. T. Whyburn: *Regular almost periodicity.* (Abstract 47-5-292-*t*.)

T. R. HOLLCROFT,
*Associate Secretary*

CALCUTTA CENTRAL LIBRARY UNIVERSITY

# BOOK REVIEWS

*Théorie Analytique des Associations Biologiques.* I. *Principes.* 1934, 45 pp. II. *Analyse Démographique avec Application Particulière à l'Espèce Humaine.* 1939, 149 pp. By A. J. Lotka. (Actualités Scientifiques et Industrielles, nos. 187, 780.) Paris, Hermann.

The first of these monographs is a general introduction to the second. It (the first) is primarily a non-mathematical discussion of what the author regards as two types of biological evolution:

(1) The ***intra-spécifique***, that which is concerned with the changes in the aggregate of distributions associated with a given biological species, brought about by mutation, genetic variation, natural selection, etc.

(2) The ***inter-spécifique***, that which deals with the changes in the numbers of individuals in each of several coexisting biological species or groups, which are brought about by different birth and death rates caused by interaction of the species with each other and with other environmental factors.

The author's main interest lies in the second type of evolution, which he discusses at some length by elaborating on the implication of chains of species regarded as food chains, and other ecological points. The problem of the variation of the numbers of individuals in a system of $n$ species is finally formulated in a system of equations $\partial x_i/\partial t = F_i(x_1, x_2, \cdots, x_n)$ $(i=1, 2, \cdots, n)$ where $x_j$ is the number of individuals in the $j$th species at time $t$. A formal solution is given for the case in which the $F_i$ are approximately homogeneous linear functions of the $x_i - c_j$ in the neighborhood of $(c_1, c_2, \cdots, c_n)$ where $(c_1, c_2, \cdots, c_n)$ are the values of $x_1, x_2, x_3, \cdots, x_n$, respectively, for some "stationary state" for which the $\partial x_i/\partial t = 0$. However, this formal solution and its properties are hardly discussed at all. Special cases of solutions for the case $n=1$ and 2 are given. For the case $n=1$, the author shows that the Malthusian and the Verhulst-Pearl-Reed logistic growth functions are obtained by respectively assuming $F_1(x)$ to be linear and then quadratic in $x$.

In the second monograph the author concerns himself with the dynamics of human populations in which there is assumed to be no immigration or emigration. The first chapter in this monograph begins with the fundamental equation

$$N(t) = \int_0^{\omega} B(t-a)p(a)da \tag{1}$$

where $B(t-a)da$ is the number of individuals born within the time interval $t-a-da$, $t-a$, $p(a)$ is the proportion of individuals born at a given time who are still alive at age $a$, while $N(t)$ is the number of individuals present in the population at time $t$, and $w$ is the maximum age attainable. $w$ is replaced by $+\infty$ in much of the work. Lotka goes into considerable detail in dealing with this equation, deriving formulas for age distribution and average age in the population, rates of birth and mortality, survival curves, vital index, and conditions for a stationary population. Particular emphasis is placed on the Malthusian case, namely that in which $N(t)=N_0 \exp(rt)$, and $B(t) = B_0 \exp(rt)$ where $r$ is the relative rate of increase in the population. A more general case is considered in which $B(t)$ (and hence $N(t)$) is a linear function of a function $\phi(t)$ and its derivatives. Particular attention is paid to the logistic function obtained by solving the equation $dN/dt=a_1N+a_2N^2$.

In the second chapter Lotka takes into account the effects of the variation of fertility with respect to age, by introducing a fertility function $m(a)$ as a multiplier in the integrand of (1), where $m(a)$ is the average number of daughters per mother borne by mothers in the age interval $a$, $a+da$. The introduction of $m(a)$ into (1) (setting $w=+\infty$) yields the integral equation

$$B(t) = \int_0^\infty B(t-a)p(a)m(a)da \tag{2}$$

which relates the number of births of daughters to the number of births of mothers. This problem and its ramifications are studied rather extensively for the Malthusian case. A formal solution of (2) for the Malthusian case of the form $B(t)=\sum_{i=1}^{\infty} Q_i \exp(r_i t)$ is obtained, where the $r_i$ are roots of the equation obtained by setting $B(t)= B_0 \exp(rt)$ in (2). Convergence and other fundamental questions which arise in connection with this solution are touched upon only lightly.

The third chapter deals with the composition of a population from the point of view of successive generations. Thus if $N$ is the number of mothers in a beginning (zeroth) generation of age $a$, the surviving female offspring which are produced during the age interval $da$ of mothers is $Np(a)m(a)da$, which is really the distribution of births of daughters in the first generation. The distribution $B_i(t)$ of births of daughters in the $i$th generation is therefore obtained by integrating $B_{i-1}(t-a)p(a)m(a)da$ from 0 to $t$, where $B_1(t)=Np(t)m(t)$. The total distribution of daughters $B(t)$ is found by summing the distribution functions for all generations, which yields an integral equation similar to (2). This same approach has been used by Lotka in his work on renewal theory.

The remaining four chapters deal with special problems such as determination of marriage rates, fertility rates and other indices and measures of natural increases in a population; determination of number of maternal, paternal and complete orphans; calculation of probabilities of extinction of a given line of descent and similar problems. The solution of the last problem, which has also been treated by R. A. Fisher, is extremely ingenious. A generation function $f(x)=\sum_{i=0}^{k}\pi_i x^i$ $(\sum_{i=0}^{k}\pi_i=1)$ is set up, where $\pi_i$ is the probability that a male will have exactly $i$ sons. $f(x)$ has many remarkable properties which enable the author to deal with the problem of descendance. For example $\partial f/\partial x]_{x=0}$ yields the average number of sons per male; the coefficient of $x^s$ in $f^r(x)$ yields the probability that $r$ males will have a total of $s$ sons; the coefficient of $x^s$ in $\pi_r f^r(x)$ which is a general term in $f(f(x))=f_2(x)$ say, is the probability that a man will have $r$ sons and $s$ grandsons. Similar interpretations can be placed on coefficients of $x$ in the iterated function

$$f\{f(f[\cdots])\} = f_n(x).$$

The second monograph is well illustrated throughout by applications of the theory to actual census data taken from the United States, England, France, Germany, and several other countries. Numerous charts and tables are given for comparing theory with facts. The monograph is an excellent account of results which have been obtained during the past quarter of a century in the theory of population dynamics. Most of the results are due to the author himself. Lotka has shown a great deal of ingenuity in formulating the problems mathematically and in reaching practical solutions of the problems. Those interested in applied mathematics in the field of biology will find these monographs well worth reading.

S. S. WILKS

*The Variate Difference Method.* By Gerhard Tintner. (Cowles Commission for Research in Economics Monograph, no. 5). Bloomington, Indiana, Principia Press, 1940. 13+175 pp.

Various mathematical statistical methods have been proposed during recent years in attempts to describe, analyze and interpret economic time series. Regression analysis and its extension to harmonic analysis, moving averages, and the variate difference method are some of the techniques which have been used. The fundamental concept adopted in these approaches is that each element $w_i$ ($i=1, 2, \cdots, N$) in a time series is assumed to consist of a sum $m_i+x_i$ where $m_i$ is the "true economic component" and $x_i$ is the "random component." The main purpose of the statistical analysis according to these methods is to estimate and describe these two components. There are, however, other approaches which use different conceptual models. As its title indicates, the present book presents a treatment (in ten chapters) of the variate difference method, although the author takes advantage of several opportunities to discuss other methods by way of comparison.

In the variate difference method it is assumed that the $m_i$ are "smooth" in the sense that the differences $\Delta^{(K)}(m_i)$ of the $m_i$ of some (finite) order $K$ vanish, while the $x_i$ are independent random variables. Under these assumptions the main problem as discussed by Tintner is that of deciding, on the basis of the $w_i$ and probability theory, for what value of $K$, $\Delta^{(K)}(m_i)=0$. The problem is therefore one of studying the probability theory of $\Delta^{(K)}(w_i)$ ($K=0, 1, 2, \cdots$) and making suitable significance tests. For large values of $N$, Tintner follows rather closely the work of Oskar Anderson, which consists in comparing the difference between the variances of two consecutive series of differences, say $\Delta^{(K)}(w_i)$ and $\Delta^{(K+1)}(w_i)$, with the variance of the difference under the assumption that $\Delta^{(K)}(m_i)=0$. The kurtosis of $\Delta^{(K)}(w_i)$ (i.e., $\mu_4' - 3\sigma^4$) is also considered in this analysis.

For the case of small $N$, Tintner proposes a significance test for testing the hypothesis that $\Delta^{(K)}(m_i)=0$, using the two series $\Delta^{(K)}(w_i)$ and $\Delta^{(K+1)}(w_i)$. If the original $x_i$ are assumed to be normally and independently distributed with variance $\sigma^2$, then it can be shown that $\Delta^{(K)}(w_i)$ are normally distributed with zero means and variance $C_{2K,K}\sigma^2$. Under the same assumptions, the terms $j+r(2K+3)$ ($r=0, 1, 2, 3, \cdots$) and terms, $j+K+1+r(2K+3)$ ($r=0, 1, 2, 3, \cdots$) will all be normally and independently distributed. Hence $\sigma^2$ can be estimated independently from the two series thus making it possible to set up an $F$-test. Such an $F$-test can be set up for $j=1, 2, 3, \cdots, 2K+3$.

The significance test based on this method of selection, while mathematically exact under the assumptions made, is very simple but not very efficient, as the author points out, in the sense that full use is not made of the data. A more efficient test would undoubtedly be very complicated.

This book is written primarily for the economic statistician who may wish to apply the variate difference method to his own economic time series. Tintner illustrates the routine application of the method in great detail, using American Wheat-Flour prices for the period

1890–1937 as his data. The author gives a large number of tables of weights and coefficients of various kinds, critical ratios of two sums of squares of differences, formulae for selections of items in various pairs of consecutive differences which would be independent, etc., to facilitate the application of the method.

A considerable amount of historical material on the variate difference method is given. One chapter is devoted to the application of Sheppard's smoothing formulae and serial correlation, and the connetion between the variate difference method and these methods is pointed out.

Several appendices are given: Appendix I is a summary of computation formulae used in the various chapters; Appendix II is devoted to the mathematics underlying the various formulae which are used; the remaining four appendices deal with special topics such as seasonal variation, normality of the random element, etc. Author and subject matter indices are provided.

The question as to whether the variate difference method is superior to other methods of time series analysis is largely a matter of opinion. Tintner has given a very good account of the method, although it appears to the reviewer that he did not concern himself enough with the problem of estimating the $m_i$ after it had been decided for which value of $K$ the value of $\Delta^{(K)}(m_i)=0$. He approached the problem of determining the $m_i$ by using the Sheppard smoothing formulae, but this, of course, is essentially another method of time series analysis.

Tintner's book is an interesting contribution to the literature of time series analysis. It is well documented by references and footnotes and reflects a great deal of work on the part of the author.

S. S. Wilks

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*Tensor Analysis of Networks.* By Gabriel Kron. New York, Wiley, 1939. 24+635 pp. $7.50.

In the introduction to this book, the author states: "It is emphasized that this book is not written by a mathematician and is not written for mathematicians. This book is written by an engineer for engineers who are interested in learning an *organized* method of attack to analyze and synthesize electrical networks."

The first two chapters provide an extremely detailed account of the tensor and matrix notation, of the multiplication of matrices and describes the impressed voltages, currents and self- and mutual impedances of a network as components of tensors $e_\alpha$, $i^\alpha$ and $Z_{\alpha\beta}$, respectively. In Chapter III the type of electromagnetic networks considered are described as mesh, junction and orthogonal networks. It is emphasized that mesh and junction networks are special cases of orthogonal networks and are due to the "special assumptions as to the nature of the impressed quantities that are known or of the response quantities that are to be found." The primitive mesh network consisting of $n$ individual coils, each short-circuited on itself is defined and observed to satisfy the generalized Ohm's law $e_m = Z_{mn} i^n$.

The central concept of the book is that of the transformation matrix, denoted by $C$. Its function is to allow one to describe any particular (mesh) network in a methodical way in terms of the primitive (mesh) network. This may be thought of as being accomplished by a linear transformation of the covariant vector $e$. The invariance of the power, $e_m i^m$, determines in the usual fashion the transformation law of the current vector and the invariance of Ohm's law then requires the impedances to be transformed as the components of a tensor having two covariant indices.

The theory is considerably complicated by the necessity for introducing equations of constraint which require the consideration of singular transformation matrices. A discussion of this, of the complications involved in combining two more-or-less interconnected networks into a single network, and a large number of detailed applications of the theory to specific problems occupy pages 117–323. In Chapter XIII the theory is extended to allow for alternating currents by permitting the current and potential vectors to have complex numbers as elements. The power input is now taken to be $e^*_m i^m$, where the star denotes the complex conjugate. Its assumed invariance leads to transformations involving $C^*$, in which case spinor indices (here denoted by the addition of a bar rather than a dot) are employed. Junction networks are next discussed, their mathematical theory being obtained from that of mesh networks by interchanging contravariant and covariant indices and adding or removing a bar. After a chapter on multielectrode-tube circuits, orthogonal networks are considered. A chapter on interlinked electric and magnetic networks precedes the introduction of a metric tensor (the inductance tensor) which is claimed to represent "the additional characteristics acquired by the electrical network owing to its linkage with an underlying magnetic network."

Chapters on Compound Networks, Symmetrical Components, and Multiple Tensors are followed by chapters on the application of the theory to the analysis and synthesis of networks. An extensive bibliography and an index conclude the book.

It is evident that a book of this kind provides difficult reading for both the mathematician and the electrical engineer. The mathematician will find the mathematical details exceedingly tedious, the derivations sometimes very obscure, and the technical terminology of the engineer unfamiliar. The engineer, on the other hand, may resent being called upon to learn a large amount of mathematical notation without the prospect of any corresponding gain in the computations he must perform. In spite of this, the book demands and is receiving attention from workers in both fields. Thus the investigation of singular transformations in tensor algebra and of tensor concepts in combinatorial topology were stimulated by this book. The introduction of concepts tending to unify special methods of approach to engineering problems will, in the long run, have an important influence on the development of engineering theory.

WALLACE GIVENS

*Fourier Series and Boundary Value Problems.* By R. V. Churchill. New York, McGraw-Hill, 1941. 206 pp. $2.50.

This book is a useful addition to the meager number of existing books of this general nature in English. Its major use will be as a textbook for students in engineering and the sciences interested in these topics.

The book contains no more subject matter than is implied by the title; that is, it leads up to and considers the solution of the usual several linear partial differential equations by series of trigonometric, Bessel, and Legendre functions. A considerable part of the book is devoted to an exposition of the concept of orthogonal sets of functions in general and Fourier series in particular.

The book contains some material of mathematical interest, but not very suitable to certain types of engineering students. However, it is so arranged that such material can be omitted.

From the point of view of mathematical preciseness the treatment is excellent. The book is also well planned for teaching purposes.

N. LEVINSON

*The Weight Field of Force of the Earth.* By William H. Roever. (Washington University Studies, New Series, Science and Technology, no. 1.) St. Louis, 1940. 84 pp. $1.50.

This monograph is an extension of the author's retiring address as Chairman of Section A of the American Association for the Advancement of Science. It deals with some statical and dynamical phenomena of the weight field of force of the earth, and is, in general, a summary of the author's earlier investigations in this field.

The earth's weight field of force is defined by the acceleration relative to the solid part of the earth which a body resting near the surface of the earth sustains, due to gravitation of the earth and the heavenly bodies, and due to the earth's motion in the absolute system.

The first three introductory sections of the monograph deal with the laws of relative motion and the effects of a moving atmosphere. In the fourth section it is shown that the tidal effects and the results of precession and nutation are negligible as compared with the gravitational attraction of the earth and the centrifugal force due to the earth's rotation. This main part of the weight field is at rest with respect to the solid part of the earth, and it possesses a potential. In the next two sections the moments actuating the Eötvös torsion balances are derived in geometrical and analytical treatment. The seventh section and the appendix are devoted to some dynamical phenomena of the weight field which lead to differential equations that can be easily integrated.

In view of the recent revival, due to the national defense program, of interest in geophysics and ballistics, this monograph may prove useful in putting some special investigations, which are capable of application in these fields, in a more available form.

MICHAEL GOLOMB

*Introduction to Abstract Algebra.* By C. C. MacDuffee. New York, Wiley, 1940. 7+303 pp. $4.00.

The rapid advances in algebra within the last few years have been largely due to an exploitation of the powerful methods of abstract algebra. Accordingly there has been a tremendous increase in the interest in this subject, so that most colleges and universities offer at least one course in abstract algebra. However, students with only an undergraduate course in the theory of equations as a background in algebra frequently have considerable difficulty with the available texts—not so much in reading the proofs as in grasping the significance of the abstract theories. The present book was written primarily as a text for beginning graduate students and is designed to fill the gap between the usual text in the theory of equations and the more advanced texts on algebra. It should prove to be a valuable addition to the growing list of texts on abstract algebra.

The material is organized in such a way that concrete instances of a theory are always at hand before the abstract formulation is made. How this is done, as well as the general scope of the book, will be indicated by brief summaries of the different chapters.

Chapter I. The Theory of Numbers. The book begins with a statement of Peano's postulates for the positive integers, followed by a rapid derivation of the simplest properties of the positive integers and an introduction of zero and the negative integers in terms of pairs of positive integers. Incidentally, rational numbers are not presented from this abstract point of view until Chapter V, although it is assumed in Chapters III and IV that the student has an intuitive knowledge of the rationals. The remainder of the chapter is devoted to a study of such topics as the fundamental theorem of arithmetic, congruences in one or more unknowns, residue classes, theorems of Fermat and Euler, primitive roots, quadratic residues and the law of quadratic reciprocity.

Chapter II. Finite Groups. After the definition of a *group* and derivations of a few properties of groups in general, the rest of the chapter deals only with finite groups. The set of automorphisms of a mathematical system is used as an important illustration of a group. A body of theorems is established, culminating in the Jordan-Hölder Theorem, which is essential for the Galois theory of the succeeding chapter. Direct products are defined and discussed briefly. Then follows the introduction of permutation groups and a proof that every finite group can be represented as a regular permutation group. The notions of transitivity and primitivity are not mentioned. The chapter closes with a derivation of the principal theorems about finite Abelian groups—existence of a basis, invariants, and related topics.

Chapter III. Algebraic Fields. The intuitive point of view is used to a considerable extent in this chapter, a more abstract treatment of fields being postponed until later. Quadratic and cubic fields are studied in some detail before more general algebraic fields are introduced. The Galois group of an equation is introduced as a group of automorphisms of its root field, and is then shown to be isomorphic to a group of permutations of the roots of the given equation. There follows a brief discussion of the Galois theory, including a derivation of Galois' criterion for solvability by means of radicals.

Chapter IV. Integral Algebraic Domains. Quadratic integral domains are studied in detail as a background for domains of higher degree. Unique factorization into primes is shown to exist in some quadratic domains but not in others, thus leading to the concept of *ideal* and the unique factorization of ideals into prime ideals. The actual determination of all prime ideals in a quadratic domain is carried out. Next the integral domain consisting of all algebraic integers in an algebraic field of degree $n$ is considered, and ideals are introduced. The arithmetic of ideals is again carried far enough to obtain the unique factorization theorem.

Chapter V. Rings and Fields. This chapter opens with abstract definitions of *ring*, *non-commutative field*, and *integral domain*. Incidentally, the reviewer is a bit unhappy at the use of a terminology according to which a commutative field is a special case of a non-commutative field. A *principal ideal ring* is defined as an integral domain having certain additional properties. Later on, when ideals in general rings have been introduced, it is shown that a principal ideal ring is an integral domain in which every ideal is principal, thus justifying the earlier terminology. The abstract construction of the quotient field of an integral domain is carried through in detail. Then follows the definition of a *prime field* and the determination of all prime fields. Polynominal rings are introduced abstractly, and various questions of factorization are discussed. Ideals are defined, and the notion of *residue class ring* is introduced, thus leading to the determination of all rings homomorphic to a given ring. A satisfactory treatment of algebraic extensions of a given field can now be carried through without use of the fundamental theorem of algebra. Galois (finite) fields are introduced and fundamental results, including a determination of the group of automorphisms, are derived. The concluding section considers the field $F(\lambda)$ obtained from $F$ by a transcendental extension and, in particular, characterizes all automorphisms of $F(\lambda)$ leaving elements of $F$ fixed.

Chapter VI. Perfect Fields. If $a$ and $b$ are elements of a field $F$, a function $\phi(a)$ is a *valuation* for $F$ if $\phi(a)$ is a positive number or 0 of some ordered field, subject to the further conditions:

$$\phi(a) > 0 \text{ for } a \neq 0,\ \phi(0) = 0,\ \phi(ab) = \phi(a)\phi(b),\ \phi(a+b) \leqq \phi(a) + \phi(b).$$

The real field is obtained as the field of regular sequences of rational numbers, with the ordinary absolute value as the valuation used in defining regularity. This field is then shown to be *perfect* with respect to this valuation, that is, it is incapable of further extension by means of regular sequences of real numbers. The field of complex numbers is obtained by a simple algebraic extension of the real field, and a proof is given that every algebraic equation with complex coefficients has a complex root. Another valuation of the rational field is obtained as follows. Let $p$ be a fixed rational prime. If $a \neq 0$ is a rational number, it is uniquely expressible in the form $(r/s)p^n$, where

$r$ and $s$ are integers prime to $p$. Then $\phi(0)=0$, $\phi(a)=p^{-n}$ is a valuation of the rational field. The Hensel $p$-adic number field $\Omega p$ is obtained by use of this valuation in the same way in which the reals are obtained from the rationals by use of the absolute value. There follows a series representation of numbers of $\Omega p$ and a characterization of the rational numbers in $\Omega p$ as those elements with periodic series. It is also shown that a $p$-adic extension of an algebraic field can always be obtained with a prime ideal playing a rôle analogous to that of the prime $p$ in the case of the rational field.

Chapter VII. Matrices. After a preliminary study of linear systems, the author introduces the concepts of *array* and *determinant* of a square array. A *matrix* is defined as an element of a total matrix algebra. After some preliminary results, the Hamilton-Cayley theorem is obtained, and also a characterization of the minimum equation of a matrix. Some ten pages are devoted to a study of matrices over a principal ideal ring, including such topics as greatest common right (left) divisor, elementary divisors and normal form. The familiar rational canonical form of a matrix over a field is then obtained, as well as the Jordan form for a matrix over the complex field. The regular representations, by matrices, of a finite algebraic field $F(\rho)$ over $F$ are given as an application of the theory of matrices, and probably also as an introduction to the more general representation theory of the next chapter. After a discussion of polynomials in a single matrix, the chapter concludes with a brief treatment of *direct products* of matrices.

Chapter VIII. Linear Associative Algebras. As a concrete introduction to the general theory, the author begins with an unusually detailed study of quaternions, including some new results, due to C. G. Latimer, on quaternion ideals and factorization. Then follows a brief discussion of division algebras, generalized quaternions and cyclic algebras. Linear algebras with unit element are presented in some detail before those without unit element. Further topics treated briefly are regular representations, direct sums, direct products, ideals and the radical of an algebra. General structure theorems are not proved although the principal ones are stated at the end of the chapter.

There are numerous exercises, several sets in each chapter, which give the student an opportunity to test his grasp of the subject, as well as furnishing additional concrete illustrations of the general theories. The author states that students should work all the problems, as they are designed to supplement the rest of the book in an essential way. At the end of each chapter there occurs a short and well-chosen list of suggested texts for further reading. At the end of the book there are tables of Greek and German letters, and a list of some of the more important symbols used in the text.

On the whole the book is well written and the typography excellent. The reviewer noticed a few misprints and even an occasional slip. However, these are mostly of a minor nature and should cause little or no confusion to the student. The book is a valuable and timely addition to the available texts on algebra.

NEAL H. MCCOY

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*Gap and Density Theorems.* By Norman Levinson. (American Mathematical Society Colloquium Publications, vol. 26.) New York, American Mathematical Society, 1940. 8+246 pp. $4.00.

In this book the author confines himself to a detailed study of a few salient topics in gap and density theory; he does not attempt to write a systematic treatise on the subject. The book is in form essentially a collection of research papers; it achieves unity principally through the author's repeated application of similar methods to a variety of problems. Most of the contributions to gap and density theory contained in the book are the author's own work, some of the most remarkable of them being published here for the first time. The principal topics treated are, on the one hand, the influence of the distribution of a sequence of numbers $\{\lambda_n\}$ on the closure properties of the sequence $\{e^{i\lambda_n x}\}$, and the closely related topic of the influence of the distribution of the $\lambda_n$ on the growth of analytic functions which vanish or are otherwise restricted at points $z=\lambda_n$; and, on the other hand, general Tauberian theorems involving gap conditions. Among the topics not treated are, for example, the Paley-Wiener theory of "pseudoperiodic" functions, and Bochner's generalizations of it. The extensive "classical" theory connecting gap or density properties of a sequence $\{\lambda_n\}$ with the position of the singularities of the function having the Dirichlet series $\sum a_n e^{-\lambda_n x}$ is represented by one theorem. The author expects his readers to be familiar with approximately the amount of information contained in Titchmarsh's *Theory of Functions*. Familiarity with the Colloquium Publication of Paley and Wiener is not prerequisite, but would be advantageous for a reader. The author collects in an appendix the auxiliary theorems which he most frequently uses. His principal tools are such things as Jensen's theorem, Carleman's theorem which is its analogue for a half-plane, Phragmén-Lindelöf theorems, and the $L^2$ theory of Fourier transforms; these he combines in ingenious and often unexpected ways.

Many of the proofs are long and involve elaborate calculations. However, the author has been helpful in many places by prefacing his longer proofs with applications of his methods to simpler problems, so that the reader can see the essential ideas unobscured by details. At times, on the other hand, he demands considerable background from the reader; for example, the "obvious" statement at the beginning of Chapter V seems to be obvious principally because it is a known true theorem.

On many of the problems which he discusses, the author says the last word, since the problems are solved by theorems which are shown by examples to be "best possible." In most cases he generalizes existing theorems by constructing entirely new proofs, rather than by generalizing existing proofs; his methods have allowed him to attain wide generality and great precision. Anyone who is interested in Fourier integrals, the growth of analytic functions, or Tauberian theorems, should profit from studying the author's methods of attack. There is no reason for supposing that their power is yet exhausted.

A chapter-by-chapter résumé follows. In Chapters I and II the central problem is the closure of a sequence $\{e^{i\lambda_n x}\}$. Such a sequence is said to be closed[1] $L^p$ if $\int_{-\pi}^{\pi} f(x)e^{i\lambda_n x}\,dx=0$ for all $n$, and $f(x)\in L^p$, imply $f(x)=0$ almost everywhere. As Szász has pointed out, studying the closure of a sequence $\{e^{i\lambda_n x}\}$ amounts to studying the zeros of entire functions of the form $\int_{-\pi}^{\pi} f(t)e^{ixt}dt$; and as Paley and Wiener observed, closure theorems have as corollaries theorems stating that a function with gaps in its sequence of Fourier coefficients cannot vanish over too long an interval without vanishing identically. The first two chapters contain a variety of closure and gap theorems, some in sharper form than when they were previously published. The few that are not due to the author in their present degree of generality are due to Miss Cartwright. Chapter III contains the theorems connecting the distribution of the zeros of an entire function of exponential type with its rate of growth along the real axis, used in proving the closure theorems of Chapters I and II. The essential idea is that if the function does not grow too fast along the real axis most of its zeros must lie near the real axis, and the sequence of zeros has a density in the right and left half-planes; the author's theorems are more precise than earlier results of this character.

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[1] The author follows Paley and Wiener, who interchange the customary terms. Most authors would write "complete in $L^{p'}$," where $p'=p/(p-1)$, instead of "closed $L^p$." Since the properties of closure in the usual sense and in the author's are equivalent, no great harm is done.

Chapter IV is devoted to non-harmonic Fourier series. If a sequence $\{e^{i\lambda_n x}\}$ is closed $L^p$, every function of $L^p$ can be represented as the limit (in the topology of $L^p$) of a sequence of linear combinations of the functions $e^{i\lambda_n x}$. It was observed by Paley and Wiener that the restriction $|\lambda_n - n| \leqq c$, with sufficiently small $c$, will ensure in additon that the sequence $\{e^{i\lambda_n x}\}$ will serve to expand functions into series preserving many of the properties of ordinary Fourier series; such series they called non-harmonic Fourier series. Paley and Wiener showed that there is an $L^2$ theory if $c < 1/\pi^2$. Levinson showed, by different methods, that there is an $L^p$ theory $(1 < p \leqq 2)$ if $c < (p-1)/(2p)$, but not necessarily if $c \leqq (p-1)/(2p)$. His results are reproduced in Chapter IV, together with an interesting inequality of Hardy and Levinson which is connected with the problem.

Chapter V contains various developments of the author's results which state (roughly) that if a function is "small at infinity" and its Fourier transform coincides with an analytic function over an interval, then its Fourier transform continues to coincide with the analytic function over the largest interval of the real axis on which that function continues to be analytic. Such results generalize gap theorems stating that if the Fourier transform vanishes over an interval it vanishes identically (the analytic function being in this case identically zero). Chapter VI contains very precise estimates for the rate of growth of an entire function of exponential type whose zeros have a density; these are applied to prove the classical theorem of Pólya connecting the density of the exponents of a Dirichlet series with the position of its singular points on its abscissa of convergence. In Chapter VII the estimates are combined with results from Chapter V to establish conditions under which the rate of growth of an analytic function on a sequence of points on or near a line determines its rate of growth along the whole line. Theorems of this character were obtained by V. Bernstein from the theory of Dirichlet series; the author proves Bernstein's theorems by simpler and more natural methods, and then develops a method of proof in which the results of Chapter V can be used to make the results much more precise.

The greater part of the material of Chapters I–VII has been published before; Chapters VIII and IX are new material. Pólya set as a problem the theorem that an entire function of order one and zero type is constant if it is bounded at the integers. The author replaces the integers by a sequence $\{\lambda_n\}$ sufficiently near the integers; in particular, if

$$\lambda_n - n = O\{n/(\log n)^{1+\delta}\}$$

as $n \to \infty$, for some positive $\delta$, the theorem remains true; the theorem of Chapter VII is still more general, but too complicated to reproduce here. Remarkably enough, the condition above is approximately "best possible," since the author shows that there are sequences $\{\lambda_n\}$ satisfying the condition $\lambda_n - n = O(n/\log n)$, but such that there exist entire functions of zero exponential type bounded on the sequence and not constant; thus the existence of a positive density for the sequence $\{\lambda_n\}$ is not enough to make the theorem true. The proof of the theorem is based on a powerful inequality for analytic functions. Let $M(x)$ be a positive even function, decreasing for increasing $|x|$, and let $M(x) \to \infty$ as $x \to 0$. Let $f(z)$ be analytic in the rectangle $|x| \leqq a$, $|y| \leqq b$, and let $|f(x+iy)| \leqq M(x)$ for $x+iy$ in the rectangle. If $\int_0^1 \log \log M(x)\, dx$ converges, then $|f(x+iy)| \leqq C$ for $|x| \leqq a$ and $|y| \leqq b(1-\delta)$, where $C$ depends only on $\delta$ and on $M(x)$. This theorem seems likely to prove useful in many problems other than that to which it is applied in this chapter. Chapter VIII is devoted to showing that the results of Chapter VII (both the theorem just quoted and the results on functions of zero type) are best possible. This is done by constructing a suitable function of zero type by means of its representation as an infinite product; this is by no means a simple task (it requires 33 pages).

Chapters X, XI, and XII are devoted to gap Tauberian theorems; Chapter X is substantially the same as a recent paper of Levinson's, while the other two chapters consist of new material. A Tauberian theorem for Dirichlet series states that if $f(x) = \sum_1^\infty a_k e^{-\mu_k x}$ for $x > 0$ and $f(x) \to s$ as $x \to 0+$, then under suitable conditions $\sum a_k = s$. Most such theorems involve restrictions on the size of the $a_k$; but a classical theorem of Hardy and Littlewood states that if $\mu_{k+1}/\mu_k \geqq \theta > 1$, no restriction on the $a_k$ is necessary. In the language of Wiener's general Tauberian theorems, one has to consider when, from

$$f(x) = \sum_1^\infty a_n \int_{\lambda_n - x}^\infty K(y)dy, \qquad \int_{-\infty}^\infty K(y)dy = 1,$$

and $f(x) \to s$ as $x \to \infty$, one can deduce that $\sum a_n = s$. [The Dirichlet series case corresponds to $K(y) = e^y \exp(-e^y)$.] In Wiener's theory the $a_n$ are restricted in size and $K(y)$ is required to have a Fourier transform which does not vanish anywhere. For a gap Tauberian theorem, one might expect to be able to omit restrictions on the $a_n$ provided that $\lambda_{n+1} - \lambda_n \geqq L > 0$. The author shows by an example that there are kernels $K(y)$ [e.g., $e^{-y^2/2}$] whose Fourier transforms do not vanish but for which such a Tauberian theorem does not hold. In Chapter X he shows that there is a gap Tauberian theorem for $K(y)$ provided that its Fourier transform satisfies sufficiently strong additional conditions. His proof, like Wiener's proof of the Hardy-Littlewood theorem, depends on reducing the Tauberian theorem to a lemma asserting the existence of a suitable set of biorthogonal functions. While in Wiener's proof this set was relatively easy to construct, in the general case it is quite difficult.

In Chapter XI it is shown that for more general kernels than those satisfying the conditions of Chapter X there are still Tauberian theorems in which the size of the $a_n$ is not restricted, provided that the gaps in the sequence $\{\lambda_n\}$ are large enough; for the kernel $e^{-y^2/2}$ the condition is essentially $\sum 1/\lambda_n = \infty$. In Chapter XII it is shown that for this kernel the original condition $\lambda_{n+1} - \lambda_n \geqq L > 0$ still ensures the existence of a Tauberian theorem if the $a_n$ are restricted in size (though only rather mildly compared with the restrictions required without the gap condition on the $\lambda_n$). It is also indicated how the method used would apply to more general kernels.

R. P. BOAS, JR.

## NOTES

A summer session of special interest is being held at Brown University this year. It is devoted to advanced instruction and research in mechanics, a program designed for the purpose of increasing the effectiveness of research in essential American industries. The summer course is approved as a part of the Engineering Defense Training Program of the U. S. Office of Education. The Carnegie Corporation of New York has made a grant toward its support. Similar sessions are planned for the period October 1, 1941–January 31, 1942, and February 11–June 6, 1942. Those who wish information about these sessions should communicate with the Dean of the Graduate School, Brown University, Providence, Rhode Island.

Among the members of the U. S. National Defense Research Committee who visited Great Britain were Dr. Warren Weaver and Professor J. L. Tate.

Professor Sydney Chapman of Imperial College, South Kensington, has been elected president of the Royal Astronomical Society. Professor H. C. Plummer of the Military College of Science, Woolwich, has been elected vice president.

Professor L. W. Cohen of the University of Kentucky has been named president of the university branch of the American Association of University Professors.

Dean L. P. Eisenhart of Princeton University has been appointed one of the directors of the National Science Fund of the National Academy of Sciences.

Professor W. D. Cairns, Secretary-Treasurer of the Mathematical Association of America, has announced that the Department of Mathematics of Brooklyn College, Brooklyn, New York, has won the first prize of $500 in the fourth annual William Lowell Putnam Mathematical Competition, held March 1, 1941. The members of the team were Richard Bellman, Peter Chiarulli, Hyman Zimmerberg. The second prize of $300 is awarded to the Department of Mathematics of the University of Pennsylvania, Philadelphia, Pennsylvania, the members of whose team were S. I. Askovitz, Hyman Kamel, P. C. Rosenbloom. The third prize of $200 is awarded to the Department of Mathematics of the Massachusetts Institute of Technology, Cambridge, Massachusetts, with team members J. R. R.

Baumberger, Eugene Calabi, W. S. Loud. In addition to these prizes to the departments of mathematics with winning teams, a prize of $50 each is awarded to the following five persons whose scores ranked highest in the six-hour examination (names listed alphabetically): R. F. Arens, University of California at Los Angeles; S. I. Askovitz, University of Pennsylvania; A. M. Gleason, Yale University; E. L. Kaplan, Carnegie Institute of Technology; P. C. Rosenbloom, University of Pennsylvania. Of these five, one will later be chosen to receive a $1000 year scholarship at Harvard University, this award to be announced later. The members of the three winning teams will receive individual cash awards according to the ranks of their teams, and all individuals receiving awards will also receive medals. Honorable mention has been awarded this year to three teams and to six individuals. The teams are from the Department of Mathematics, Carnegie Institute of Technology, Pittsburgh, members of the team being R. E. Beatty, E. L. Kaplan, N. H. Painter; the Department of Mathematics, Cooper Union Institute of Technology, New York, members of the team being Murray Klamkin, Benjamin Lax, Samuel Manson; and the Department of Mathematics, Yale University, New Haven, the team members being A. M. Gleason, G. R. MacLane, D. M. Merrill. The five individuals receiving honorable mention are Richard Bellman, Brooklyn College; Harvey Cohn, College of the City of New York; W. S. Loud, Massachusetts Institute of Technology; G. R. MacLane, Yale University; Samuel Manson, Cooper Union Institute of Technology; Hyman Zimmerberg, Brooklyn College.

Dr. C. A. Coulson of University College, Dundee, has been elected a fellow of the Royal Society of Edinburgh.

Professor T. G. Room of the University of Sydney has been elected to fellowship in the Royal Society, London.

Professor G. A. Bliss of the University of Chicago will retire on September 30 with the title of professor emeritus.

Professor G. C. Evans of the University of California and Professor J. R. Kline of the University of Pennsylvania have been elected to membership in the American Philosophical Society.

Professor E. V. Huntington of Harvard University has been given the title of professor emeritus.

Dr. Joseph Slepian of the Westinghouse Electric and Manufacturing Company and Professor T. Y. Thomas of the University of

California at Los Angeles have been elected to membership in the National Academy of Sciences.

Professor H. S. Uhler of Yale University has been given the title of professor emeritus.

Professor Mabel M. Young of Wellesley College has been given the title of professor emeritus.

Dr. Theodor Esterman and Mr. A. C. Stevenson have been appointed readers in mathematics at University College, London.

Associate Professor A. A. Albert of the University of Chicago has been promoted to a professorship.

Associate Professor W. L. Ayres of the University of Michigan has been appointed head of the department of mathematics at Purdue University.

Mr. J. J. Barron of St. Joseph College, Hartford, Connecticut, has been promoted to an associate professorship.

Associate Professor M. A. Basoco of the University of Nebraska has been promoted to a professorship.

Dr. C. B. Boyer of Brooklyn College has been promoted to an assistant professorship.

Assistant Professor Richard Brauer of the University of Toronto has been on leave of absence at the University of Wisconsin as visiting lecturer during the past semester.

Assistant Professor B. L. Brown of Amherst College has been promoted to an associate professorship.

Assistant Professor O. E. Brown of Case School of Applied Science has been promoted to an associate professorship.

Assistant Professor I. S. Carroll of Syracuse University has been promoted to an associate professorship.

Dr. A. H. Clifford of Massachusetts Institute of Technology has been promoted to an assistant professorship.

Dr. S. H. Gould of Victoria College, Toronto, has been promoted to an assistant professorship in classics.

Dr. Harriet M. Griffin of Brooklyn College has been promoted to an assistant professorship.

Assistant Professor D. C. Lewis of the University of New Hampshire has been promoted to an associate professorship.

Assistant Professor Saunders MacLane of Harvard University has been promoted to an associate professorship.

Associate Professor C. J. Rees of the University of Delaware has been promoted to a professorship.

Dr. R. F. Rinehart of Case School of Applied Science has been promoted to an assistant professorship.

Assistant Professor J. H. Roberts of Duke University has been promoted to an associate professorship.

Assistant Professor P. J. Rulon of Harvard University has been promoted to an associate professorship.

Assistant Professor C. H. W. Sedgewick of the University of Connecticut has been promoted to an associate professorship.

Associate Professor G. W. Starcher of Ohio University has been promoted to a professorship.

Dr. C. C. Torrance of Case School of Applied Science has been promoted to an assistant professorship.

Dr. J. W. Tukey of Princeton University has been promoted to an assistant professorship.

Mr. R. L. Wilson of the University of Wisconsin has been appointed to an assistant professorship in military science and tactics at the Alabama Polytechnic Institute.

Associate Professor H. N. Wright of the College of the City of New York has been appointed acting president of the College in succession to Dr. N. P. Mead.

Dr. G. W. Whaples of Indiana University has been made an assistant in mathematics at the Institute for Advanced Study.

The following appointments to instructorships are announced: Cornell University: Dr. E. H. Hadlock; Fenn College, Cleveland, Ohio: Mr. D. H. Erkiletian; Harvard University: Mr. L. H. Loomis; The Hill School, Pottstown, Pennsylvania: Mr. W. N. Huff; Johns Hopkins University: Mr. E. A. Coddington; University of Kansas City: Mr. D. E. Kibbey; University of Maryland: Dr. Martha H. Williams; Northwestern University: Dr. J. W. Givens; University of

Pennsylvania: Dr. A. D. Wallace, Mr. P. M. Whitman; Yale University: Mr. J. G. Herriot.

In Nature has appeared an announcement of the death of Dr. E. L. Ince, lecturer on technical mathematics at the University of Edinburgh, on March 16, 1941, at the age of forty-nine years.

Assistant Professor G. M. Hayes of the College of the City of New York died January 2, 1941.

Dr. Ethel I. Moody of Pennsylvania State College was killed in an automobile accident April 11, 1941.

Professor Emeritus J. E. Trevor of Cornell University died May 4 1941, at the age of seventy-six years. He had been a member of the American Mathematical Society since 1921.

# ABSTRACTS OF PAPERS

## SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

## ALGEBRA AND THEORY OF NUMBERS

### 293. A. H. Clifford: *Matrix representations of completely simple semigroups.*

By a representation $\mathfrak{T}^*$ of degree $n$ of a semigroup $S$ is meant a mapping $a \rightarrow T^*(a)$ of $S$ into the set of all square matrices of $n$ rows and columns such that $T^*(ab) = T^*(a)T^*(b)$. A. Suschkewitsch (Communications de la Société Mathématique de Kharkow, vol. 6 (1933), pp. 27–38) partially solved the problem of determining all representations of a type of semigroup which he calls a "Kerngruppe." In the terminology of D. Rees (Proceedings of the Cambridge Philosophical Society, vol. 36 (1940), pp. 387–400), the latter is a "completely simple semigroup without zero." The present paper completes the solution, at the same time allowing $S$ to be any completely simple semigroup. To any such $S$ belongs a "structure group" $G$, and every representation $\mathfrak{T}^*$ of $S$ is an "extension" of a representation $\mathfrak{T}$ of $G$. To any given $\mathfrak{T}$ corresponds a certain matrix $H$, possibly infinite. $\mathfrak{T}$ admits an extension $\mathfrak{T}^*$ of finite degree if and only if $H$ has finite rank $h$. If $\mathfrak{T}$ has degree $n$, there exists a "basic" extension $\mathfrak{T}_0^*$ of degree $n+h$, and every extension $\mathfrak{T}^*$ of $\mathfrak{T}$, even though it may be indecomposable, reduces into $\mathfrak{T}_0^*$ and null representations. (Received March 31, 1941.)

### 294. C. J. Everett: *Vector spaces over rings.*

Every basis of a vector space $V_n$ (with $n$ basis elements) over a ring $K$ with right ideal maximal condition has $n$ elements. A ring is exhibited over which any $V_n$ is always a $V_1$. Every submodul $M$ of a $V_n$ over ring $K$ has a basis of $b(M) \leqq n$ elements if and only if (property $P$) every right ideal of $K$ is principal of type $rK$, where $r$ is a non-left-zero-divisor. For $V_n$ over $K$ with property $P$, $b(M)$ is a positive modular functional (G. Birkhoff, *Lattice Theory*, American Mathematical Society Colloquium Publications, vol. 25, p. 40). If $K$ has property $P$ and is without zero-divisors, the metric homomorph (loc. cit., Theorem 3.10) of the lattice of submoduls of $V_n$ over $K$ is lattice-isomorphic with the lattice of submoduls of $V_n$ over the quotient field of $K$. For $V_n$ over $K$ of type $P$, $b(M)$ is sharply positive (loc. cit., p. 41) if and only if $K$ is a quasi-field. (Received May 26, 1941.)

### 295. A. W. Jones: *On the characterization of groups whose lattices satisfy specified lattice identities.*

It is shown that the lattice of a group is simple, complemented, and modular (that is, capable of abstractly representing a finite projective geometry) if and only if the group is either: (1) abelian of prime power order and with prime order elements only, or (2) a non-abelian group of order $pq^n$ generated by the elements $a$ and $b_i$ ($i=1, 2, \cdots, n$) which satisfy the relations $a^p=b_i^q=1$ and $a^{-1}b_ia=b_i^{u}$ ($p$ divides $q-1$ and $q$ divides $u^p-1\neq 0$). A group lattice is complemented and modular if and only if the group is the direct product of groups of relatively prime orders and of types (1) or (2) above. (Received May 21, 1941.)

### 296. Leonard Carlitz: *An analogue of the Bernoulli polynomials.*

The polynomials in question are defined by means of $\psi(tu)/u\psi(t)=\sum\beta_m(u)t^m/g_m$, which may be compared with $t/\psi(t)=\sum B_mt^m/g_m$. (For the quantities involved see *An analog of the Staudt-Clausen theorem*, Duke Mathematical Journal, vol. 3 (1937), p. 503; also, vol. 7 (1940), p. 62.) If $u$ is put equal to $U$, a polynomial in $x$ over $GF(p^n)$, then $\beta_m(U)$ becomes a rational fraction in $x$ alone. This paper contains various theorems of an arithmetic nature on $\beta_m(U)$, in particular a theorem of the Staudt-Clausen type on the fractional part. (Received April 11, 1941.)

### 297. Mark Kac: *Remark on the distribution of values of the arithmetic function d(m).*

Let $d(m)$ denote the number of different divisors of $m$ (1 and $m$ included) and $r_n(\omega)$ the number of positive integers less than or equal to $n$ for which $d(m)\leqq 2^{\log\log n+\omega(2\log\log n)^{1/2}}$. Then $\lim_{n\to\infty}r_n(\omega)/n=\pi^{-1/2}\int_{-\infty}^{\omega}e^{-u^2}du$. (Received May 2, 1941.)

### 298. Joseph Lehner: *A partition function connected with the modulus five.*

Let $p_1(n)$, $p_2(n)$ denote the number of partitions of a positive integer $n$ into summands of the form $5l\pm 1$ and $5l\pm 2$, respectively. These functions occur in identities of I. Schur, in which they are related to the number of partitions of $n$ into summands which differ by at least two. Convergent series for $p_1(n)$ and $p_2(n)$ are derived by using the Hardy-Littlewood method in the strengthened form of Kloosterman-Rademacher. The transformation equations of the generating functions of $p_1(n)$, $p_2(n)$, which are closely related to certain modular forms of dimension zero, yield new relations in the theory of theta-functions. The familiar exponential sums which arise from the application of the Hardy-Littlewood method are reduced by a number-theoretic method to Kloosterman sums, thereby permitting the sharper estimate $O(n^{1/3}k^{2/3+\epsilon})$ which is needed for the application of Rademacher's method to modular forms of dimension zero. (Received April 2, 1941.)

### 299. J. B. Rosser: *A generalization of the euclidean algorithm to several dimensions.*

If $Y=\sum a_iX_i$, the $a$'s being integers, $Y$ is said to be an I.L.C. (integral linear combination) of the $X$'s. A set of vectors $U_1, \cdots, U_r$ is called a G.C.F. of a set of vectors $V_1, \cdots, V_s$ if: I. The $U$'s are linearly independent. II. Each $U$ is an I.L.C. of the $V$'s. III. Each $V$ is an I.L.C. of the $U$'s. A set of $V$'s is said to be commensurable if the set of I.L.C.'s of the $V$'s has no limit point. It is proved that every commensurable set of vectors has a G.C.F. The G.C.F. of two commensurable collinear vectors is a single vector, which can be found by the euclidean algorithm. Generalizing the euclidean algorithm in a rather obvious fashion, one has a procedure for finding a G.C.F. of more general sets of vectors. This algorithm solves numerous problems relative to positive definite quadratic forms and Diophantine equations. For two incommensurable collinear vectors, the euclidean algorithm becomes the continued fraction algorithm, a powerful tool in approximation problems. It is conjectured that the generalized algorithm is as effective for problems of simultaneous approximation as the continued fraction algorithm is for simple approximation problems. To support this conjecture, satisfactory solutions of such problems are obtained by use of the generalized algorithm. (Received April 24, 1941.)

### 300. Albert Whiteman: *Sums connected with the partition function.*

The sum $A_k(n)=\sum \exp\,(\pi i s(h,\,k)-2\pi i hn/k)$, where $h$ runs over a reduced residue system with respect to the modulus $k$ and $s(h,\,k)$ is a Dedekind sum, appears in Rademacher's formula for the number of partitions of $n$. On the basis of a different expression for this sum Lehmer (Transactions of this Society, vol. 43 (1938), pp. 271–295) factored the $A_k(n)$ according to the prime number powers contained in $k$, and evaluated the $A_k(n)$ in the case in which $k$ is a prime or a power of a prime. A new approach to the first of these results has recently been given in a paper by Rademacher and Whiteman (American Journal of Mathematics, vol. 63 (1941), pp. 377–407). In the present paper the second of these results is derived by a method which is considerably simpler than Lehmer's. The paper also contains a new method for evaluating certain generalized Kloosterman sums. (Received May 29, 1941.)

## ANALYSIS

### 301. R. P. Agnew: *On limits of integrals.*

It is shown that existence of $\lim_{A\to\infty}\int_A^{A+\lambda} f(t)dt$, for each $\lambda$ in some set having positive measure, implies that the limit exists and is uniform over each finite interval of values of $\lambda$. The result is applied to prove two theorems of Iyengar (Proceedings of the Cambridge Philosophical Society, vol. 37 (1941), pp. 9–13) and the following Tauberian theorem. If $F(t)$ is absolutely continuous over each finite interval, if $\lim_{A\to\infty}\int_A^{A+\lambda}[F(t)-F'(t)]dt=0$ for each real $\lambda$, and if $\lim_{t\to\infty}e^{-t}F(t)=0$, then $\lim_{t\to\infty}F(t)=0$. (Received April 11, 1941.)

### 302. Stefan Bergman: *A method for summation of series of orthogonal functions of two variables.*

Suppose $\{\Phi_\nu(\phi)\}$ $[(\phi)=(\phi_1,\ \phi_2)]$ is a system of O.N. functions $[0\leqq\phi_k\leqq 2\pi$, $\Phi_k(\phi_1+2\pi,\ \phi_2)=\Phi_k(\phi_1,\ \phi_2+2\pi)=\Phi_k(\phi_1,\ \phi_2)$, $k=1,\ 2]$, $\sigma(e)$ a completely additive set function, $f(\phi_1,\phi_2)\in L^{1+p}\,(p>0)$, $a_\nu=\int_0^{2\pi}\int_0^{2\pi}\Phi_\nu d\sigma$, $b_\nu=\int_0^{2\pi}\int_0^{2\pi}\Phi_\nu f d\phi_1 d\phi_2$. Consider the series (1) $\sum_{\nu=0}^{\infty} a_\nu\Phi_\nu(\phi_1,\ \phi_2)$ and (2) $\sum_{\nu=0}^{\infty} b_\nu\Phi_\nu(\phi_1,\ \phi_2)$. Let $\mathfrak{M}$ be a four-dimensional domain of the type described in Mathematische Annalen, vol. 104 (1931), pp. 611–636, with the distinguished boundary surface $\mathfrak{F}=\mathrm{E}[z_k=h_k(\phi_1,\ \phi_2),\ k=1,\ 2]$. Let $\Psi_\nu(z_1,\ z_2)$, $\nu=1$, $2,\cdots$, be the functions of the extended class which assume the values $\Phi_\nu$ on $\mathfrak{F}$, and (3) $S(z_1,\ z_2)=\sum_{\nu=0}^{\infty} a_\nu\Psi_\nu(z_1,\ z_2)$, (4) $F(z_1,\ z_2)=\sum_{\nu=0}^{\infty} b_\nu\Psi_\nu(z_1,\ z_2)$. The series (4) converges absolutely and uniformly in every closed subdomain of $\mathfrak{M}$ for $p>1$. Using the results of Jessen, Marcinkiewicz and Zygmund (Fundamenta Mathematicae, vol. 25 (1935), pp. 217–234), Bergman and Marcinkiewicz (Fundamenta Mathematicae, vol. 33 (1939), pp. 75–94) and Bers (Comptes Rendus de l'Académie des Sciences, Paris, vol. 208 (1939), pp. 1273–1275 and 1475–1477) it is shown that $S$ possesses a finite sectorial limit at the point $\{h_1(\phi),\ h_2(\phi)\}$ if $\sigma$ possesses a finite strong derivative at $(\phi)$, and $F$ possesses the sectorial limit $f$ almost everywhere on $\mathfrak{F}$. (Received May 2, 1941.)

### 303. R. P. Boas: *Generalized Laplace integrals.*

The generalized Laplace integrals discussed have the form $f(s)=\int_0^\infty g(s, t)\phi(t)dt$, where $g(s, t)$ is in some sense "nearly" equal to the kernel $e^{-st}$. The conditions imposed on $g(s, t)$ are analogous to those which have been imposed on functions $g_n(s)$ to make the series $\sum a_n g_n(s)$ behave like a power series (cf. Boas, Transactions of this Society, vol. 48 (1940), pp. 467–487). An application is made to the kernel $g(s, t)=(2st/\pi)^{1/2}K_\nu(s, t)$ recently discussed by Meijer (Proceedings of the Section of Sciences, Koninklijke Akademie van Wetenschappen te Amsterdam, vol. 43 (1940), pp. 599–608, 702–711; see Mathematical Reviews, vol. 2 (1941), p. 96); here $-\frac{1}{2}<\Re(\nu)<\frac{1}{2}$, and $K_\nu(s)$ is the usual notation for a Bessel function of imaginary argument; when $\nu=\pm\frac{1}{2}$, $g(s, t)=e^{-st}$. (Received May 19, 1940.)

### 304. D. G. Bourgin: *On reflexive Banach spaces.*

Several theorems are obtained generalizing a result of Ghantmacher and Smulian (Comptes Rendus de l'Académie des Sciences de l'URSS, vol. 17 (1937), p. 91). For instance: *If the unit sphere of $E$ is sectionally compact of order $\aleph_\alpha$ in the weak topology of $E$ as elements and if $E^*$ contains a dense set, of power less than or equal to $\aleph_\alpha$, in the sense of the weak topology of $E^*$ as functionals, then $E$ is reflexive.* Some incidental results are also available. Thus a theorem implying the possibility of interchange of order of certain Moore-Smith limiting operations is: *If* (a) $\{f^{(l)}\}$ *is a $w^*(E^*)$ dense set of power less than or equal to $\aleph_\alpha$ and* (b) $\{x^\rho\}$, *$\rho$ a directed set of power less than or equal to $\aleph_\alpha$, is contained in a $w(E)$ compact (of order $\aleph_\alpha$) set, $Q$, and* (c) *$f^l(x^\rho)$ converges in the Moore-Smith sense to $f^l(x_0)$ for each $l$, then $\{x^\rho\}$ is $w(E)$ convergent to $x_0$ in the Moore-Smith sense.* (Received April 2, 1941.)

### 305. J. H. Curtiss: *Riemann sums and the fundamental polynomials of Lagrange interpolation.*

Let $C$ denote an arbitrary Jordan curve of the $z$-plane and let $z=\phi(w)$ be an analytic function which maps the exterior $K$ of $C$ conformally onto the region $|w|>1$ so that the points at infinity correspond. Let $\phi'(\infty)=c$. Let $\omega_n(z)=\prod_{k=1}^{n}[z-\phi(e^{2\pi ik/n})]$, $n=1, 2, \cdots$. Then if $C$ is rectifiable, $\lim_{n\to\infty}\omega_n(z)/(-c^n)=1$ uniformly for $z$ on any closed point set of the region interior to $C$, and $\lim_{n\to\infty}\omega_n(z)/c^n(w^n-1)=1$ uniformly for $z$ on any closed point set in $K$. Furthermore, the last equation is true uniformly for $z$ in $C+K$ if $\phi'(w)$ is nonvanishing and of bounded variation for $|w|=1$. Auxiliary results used in the proof include an extension of a known result on Riemann sums and a proof that if $\phi'(w)$ is absolutely continuous for $|w|=1$, then $[\phi(w)-\phi(\bar{w})]/[w-\bar{w}]$ is uniformly absolutely continuous in either $w$ or $\bar{w}$ for $|w|=1$. (Received April 18, 1941.)

### 306. M. H. Heins: *A generalization of the Aumann-Carathéodory "Starrheitssatz."*

Let $F$ denote a Riemann surface in the sense of Weyl-Radó with the property that its fundamental group is non-abelian. Then the identical map $W\equiv w$ of $F$ onto itself is isolated in the family of analytic maps $\{f(w)\}$ of $F$ into itself in the sense that there exists no sequence of maps of the family $\{f_n(w)\}$ $(f_n\not\equiv w,\ n=1, 2, \cdots)$ which converges pointwise to $w$ as $n\to\infty$. The proof is based upon a study of the transforms of $f(w)$ with respect to a uniformizing map by means of Julia's lemma.

This theorem implies the following immediate corollaries: (a) the theorem of Klein and Poincaré concerning the proper discontinuity of the group of (1, 1) conformal maps of $F$ onto itself where $F$ is a surface of the above type; (b) the theorem of Koebe stating that the number of (1, 1) conformal maps of a plane region of finite connectivity greater than two onto itself is finite; (c) the "Starrheitssatz" of Aumann and Carathéodory. (Received April 2, 1941.)

### 307. Einar Hille: *Notes on linear transformations.* III. *Semi-groups on Lebesgue spaces.*

This paper deals with the properties of semi-groups on $L_1(-\pi, \pi)$ and $L_1(-\infty, \infty)$, essentially those which are multiplicative in the representation by Fourier series, Fourier transforms or Hermitian series. The first two types coincide with the semi-groups commuting with real translations. The discussion requires the solution of the factor problems of type $(L, L)$ for Fourier transforms and for Hermitian series. Conditions are found for the convergence of $T_\alpha$ to $T_0$ in various topologies and the best degree of approximation is discussed. It is shown that any convex domain, invariant under addition, of the parameter plane can be the domain of existence of an analytic semi-group in $L_1(-\pi, \pi)$ and the rate of growth of $\|T_\alpha\|$ is studied when the domain is $\Re(\alpha)>0$. (Received May 5, 1941.)

### 308. Walter Leighton and W. J. Thron: *Convergence criteria for continued fractions.*

In the $z=x+iy$ plane let alternate coordinate axes $x'$, $y'$ be obtained by rotating the $x$, $y$ axes through an arbitrary angle $0\leqq\beta<\pi/2$. Let $R(\beta)$ be the open convex region containing $z=0$ whose boundary is the broken line contour consisting of the two arcs of the parabola $y'^2=\cos\beta\cdot(x'+\frac{1}{4}\cos\beta)$ from infinity to the points of tangency of this parabola with the tangent rays drawn from the point $z=-\frac{1}{4}$ and completed by the segments of these tangent rays extending from $z=-\frac{1}{4}$ to the points of tangency. If the $a_n$ are any complex numbers lying in any closed bounded region in $R(\beta)$, the continued fraction $1+K(a_n/1)$ converges. Further, for $\beta=\pi/2$ let $R_1$ be the open region defined by $-\frac{1}{4}<\Re(z)<0$, $\Im(z)>0$. If the $a_n$ lie in a bounded closed region in $R_1$, the given continued fraction converges. In this theorem $R_1$ may be replaced by $R_2$: $-\frac{1}{4}<\Re(z)<0$, $\Im(z)<0$. When $\beta=0$ the region described becomes the parabola of Wall and Scott. (Received May 9, 1941.)

### 309. E. J. McShane: *Sufficient conditions for a weak relative minimum in the problem of Bolza.*

In the problem of Bolza there remains one large gap between the necessary conditions for a minimum and the sufficient conditions. For anormal problems, the condition on the positiveness of the second variation which figures in the sufficient conditions can not be established among the necessary conditions. Here a substitute is proposed for the condition of positiveness of the second variation which is equivalent for normal problems but weaker in anormal problems. It is shown that with this weakened assumption the sufficiency theorem for weak relative minima can still be established. It is conjectured that the new condition (with positiveness replaced by non-negativeness) can be shown to be necessary for a minimum. (Received April 3, 1941.)

### 310. J. D. Mancill: *On the solutions of a certain class of partial differential equations.*

It is shown in this note that the most general solution of the partial differential equation $(\sum_{i=1}^{n} x_i \partial/\partial x_i)^k F(x) = mF(x)$, $i=1, 2, \cdots, n$, where $k$ is a positive integer and $m$ is any constant, is the sum of $p$ functions $F_j(x)$, $j=1, 2, \cdots, p$, $p \leqq k$, homogeneous of degree $r_j$ but otherwise arbitrary; the $r_j$ being the distinct roots of the equation $r(r-1)(r-2) \cdots (r-k+1) = m$. (Received May 19, 1941.)

### 311. E. N. Oberg: *Notes on the approximation of a function by sums of orthonormal functions.* Preliminary report.

Let $\phi_0, \phi_1, \phi_2, \cdots, \phi_n$ be a set of orthonormal functions on an interval $(a, b)$, each having a derivative at every point. If $S_n(x) = a_0\phi_0 + a_1\phi_1 + \cdots + a_n\phi_n$ is an arbitrary sum of the $\phi$'s, it is shown that a bound for $|(d/dx)S_n(x)|$ exists in terms of $S_n(x)$. A study is made of the degree of convergence of sums of the form $S_n(x)$ to a function $f(x)$ on $(a, b)$. Lastly, if $\Phi_n(x)$ is a sum of the $\phi$'s that minimizes the integral of $|f(x) - \Phi(x)|^p dx$, $p > 0$, then under certain hypotheses on $f(x)$, $\Phi_n(x)$ converges uniformly to $f(x)$ as $n \to \infty$. (Received April 7, 1941.)

### 312. George Polya and Norbert Wiener: *On the oscillation of the derivatives of a periodic function.*

Let $f(x)$ be periodic, real valued and infinitely derivable. Denote by $N_k$ the number of changes of sign of the $k$th derivative of $f(x)$, in a period. Let $k \to \infty$. *If $N_k$ is bounded, $f(x)$ is a trigonometric polynomial. If $N_k = o(k^{1/2})$, $f(x)$ is an entire function. If $N_k = O(k^{(1-\alpha)/2})$, where $0 < \alpha < 1$, $f(x)$ is an entire function of finite order not exceeding* $(1+\alpha^{-1})/2$. (Received April 19, 1941.)

### 313. W. T. Reid: *Green's lemma and related results.*

This paper is concerned with a proof of Green's lemma for the general case of a region $R$ which is the interior of a simply closed rectifiable curve $J$, and where the continuity and differentiability conditions imposed on the functions involved are merely on $R+J$ and $R$, respectively. There is also established a general form of Cauchy's theorem for a function $f(z)$ which is holomorphic on $R$, and merely continuous on $R+J$. Finally, there is considered an associated result which is useful in the derivation of edge conditions for double integral problems of the calculus of variations. (Received April 11, 1941.)

### 314. M. S. Robertson: *The partial sums of multivalently star-like functions.*

Let $f(z) = z^p + a_{p+1}z^{p+1} + \cdots + a_n z^n + \cdots$ be regular and multivalently star-like of order $p$ with respect to the unit circle $|z| < 1$. This means that within the unit circle $f(z)$ assumes no value more than $p$ times, at least one value exactly $p$ times, and in addition is star-like, that is, $\Re[zf'(z)/f(z)] > 0$, $|z| < 1$. When $p = 1$, $f(z)$ is univalently star-like. The author shows that for $n > n_0(p)$ the $n$th partial sum $f_n(z) = z^p + a_{p+1}z^{p+1} + \cdots + a_n z^n$ is also multivalently star-like of order $p$ for $|z| < 1 - A(p) \log n/n$ where $2p+1 \leqq A(p) \leqq 2p+2$. The order $\log n/n$ is best possible since the $n$th partial sum of the power series for $z^p(1-z)^{-2p}$ is multivalently star-like of order $p$ in a circle about the origin whose radius is at most $R_n$ where $R_n \geqq 1 - (2p+1) \log n/n$ and $\limsup_{n\to\infty}(1 - R_n/n^{-1} \log n) = 2p+1$. (Received April 3, 1941.)

### 315. L. B. Robinson: *Some complete systems of semitensors.*

The first problem is to find a complete system of semitensors of order one associated with the system of linear homogeneous differential equations (A) $y_i'' + \sum_{j=1}^{3} p_{ij} y_j' + \sum_{j=1}^{3} q_{ij} y_j = 0$ $(i = 1, 2, 3)$. The semitensors are transformed thus: $\bar{I} = \Delta_{1i} I_1 + \Delta_{2i} I_2 + \Delta_{3i} I_3$ where the $\Delta_{ji}$ are minors of the determinant of the transformation. Write $V_i \equiv D_{i1} I_1 + D_{i2} I_2 + D_{i3} I_3 \equiv f_i$, in which the $D_{ij}$ are minors of the determinant $|Z\ Y\ y|$ and the $f_i$ arbitrary functions of the covariants of A. Solve with respect to the $I_j$ and the complete system of semitensors results. To get semitensors of the second order, write $W_{ij} \equiv V_i \cdot V_j = f_{ij}$, wherein $f_{ij}$ is an arbitrary function of the covariants, assume $I_i I_j \equiv I_{ij} \equiv I_{ji}$, solve with respect to $I_{ij}$ and the complete system of semitensors is obtained. (Received April 7, 1941.)

### 316. W. J. Trjitzinsky: *Singular Lebesgue-Stieltjes integral equations.*

Integral equations, involving Stieltjes integrals, have been studied by a number of authors and, quite extensively, by N. Gunther. The kernels considered by Gunther are sufficiently "regular" to secure results resembling those of Fredholm and in the case of "symmetry," suitable defined, resembling those of Schmidt. The present work develops comprehensive theories of several types of integral equations, involving "symmetric" kernels essentially more general than those of Gunther; the kernels of the present work are representable as limits, in a suitable sense, of "regular" kernels. Our theory is based on Lebesgue-Stieltjes (Radon) integration, which appears to be an appropriate (in fact, necessary) tool for such problems. The developments are not of Fredholm type and they involve, in a very essential way, the use of an appropriate spectral theory. (Received April 3, 1941.)

## APPLIED MATHEMATICS

### 317. K. O. Friedrichs: *On the mathematical theory of the buckling of spherical shells.*

This paper is a contribution to von Kármán's recent nonlinear theory of buckling of spherical shells. The physically important state of equilibrium corresponds to a stationary value of the energy functional which is not a minimum but is of a degenerate character (no type number). The fact that buckling occurs only in the vicinity of one point is explained as a boundary layer phenomenon. For the analysis of the problem the Ritz method is employed. (Received April 2, 1941.)

### 318. F. G. Gravalos: *The algebraic integrals of Hill's equations.*

Hill's equations are $x'' - 2y' = 3x - \mu x/r^3$, $y'' + 2x' = -\mu y/r^3$, $r = (x^2 + y^2)^{1/2}$, $\mu$ a parameter. Such a system of differential equations admits of Jacobi's integral $\frac{1}{2}(x'^2 + y'^2) - \frac{3}{2}x^2 - \mu/r = c$. In this paper the theorem is proved: All the algebraic integrals of Hill's equations are functions of Jacobi's integral. The proof is done by parts in the form of three theorems. First, it is proved that all the integrals in the field $\Gamma(x', y', x, y, r)$ are reducible from Jacobi's. Second, the existence of integrals containing $t$ rationally is ruled out. Finally, the above theorem is proved. Some parts of the proof are the same as those in a paper by C. L. Siegel on the restricted problem of three bodies (Transactions of this Society, vol. 39 (1936), pp. 225–237). (Received April 8, 1941.)

## GEOMETRY

### 319. J. J. DeCicco: *Equilong geometry of series of collineal third order differential elements.*

The direct equilong group induces at a fixed line $(u, v)$ of the plane a six-parameter group $G_6$ between the differential elements of third order. Any three elements $(C_1, C_2, C_3)$ possess the fundamental invariant $(\delta_1 r_1+\delta_2 r_2+\delta_3 r_3)^2/(\delta_1 s_1+\delta_2 s_2+\delta_3 s_3)$, where $\delta_k$ is the distance from the point of $C_i$ to that of $C_j$, $r_i$ is the radius of curvature of $C_i$ and $s_i=dr_i/d\theta_i$ is the rate of variation of the radius of curvature per unit radian measure of the inclination $\theta_i$ at the fixed line. In general, $n$ elements possess $3n-6$ independent invariants, of which $n-1$ are distances, and the remaining $2n-5$ are of the new type given above. Any other invariant of these $n$ elements must be a function of these $3n-6$ independent invariants. A series $r=r(w)$, $s=s(w)$ possesses the two fundamental differential invariants $k_1=(d^2r/dw^2)/(d^3r/dw^3)$, and $k_2=(d^2r/dw^2)^2+(d^2s/dw^2)$. Any two series with their curvatures the same functions of the distance $w$ are equivalent under our group $G_6$. (Received May 21, 1941.)

### 320. Edward Kasner and J. J. DeCicco: *Conformal geometry of series of third order differential elements.*

Kasner and Comenetz have shown that at a fixed point of the plane the direct conformal group induces a six-parameter group $G_6$ between the differential elements of third order. In this paper it is found that three elements $(C_1, C_2, C_3)$ possess the fundamental invariant $(k_1 \sin a_1+k_2 \sin a_2+k_3 \sin a_3)^2/(l_1 \sin 2a_1+l_2 \sin 2a_2+l_3 \sin 2a_3)$, where $a_\lambda$ is the angle from $C_\mu$ to $C_\nu$, $k_\lambda$ is the curvature of $C_\lambda$, and $l_\lambda=dk_\lambda/ds_\lambda$ is the rate of variation of the curvature per unit length of arc at the fixed point. In general, $n$ elements possess $3n-6$ independent invariants, of which $n-1$ are angles and the remaining $2n-5$ are of the new type given above. Any other invariant of these $n$ elements must be a function of these $3n-6$ independent invariants. A series $k=k(\theta)$, $l=l(\theta)$ possesses the two fundamental differential invariants $\rho_1=(d^2k/d\theta^2+k)\div(d^3k/d\theta^3+dk/d\theta)$ and $\rho_2=(d^2k/d\theta^2+k)^2/(d^2l/d\theta^2+4l)$. Any two series with the curvatures $\rho_1$ and $\rho_2$ the same functions of the inclination $\theta$ are equivalent under the Kasner-Comenetz group $G_6$. (Received May 21, 1941.)

### 321. Edward Kasner and J. J. DeCicco: *General differential geometry of second order differential elements.*

Kasner (American Journal of Mathematics, vol. 28, pp. 203–213) showed that at a fixed point of the plane the entire group of arbitrary point transformations induces an eight-parameter group $G_8$ between differential elements of second order. He gave a complete discussion of all the invariants of $n$ elements with all the appropriate geometric interpretations. This new paper considers the differential geometry of a series $q=q(p)$ under this group $G_8$, where $p=dy/dx$ and $q=d^2y/dx^2$. The length of arc of a series is $s=\int[5q^{iv}q^{vi}-6(q^{v})^2]^{1/2}/q^{iv}dp$, and the curvature is $K=[25(q^{iv})^2q^{vii}-105q^{iv}q^{v}q^{vi}+84(q^{v})^3]/[5q^{iv}q^{vi}-6(q^{v})^2]^{3/2}$, where the superscripts denote differentiation with respect to $p$. Any two series with their curvatures $K$ the same functions of the arc length $s$ are equivalent under the group $G_8$. (Received May 21, 1941.)

### 322. D. T. McClay: *Clifford numbers.*

The system of numbers $a+b\Lambda$, with $a$ and $b$ real and $\Lambda^2=1$, first used by Clifford, is an algebra of Weierstrass. If these "Clifford numbers" are represented by points

$(a, b)$ in a rectangular coordinate system, the analytic functions correspond to transformations analogous to conformal transformations, the "angle" between two directions of slopes $m_1$, $m_2$ being defined as $\tanh^{-1}(m_2-m_1)/(1-m_1m_2)$. Lorentz transformations are a particular case. By adjoining suitably two intersecting lines at infinity, he plane of Clifford numbers may be mapped on a ring-shaped second order surface projective 3-space just as complex numbers are mapped stereographically on a here. The linear transformations of a Clifford variable give geometrical transforma-ns similar to the linear transformations of a complex variable, the metric $dx^2-dy^2$ acing the euclidean. The exponential function $e^{x+y\Lambda}=e^x(\cosh y+\Lambda \sinh y)$ is not odic, whereas $\sin (x+y\Lambda)=\sin x \cos y+\Lambda \cos x \sin y$ and the other trigonometric tions are doubly periodic. Cauchy's integral theorem and other complex variable orems hold unchanged, or with slight modifications. (Received April 2, 1941.)

### 23. Nelson Robinson: *A transformation between osculating curves a rational normal curve in an odd dimensional space.*

The equations of a rational normal curve $\Gamma_n$ in a linear space of $n$ dimensions are uced to a canonical form. Hyperquadrics $Q$, containing $\Gamma_n$, and osculating curves $_j$ of $\Gamma_n$ at a point $P$ are defined geometrically and their equations derived. By use a hyperquadric $Q$, a transformation involving a generalized null system is set up between osculating curves. This correspondence is proved to exist if and only if the mbient space is of odd dimensions. (Received May 14, 1941.)

## STATISTICS AND PROBABILITY

### 324. Alfred Basch: *A contribution to the theory of multiple correlation.*

If $r_{xy}$ and $r_{xz}$ are the correlation coefficients between $x$ and $y$, and $x$ and $z$, then $r_{yz}$ lies between the limits $r_{xy}r_{xz}\pm k_{xy}k_{xz}$, where $k_{xy}=(1-r_{xy}^2)^{1/2}$, $k_{xz}=(1-r_{xz})^{1/2}$ are the alienation coefficients. In a geometrical representation where $r_{xy}$ and $r_{xz}$ are the rectangular coordinates, the ellipses inscribed in the square with the sides $r_{xy}=\pm 1$, $r_{xz}=\pm 1$ are the loci of constant lower and constant upper limits of $r_{yz}$. Rays parallel to the three coordinate axes give the three contour ellipses of the standard ellipsoid. From these three contour ellipses can be obtained the three correlation coefficients. If two of the contour ellipses are given, then the third must satisfy restricting conditions in agreement with the limits given for the third correlation coefficient. The intersection ellipses of the standard ellipsoid with the coordinate planes are characteristic for the coefficients of the correlation between two variables, freed from the influence of the third. The "limit cases" correspond to the degeneration of the standard llipsoid, $r_{yz,\,x}=r_{xy,\,z}=r_{xz,\,y}$. In these limit cases there exists a functional relation between the three variables. The "middle case," $r_{yz}=r_{xy}r_{xz}$ is equivalent to $r_{yz,\,x}=0$. The standard ellipsoid intersects in this case the $yz$-plane in an ellipse, whose chief xes are the coordinate axes. (Received April 3, 1941.)

## TOPOLOGY

### 5. H. A. Arnold: *Homology in set-intersections, with an applica- r-regular convergence.*

$B$ are closed subsets of compact space $R$. Using Vietoris cycles the following proved: (1) If the complete $r$-cycle $\gamma^r$, carried by $A$ is $\sim 0$ in $A+B$, then

$A \cdot B$ contains a complete $r$-cycle $\mu^r$ (which may be the null cycle) with $\gamma^r \sim \mu^r$ in
(2) If the complete $r$-cycle $\gamma^r$ in $A \cdot B$ is $\sim 0$ in $A$ and $\sim 0$ in $B$, and the $(r-1)$st Be
number of $A+B$ is 0, then $\gamma^r \sim 0$ in $A \cdot B$. These and other similar lemmas are
to prove that if a sequence of continua has a $\gamma^1$-continuum as a limit, then there
be no isolated points of non 0-regular convergence. (For 0-regular convergence,
G. T. Whyburn, Fundamenta Mathematicae, vol. 25 (1935), pp. 408–426.) A poi
non 0-regular convergence is defined by the present writer to be a point of the
set, on every neighborhood of which the condition of 0-regular convergence is vio
This theorem is extended to the cases $r>1$. (Received April 3, 1941.)

### 326. H. A. Arnold: *On r-regular convergence of sets.*

Using the theory of $r$-regular convergence, created by G. T. Whyburn (see
menta Mathematicae, vol. 25 (1935), pp. 408–426), the following theore
proved for compact spaces: (1) If the sequence $\{M_n\}$ of closed $\gamma^r$-continua $M$
verges $s$-regularly to the limit set $M$, for all $s \leqq r$, then $M$ is a $\gamma^r$-continuum. (2)
sequence $\{M_n\}$ of closed sets $M_n$ converges $r$-regularly to the limit set $M$, th
every sequence of decompositions $M_{n_i} = A_{n_i} + B_{n_i}$ into closed sets, such th
sequence $\{X_i\} = \{A_{n_i} \cdot B_{n_i}\}$ converges to $X$ and the sequences $\{B_{n_i}\}$, $\{A_{n_i}\}$ co
$(r-1)$-regularly to $B$ and $A$ respectively, then $\{X_i\}$ converges $(r-1)$-regularly
(3) In the above notation, if $\{M_n\}$ converges to $M$, and $\{A_n \cdot B_n\}$ converges
$\{A \cdot B\}$, both $r$-regularly, then $\{A_n\}$ converges to $A$, and $\{B_n\}$ converges to $B$, b
$r$-regularly. (Received April 3, 1941.)

### 327. T. A. Botts: *On convex sets in linear normed spaces.*

This note contains a simple proof of the theorem that in a linear normed space convex bodies without common inner points are separated by a plane. (Rece
May 8, 1941.)

### 328. F. A. Ficken: *Certain systems of subsets of quasi and partia ordered sets.*

Let $S$ be a set which is quasi ordered, partially ordered, or a lattice, and let
denote the set of subsets of $S$. Assuming the existence of the necessary meets and jo
a subset $T$ which is a lattice is said to be a strict sublattice if both joins and mee
$T$ of appropriate subsets of $T$ agree with joins and meets in $S$, to be a $JS$-subla
($MS$-sublattice) if joins but not meets (meets but not joins) agree, and otherwi
be a loose sublattice. Subsystems of $U(S)$ are investigated which consist of those
sets of $S$ which are: lower-convex, lower-normal, convex, strict-sublattices, $MS$-
lattices, and so on. Among the many sublattices $L$ of $U(S)$ thus found are
which are $MU$-sublattices. Many of the $L$ are sublattices of others of the $L$.
been shown by Garrett Birkhoff that each quasi ordering of $S$ determines and
termined by a complete strict sublattice of $U(S)$. The present discussion yi
specialization of this result to partial orderings, lattices, and so on. (Received
1941.)

### 329. Witold Hurewicz: *On duality theorems.*

Let $A$ be a locally compact space, $B$ a closed subset of $A$, and $H^n($
$H^n(A-B)$ the $n$-dimensional cohomology groups of the sets $A$, $B$ and
integers as coefficients). Consider "natural homomorphisms" $H^n($

$^{n+1}(A-B)\to H^{n+1}(A)\to H^{n+1}(A-B)$. It can be shown that the kernel of each of these
momorphisms is the image of the preceding homomorphism. This statement con-
Kolmogoroff's generalization of Alexander's duality theorem and has many
ications. Using the preceding theorem one can prove: If $A$ and $B$ are compact
t es of dimensions $n$ and $m$ respectively, the necessary and sufficient condition that
i opological product $A\times B$ be of dimension $n+m$ is the existence of an open set
sp $A$ and an open set $V\subset B$ such that $H^n(U)$ and $H^m(V)$ contain elements $\alpha$ and
ti fying the following conditions: If the integer $d$ is a factor of the order of $\alpha$,
repl $\neq 0$ modulo $d$ (that is, there is no element $\gamma$ of $H^m(\dot{V})$ satisfying $\beta=d\gamma$); if the
peri $e$ is a factor of the order of $\beta$, then $\alpha\neq 0$ modulo $e$. (Received May 3, 1941.)
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# NEW PUBLICATIONS

Beatley, R. See Birkhoff, G. D.

Bergmann, S. Sur les Fonctions Orthogonales de Plusieurs Variables Complexes ave les Applications à la Théorie des Fonctions Analytiques. New York, Interscienc 1941. 62 pp. $1.50.

Birkhoff, G. D., and Beatley, R. Basic Geometry. Chicago, Scott, Foresman, 194 294 pp. $1.32.

Cameron, E. A. Brief Trigonometry. New York, Reynal and Hitchcock, 19 8+121+17 pp. $1.25.

Candy, A. L. Pandiagonal Magic Squares of Composite Order. Lincoln, Nebr A. L. Candy, 1941. 10+155 pp.

Coulson, C. A. Waves. A Mathematical Account of the Common Types of V Motion. New York, Interscience, 1941. 156 pp. $1.50.

Cowden, D. J. See Croxton, F. E.

Crathorne, A. R., and Moore, G. E. Brief Trigonometry. A Text in Twenty Assignments. New York, Holt, 1941. 5+121 pp. $1.20.

Croxton, F. E., and Cowden, D. J. Applied General Statistics. New York, Prentice-Hall, 1940. 18+944+13 pp. $4.00.

Davis, H. T. The Theory of Econometrics. Bloomington, Indiana, Principia Press, 1941. 14+482 pp. $4.00.

Föppl, A. Vorlesungen über technische Mechanik. Vol. 3. Festigkeitslehre. 12th edition. Munich, Oldenbourg, 1940. 465 pp. RM 12.

Fulton, J. F. See Ore, O.

Gale, A. R., and Watkeys, C. W. Elementary Functions and Applications. Revised New York, Holt, 1941. 21+409 pp. $2.25.

Longwell, C. R. See Ore, O.

Margenau, H. See Ore, O.

Miles, W. R. See Ore, O.

Mittmann, O. Erbbiologische Fragen in mathematischer Behandlung. Berlin, Gruyter, 1940. 265 pp. RM 22.

Moore, G. E. See Crathorne, A. R.

Oberdorfer, G. Lehrbuch der Elektrotechnik. Vol. 2. Rechenverfahren und al meine Theorien der Elektrotechnik. Munich, Oldenbourg, 1940. 377 pp. 18.50.

Ore, O., Schlesinger, F., Margenau, H., Timm, J. A., Longwell, C. R., W ruff, L. L., Miles, W. R., and Fulton, J. F. Development of the Sci Second Series. New Haven, Yale University Press; London, Humphrey Mil and Oxford University Press, 1941. 10+336 pp. $3.00.

Richardson, M. Fundamentals of Mathematics. New York, Macmillan, 194 +525 pp. $3.25.

Schlesinger, F. See Ore, O.

Timm, J. A. See Ore, O.

Watkeys, C. W. See Gale, A. R.

Wintner, A. The Analytical Foundations of Celestial Mechanics. (Princ Mathematical Series, no. 5.) Princeton, University Press; London, Hum Milford and Oxford University Press, 1941. 12+448 pp. $6.00.

Woodruff, L. L. See Ore, O.

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